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Constraints in Channel Shortening Equalizer Design for DMT-based Systems[1]

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Abstract

In discrete multitone receivers, a time domain equalizer (TEQ) is used to shorten the channel impulse response, so that the equalized channel impulse response is shorter than the inserted prefix. The aim of this paper is to show that the MMSE channel shortening problem with two different energy constraints, remarkably, lead to the same TEQ coefficients, up to a scaling factor. Moreover, implying the two energy constraints together in the MMSE optimization again yields the same result and comes down to a canonical correlation analysis between the subspace spanned by the transmitted samples and the received samples respectively. Hence, the TEQ obtained by these three distinct MMSE cases yields the same performance in terms of bit rate.

Since the resulting problem can easily be reformulated as a maximization problem, an iterative procedure based on power iterations can be devised to reduce the computational complexity.

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Lower case bold–faced letters are used to denote vectors and upper case bold–faced letters to denote matrices. In addition the following notation is used throughout the paper :

$\mathbf{A}^{\mathrm{T}}$	transpose of $\mathbf{A}$
$\mathbf{A}^{1/2}$	Cholesky factor of $\mathbf{A}$, i.e. $\mathbf{A} = \mathbf{A}^{\mathrm{T}/2}\mathbf{A}^{1/2}$
J	cost function
$\mathbf{e}_k$	zero vector with 1 on the kth position
$\nabla_{\mathbf{a}} J$	gradient of cost function J with respect to $\mathbf{a}$
$\mathcal{E}\{\cdot\}$	expected value
$\|\|\mathbf{a}\|\|_2$	Euclidean norm of $\mathbf{a}$
$\mathbf{I}$	identity matrix
$\lfloor a \rfloor$	floor(a)

Other notation is explained in the text.

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Abstract

In discrete multitone receivers, a time domain equalizer (TEQ) is used to shorten the channel impulse response, so that the equalized channel impulse response is shorter than the inserted prefix. The aim of this paper is to show that the MMSE channel shortening problem with two different energy constraints, remarkably, lead to the same TEQ coefficients, up to a scaling factor. Moreover, implying the two energy constraints together in the MMSE optimization again yields the same result and comes down to a canonical correlation analysis between the subspace spanned by the transmitted samples and the received samples respectively. Hence, the TEQ obtained by these three distinct MMSE cases yields the same performance in terms of bit rate.

Since the resulting problem can easily be reformulated as a maximization problem, an iterative procedure based on power iterations can be devised to reduce the computational complexity.

1 Introduction

In discrete multitone (DMT) transceivers, a guard time sequence, called a cyclic prefix ν, is inserted between symbols to ensure that samples from one symbol do not interfere with the samples from another symbol. The length of the cyclic prefix is determined by the length of the impulse response of the channel. Using a long prefix, however, reduces the throughput of the receiver, resulting in a loss of useful capacity. To avoid using large prefixes, a time domain equalizer (TEQ) is used to *shorten* the effective channel impulse response and the remaining amplitude and phase distortion in the frequency domain is solved by one tap frequency domain equalizers (FEQ).

The TEQ design problem has been addressed in several papers, see e.g. [1][2][7][3]. More in particular we will focus on Minimum Mean Square Error (MMSE) TEQ design that is based on the following principle [1][2]: the transmitted signal[1] $\mathbf{x}_k$ is passed through the channel $\mathbf{h}$ and is equalized by the TEQ $\mathbf{w}$ with T coefficients. This equalized signal is compared with the input signal filtered by a $\nu + 1$ taps target impulse response (TIR) $\mathbf{b}$ and delayed with Δ (see **Fig. 1**). The difference e_k is then minimized in mean square sense with respect to the unknowns $\mathbf{w}$ and $\mathbf{b}$. For further details, we refer to [1][2].

[1] $\mathbf{x}_k$ indicates a vector of $\nu + 1$ consecutive transmitted samples, while $\mathbf{y}_k$ determines a vector of T received samples.

Referring to **Fig. 1**, the TEQ coefficients can be calculated by minimizing the error sequence. The mean square error (MSE) can be expressed as

$$\begin{aligned} J(\mathbf{w},\mathbf{b}) &= \mathcal{E}\{(e_k)^2\} && (1)\\ &= \mathcal{E}\{(\mathbf{w}^{\mathrm{T}}\mathbf{y}_k - \mathbf{b}^{\mathrm{T}}\mathbf{x}_{k-\Delta})^2\} && (2)\\ &= \mathbf{w}^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w} + \mathbf{b}^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b} - 2\mathbf{b}^{\mathrm{T}}\mathbf{R}_{yx}(\Delta)^{\mathrm{T}}\mathbf{w} && (3)\end{aligned}$$

where $\mathbf{R}_{xx}$, $\mathbf{R}_{yy}$ are the autocorrelation matrices for $\mathbf{x}_k$ and $\mathbf{y}_k$ respectively and $\mathbf{R}_{yx}(\Delta)$ is the corresponding cross-correlation matrix which is a function of the delay parameter Δ. For the sake of compact notation we will use $\mathbf{R}_{yx}$ instead of $\mathbf{R}_{yx}(\Delta)$. It is clear that the trivial solution $\mathbf{w} = \mathbf{b} = \mathbf{0}$ should be avoided by adding a constraint, e.g. unit-norm constraints $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 1$, $\mathbf{b}^{\mathrm{T}}\mathbf{b} = 1$; unit-tap constraints $\mathbf{e}_1^{\mathrm{T}}\mathbf{w} = 1$, $\mathbf{e}_1^{\mathrm{T}}\mathbf{b} = 1$; unit-energy constraints $\mathbf{b}^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b} = 1$, $\mathbf{w}^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w} = 1$, **...**

In [2] only unit-tap and unit-norm constraints were treated in an extensive way. Here, we will investigate MMSE TEQ design subject to intuitively appealing unit-energy constraints and prove a number of properties of the solution vectors. According to the authors of [2] a unit-norm constraint has better performance than a unit-tap constraint since a TEQ optimization with a unit-norm constraint reaches a lower MMSE value. However, in DSL only the achievable bit rate is of real importance, which is not addressed in [2]. Since the MMSE value and the bit rate are not related in a linear way different constraints can lead to optimal performance.

Our focus lies on the unit-energy constraints since they lead to high performance for almost all DSL loops, despite the unpredictable behaviour of all convex TEQ algorithms. The aim of this paper is not to device a new TEQ criterion which truly optimizes channel capacity or robustness, but rather to give an insight in the relations among unit-energy constraints in the case Minimum Mean Square Error (MMSE) TEQ design is considered. These relations lead to a common optimization strategy - based on power iterations - for the different constraints. The obtained algorithm requires less iterations for convergence compared to ordinary LMS on TEQ and TIR coefficients.

More advanced equalizer structures include e.g. the per tone equalizers [9]. Despite its optimal performance, the per tone equalizers have much higher memory requirements compared to a conventional TEQ. Hence, research for cheap TEQ design is still ongoing. Other TEQ design methods try to optimize the bit rate [1] but they rely on unattractive non convex optimization. Melsa *et al* [7] proposed a cost function based on the Maximum Shortening SNR

(MSSNR) to shorten the effective channel such that inter symbol interference (ISI) is minimized. This cost function does not take into account the channel noise often resulting in 'noise pick up' by the demodulating FFT sidelobes. More recent results [3] generalize this MSSNR method and claim to achieve near optimal bitrate performance. However, the sidelobe effects of the FFT after filtering the received signal with the TEQ are neglected, leading to an incorrect bit rate calculation. In this paper, simulation results will indicate that MMSE TEQ design with unit-energy constraints can achieve higher bit rates than the MSSNR method and its generalization.

The paper is organized as follows: section 2 treats the MMSE TEQ design solutions for two different unit-energy constraints as well as for the joint application of these constraints. A cheap iterative procedure is presented in section 3. Simulation results and conclusions are presented in the final sections.

2 MMSE solution with unit-energy constraints

Here, we will investigate three unit-energy constrained cases, namely

- case 1: $\min_{\mathbf{w}_1,\mathbf{b}_1} J(\mathbf{w}_1,\mathbf{b}_1)$ subject to $\mathbf{b}_1^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b}_1 = 1$
- case 2: $\min_{\mathbf{w}_2,\mathbf{b}_2} J(\mathbf{w}_2,\mathbf{b}_2)$ subject to $\mathbf{w}_2^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w}_2 = 1$
- case 3: $\min_{\mathbf{w}_3,\mathbf{b}_3} J(\mathbf{w}_3,\mathbf{b}_3)$ subject to $\mathbf{w}_3^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w}_3 = 1$ and $\mathbf{b}_3^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b}_3 = 1$

In appendix A the following remarkable properties for the optimal solutions are demonstrated:

$$\mathbf{w}_1 = \sigma \cdot \mathbf{w}_2 \tag{4}$$
$$\mathbf{b}_1 = \sigma^{-1} \cdot \mathbf{b}_2 \tag{5}$$
$$\mathbf{w}_2 = \mathbf{w}_3 \tag{6}$$
$$\mathbf{b}_1 = \mathbf{b}_3 \tag{7}$$
$$J_{min}(\mathbf{w}_1,\mathbf{b}_1) = J_{min}(\mathbf{w}_2,\mathbf{b}_2) \leq J_{min}(\mathbf{w}_3,\mathbf{b}_3) \tag{8}$$

where σ is a scaling factor and J_{min} is the MMSE. Remarkably, the three cases lead to the same TEQ coefficients up to a scaling factor. Hence, the performance in terms of achievable bit rates will be equal since the difference in scaling will be adopted by the one tap FEQs. By this, we also illustrate that $\mathbf{w}_3$ is not inferior to the other two solutions, despite its higher MMSE value.

In appendix A it is also shown that the unit-energy solutions $\mathbf{w}_1 = \sigma\mathbf{w}_2 = \sigma\mathbf{w}_3$ can be obtained (up to a scalar) by taking the most dominant eigenvector of one and the same matrix, i.e. $\mathbf{B} = \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}}$. This will lead to a unified iterative procedure to obtain the TEQ coefficients for the three cases.

3 Iterative procedure

Since the TEQs for the three cases can be obtained by taking the most dominant eigenvector of $\mathbf{B}$, a simple iterative procedure based on *power iterations* [6] exists to calculate the TEQs. As the scaling factor between $\mathbf{w}_1$ and $\mathbf{w}_2$ is irrelevant in view of the obtained performance, the solution we actually select might as well be $\mathbf{w} = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}$. The iterative algorithm can be described as:

$$
\begin{aligned}
&\mathbf{w}^{(0)} = \mathbf{w}_{init};\\
&\mathbf{B} = \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}} = \underbrace{(\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx})}_{\mathbf{B}_1}\cdot\underbrace{(\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}})}_{\mathbf{B}_2};\\
&\text{for} \quad i = 1 : N_{iter}\\
&\qquad \mathbf{w}^{(i)} = \mathbf{B}\mathbf{w}^{(i-1)};\\
&\qquad \mathbf{w}^{(i)} = \mathbf{w}^{(i)}/\sqrt{\mathbf{w}^{(i)\mathrm{T}}\mathbf{w}^{(i)}};\\
&\text{end.}
\end{aligned}
\tag{9}
$$

A similar procedure may be set up for the computation of $\mathbf{b}_1$ and $\mathbf{b}_2$, based on a matrix $\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx} = \mathbf{B}_2\mathbf{B}_1$. Finally, the two procedures may then be combined as follows

$$
\begin{aligned}
&\mathbf{w}^{(0)} = \mathbf{w}_{init};\\
&\mathbf{B}_1 = \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx};\\
&\mathbf{B}_2 = \mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}};\\
&\text{for} \quad i = 1 : N_{iter}\\
&\qquad \mathbf{b}^{(i)} = \mathbf{B}_2\mathbf{w}^{(i-1)}; \qquad \text{(similar to (13))}\\
&\qquad \mathbf{w}^{(i)} = \mathbf{B}_1\mathbf{b}^{(i)}; \qquad \text{(see (13))}\\
&\qquad \mathbf{w}^{(i)} = \mathbf{w}^{(i)}/\sqrt{\mathbf{w}^{(i)\mathrm{T}}\mathbf{w}^{(i)}};\\
&\text{end.}
\end{aligned}
\tag{10}
$$

This can easily be implemented since $\mathbf{R}_{xx}$ and $\mathbf{R}_{yy}$ are Toeplitz matrices for which cheap matrix inversion algorithms exist (e.g. the Trench algorithm [6]). Moreover, if chip memory is limited, $\mathbf{R}_{yy}^{-1}$ (size $T \times T$) and $\mathbf{R}_{xx}^{-1}$ (size $(\nu+1) \times (\nu+1)$) don't need to be stored since they can be recomputed (implicitly[2]) in

[2]Instead of computing $\mathbf{b}^{(i)} = \mathbf{B}_2\mathbf{w}^{(i-1)}$ and $\mathbf{w}^{(i)} = \mathbf{B}_1\mathbf{b}^{(i)}$ in every iteration, it is cheaper to solve the linear sets of equations: $\mathbf{R}_{xx}\mathbf{b}^{(i)} = \mathbf{R}_{yx}^{\mathrm{T}}\mathbf{w}^{(i-1)}$ and $\mathbf{R}_{yy}\mathbf{w}^{(i)} = \mathbf{R}_{yx}\mathbf{b}^{(i)}$.

every iteration. Of course computational complexity will increase leading to a trade-off memory versus complexity.

Simulation results will show that the obtained algorithm requires a few hundreds of iterations for complete convergence whereas [11] reported that an ordinary LMS on TEQ and TIR coefficients needs thousands of iterations to achieve the optimal performance.

4 Simulation Results

We present downstream simulation results to analyze the presented theory and to compare the performance of the MMSE method with unit-energy constraints with the MMSE method with other constraints. The eight standard carrier-serving-area (CSA) loops are used as test channels. For a configuration of these channels, we refer to [4]. The channel impulse responses consist of 512 samples at 2.208MHz. Analog and digital front end filters plus POTS splitter are included in the channel model.

The sampling rate F_s at transmitter and receiver is 2.208 MHz, transmit IFFT and receive FFT size is 512, cyclic prefix (ν) is 32. Additive white Gaussian noise of -140dBm/Hz, echo noise and near-end-crosstalk (NEXT) of 24 ADSL disturbers were added. The transmit signal power equals -40dBm/Hz and is equally distributed over all used subchannels (tones 38 to 255). SNR calculations are based on the signal model in [9]. The achieved bit rate for a particular TEQ design can then be written as

$$\text{bitrate} = \frac{F_s}{N+\nu} \sum_{i=38:255} \lfloor \log_2 \left(1 + \frac{\text{SNR}_i}{\Gamma}\right) \rfloor \tag{11}$$

with Γ the SNR gap of $9.8+6-3 = 12.8$dB for a bit error probability $P_e = 10^{-7}$. This corresponds to a system margin of 6dB and a coding gain of 3dB.

4.1 Bit rate performance

To indicate the importance of unit-energy constraints in MMSE TEQ design, simulations were done with several possible constraints for the different reference loops. The MMSE channel shortening combined with one of these constraints has no direct relation to the capacity of the system. Hence, it is difficult to predict which constraint will be the best for a certain channel. Although the channel shortening cost function with unit-energy constraints does not optimize capacity, it still results in an acceptably large bit rate for a large number of

channels. This is indicated in **Fig. 2.** The obtained bit rate is plotted as a function of the used CSA reference loop for $T = 16$. The used constraints are: unit-energy constraints $\mathbf{b}^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b} = 1$ and/or $\mathbf{w}^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w} = 1$, unit-norm constraints $\mathbf{b}^{\mathrm{T}}\mathbf{b} = 1$ and $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 1$ and unit-tap constraints $\mathbf{e}_{\mathrm{I}}^{\mathrm{T}}\mathbf{b} = 1$ and $\mathbf{e}_{\mathrm{I}}^{\mathrm{T}}\mathbf{w} = 1$.

As benchmark the performance of the optimal SSNR TEQ design method [7] and the minimum ISI method [4] is also indicated. Delay optimization has been applied by running all methods over a large range of delays. The maximum bit rate over this delay range is depicted.

Referring to **Fig. 2,** the bit rates achieved with MMSE TEQ design are for all loops sufficiently high. It can be seen that $\mathbf{b}^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b} = 1$ and $\mathbf{w}^{\mathrm{T}}\mathbf{R}_{yy}\mathbf{w} = 1$ are useful constraints when MMSE TEQ design is under consideration. It is also noticed that the optimal SSNR is possibly not the best measure for TEQ design since it doesn't take into account the channel noise. Even the minimum ISI method does not always outperform the MMSE method – as claimed in [4] – since inter-carrier-interference (ICI) introduced by the FFT sidelobes is not taken into account in the design criterium.

4.2 Iterative procedure for unit-energy constraints

In **Fig. 3** the convergence for the iterative procedure of (10) is given for the different CSA loops. The bit rate as a function of the number of power iterations is plotted. The simulation was performed with $T = 16$ for optimal (i.e. in the sense of maximum bit rate) delay settings, $\mathbf{w}_{init} = [0 \ldots 0 \quad 1]^{\mathrm{T}}$. Hence, the optimal achievable bit rate can be found in **Fig. 2. Fig. 3** shows that initial convergence is fast, but convergence slows down to reach eventually the true optimal bit rate, obtained with an eigenvalue decomposition.

5 Conclusions

In this paper we investigated the MMSE criterion for TEQ design under three distinct energy constraints: unit energy constraint for the target signal, unit energy for the equalized signal and both the first and the second together. We have shown that the solution for all cases are essentially identical (up to a scaling factor) and hence yield the same performance in terms of bit rate. The mathematical derivation revealed a cheap and common iterative method to obtain the solutions for all three cases. Simulation results showed that, when the

MMSE criterion is under consideration for TEQ design, unit energy constraints lead in general to very good performance with high bit rates.

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A Proofs

- In case 1, the gradient with respect to $\mathbf{w}_1$ is equal to zero at the optimum of J, i.e.

$$\begin{aligned} \nabla_{\mathbf{w}_1} J &= 2(\mathbf{R}_{yy}\mathbf{w}_1 - \mathbf{R}_{yx}\mathbf{b}_1) = 0 && (12)\\ \Rightarrow \mathbf{w}_1 &= \mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{b}_1. && (13) \end{aligned}$$

With this relation, $\mathbf{w}_1$ can be eliminated from the expression for J

$$\begin{aligned} \min_{\mathbf{b}_1} J = & \min_{\mathbf{b}_1} \mathbf{b}_1^{\mathrm{T}} \cdot (\mathbf{R}_{xx} - \mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}) \cdot \mathbf{b}_1 \quad \text{such that} \quad \mathbf{b}_1^{\mathrm{T}}\mathbf{R}_{xx}\mathbf{b}_1 = 1 && (14)\\ = & 1 - \max_{\tilde{\mathbf{b}}_1} \tilde{\mathbf{b}}_1^{\mathrm{T}} \cdot (\mathbf{R}_{xx}^{-\mathrm{T}/2}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1/2}) \cdot \tilde{\mathbf{b}}_1 && (15)\\ = & 1 - \max_{\tilde{\mathbf{b}}_1} \tilde{\mathbf{b}}_1^{\mathrm{T}} \cdot (\mathbf{A}^{\mathrm{T}}\mathbf{A}) \cdot \tilde{\mathbf{b}}_1 \quad \text{such that} \quad \tilde{\mathbf{b}}_1^{\mathrm{T}}\tilde{\mathbf{b}}_1 = 1 && (16) \end{aligned}$$

where $\tilde{\mathbf{b}}_1$ is a transformed version of $\mathbf{b}_1$,

$$\tilde{\mathbf{b}}_1 = \mathbf{R}_{xx}^{1/2}\mathbf{b}_1, \qquad (17)$$

Hence, the optimal $\tilde{\mathbf{b}}_1$ can be found as the eigenvector corresponding to the maximum eigenvalue of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ with $\mathbf{A} = \mathbf{R}_{yy}^{-\mathrm{T}/2}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1/2}$. Using (17) results in the optimal TIR.

- For case 2, one can proof (similarly to case 1) that $\mathbf{w}_2$ is the solution of

$$\min_{\mathbf{w}_2} J = 1 - \max_{\tilde{\mathbf{w}}_2} \tilde{\mathbf{w}}_2^{\mathrm{T}} \cdot (\mathbf{A}\mathbf{A}^{\mathrm{T}}) \cdot \tilde{\mathbf{w}}_2 \qquad \text{such that} \quad \tilde{\mathbf{w}}_2^{\mathrm{T}} \tilde{\mathbf{w}}_2 = 1 \quad (18)$$

where $\tilde{\mathbf{w}}_2 = \mathbf{R}_{yy}^{1/2} \mathbf{w}_2$. This time, the optimal TEQ can be obtained from the eigenvector corresponding to the maximum eigenvalue of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$. Since the eigenvalues of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ are equal, both unit energy constraints result in the same MMSE, i.e.

$$J_{min}(\mathbf{w}_1, \mathbf{b}_1) = J_{min}(\mathbf{w}_2, \mathbf{b}_2) = 1 - \lambda_{max}, \quad (19)$$

where λ_{max} is the largest eigenvalue of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

- If λ_{max} is the largest eigenvalue of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\tilde{\mathbf{b}}_1$ the corresponding eigenvector then the following holds

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} \cdot \tilde{\mathbf{b}}_1 = \lambda_{max} \cdot \tilde{\mathbf{b}}_1. \quad (20)$$

By multiplying each side with $\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1/2}$ and by using (17) and (13), we obtain

$$\underbrace{\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}}}_{\mathbf{B}} \cdot \mathbf{w}_1 = \lambda_{max} \cdot \mathbf{w}_1 \quad (21)$$

In a similar way, vector $\tilde{\mathbf{w}}_2$ is such that

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} \cdot \tilde{\mathbf{w}}_2 = \lambda_{max} \cdot \tilde{\mathbf{w}}_2 \quad (22)$$

$$\underbrace{\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}}}_{\mathbf{B}} \cdot \mathbf{w}_2 = \lambda_{max} \cdot \mathbf{w}_2 \quad (23)$$

with the same value of λ_{max} as in formulas (20) and (21). From (21) and (23), it is seen that the TEQs of the two different optimization problems are eigenvectors of the same matrix $\mathbf{B}$ corresponding to the same eigenvalue[3] λ_{max}.

- In case 3, it is easy to show that the optimal TEQ and TIR can be obtained from a generalized eigenvalue problem [6].

$$\begin{bmatrix} \mathbf{0} & \mathbf{R}_{yx} \\ \mathbf{R}_{yx}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w}_3 \\ \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{yy} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{xx} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w}_3 \\ \mathbf{b}_3 \end{bmatrix} \cdot (1 - \alpha). \quad (24)$$

[3]The contrived case of an algebraic multiplicity larger than one is not considered here.

Since the cost function can be simplified into

$$J(\mathbf{w}_3, \mathbf{b}_3) \quad = \quad 2 - 2\mathbf{b}_3^{\mathrm{T}}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{w}_3 = 2 \cdot \alpha, \tag{25}$$

the minimum is reached for the smallest possible α or the optimal solution corresponds to the generalized eigenvector belonging to the largest generalized eigenvalue.

- The scaling factor between $\mathbf{w}_1$ and $\mathbf{w}_2$ can be computed as follows. Consider first the singular value decomposition of $\mathbf{A}$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}. \tag{26}$$

Then we know from (20) and (22) that $\tilde{\mathbf{b}}_1 = \mathbf{V}(:,1)$ and $\tilde{\mathbf{w}}_2 = \mathbf{U}(:,1)$. By applying (13) and (17), the norm of $\mathbf{w}_1$ becomes

$$\begin{aligned}
\mathbf{w}_1^{\mathrm{T}}\mathbf{w}_1 &= \tilde{\mathbf{b}}_1^{\mathrm{T}} \cdot \mathbf{R}_{xx}^{-\mathrm{T}/2}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-\mathrm{T}}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1/2} \cdot \tilde{\mathbf{b}}_1 && (27)\\
&= \sigma_{max} \cdot \mathbf{U}(:,1)^{\mathrm{T}} \cdot \mathbf{R}_{yy}^{-\mathrm{T}/2}\mathbf{R}_{yy}^{-1/2} \cdot \mathbf{U}(:,1) \cdot \sigma_{max} && (28)\\
&= \sigma_{max}^2 \cdot \mathbf{w}_2^{\mathrm{T}}\mathbf{w}_2 && (29)
\end{aligned}$$

In conclusion, $\mathbf{w}_1 = \pm\sigma_{max} \cdot \mathbf{w}_2$, i.e. the scaling factor between the TEQs is the largest singular value, σ_{max}, of $\mathbf{A}$.

- A similar derivation can be given for the TIRs. It can be proven that $\mathbf{b}_1$ and $\mathbf{b}_2$ are eigenvectors belonging to the largest eigenvalue of matrix $\mathbf{R}_{xx}^{-1}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-1}\mathbf{R}_{yx}$ and that $\mathbf{b}_2^{\mathrm{T}}\mathbf{b}_2 = \sigma_{max}^2 \cdot \mathbf{b}_1^{\mathrm{T}}\mathbf{b}_1$, so that $\mathbf{b}_1 = \pm\sigma_{max}^{-1} \cdot \mathbf{b}_2$.

- It is also possible to relate the solution obtained in case 3, with the solutions computed in case 1 and 2. Since $\mathbf{R}_{yy} = \mathbf{R}_{yy}^{\mathrm{T}/2}\mathbf{R}_{yy}^{1/2}$ and $\mathbf{R}_{xx} = \mathbf{R}_{xx}^{\mathrm{T}/2}\mathbf{R}_{xx}^{1/2}$ we can turn the generalized eigenvalue problem into an ordinary eigenvalue problem:

$$\begin{bmatrix} \mathbf{0} & \mathbf{R}_{yy}^{-\mathrm{T}/2}\mathbf{R}_{yx}\mathbf{R}_{xx}^{-1/2} \\ \mathbf{R}_{xx}^{-\mathrm{T}/2}\mathbf{R}_{yx}^{\mathrm{T}}\mathbf{R}_{yy}^{-1/2} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{R}_{yy}^{1/2}\mathbf{w}_3 \\ \mathbf{R}_{xx}^{1/2}\mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{yy}^{1/2}\mathbf{w}_3 \\ \mathbf{R}_{xx}^{1/2}\mathbf{b}_3 \end{bmatrix} \cdot (1-\alpha) \tag{30}$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \tilde{\mathbf{w}}_3 \\ \tilde{\mathbf{b}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{w}}_3 \\ \tilde{\mathbf{b}}_3 \end{bmatrix} \cdot (1-\alpha) \tag{31}$$

$$\tag{32}$$

where $\tilde{\mathbf{w}}_3$ and $\tilde{\mathbf{b}}_3$ are defined in the same way as $\tilde{\mathbf{b}}_1$ and $\tilde{\mathbf{w}}_2$. With the singular value decomposition of $\mathbf{A}$ (Eq. (26)), the optimal solution becomes $\tilde{\mathbf{w}}_3 = \mathbf{V}(:,1)$ and $\tilde{\mathbf{b}}_3 = \mathbf{U}(:,1)$, which means that $\mathbf{w}_3 = \mathbf{w}_2$ and $\mathbf{b}_3 = \mathbf{b}_1$.

It can easily be seen that the MMSE from (25) can be redefined in terms of the largest singular value of $\mathbf{A}$

$$J_{min}(\mathbf{w}_3, \mathbf{b}_3) = 2 \cdot (1 - \sigma_{max}). \tag{33}$$

Since $J_{min}(\mathbf{w}_1, \mathbf{b}_1) = J_{min}(\mathbf{w}_2, \mathbf{b}_2) = 1 - \sigma_{max}^2 \geq 0$, $J_{min}(\mathbf{w}_3, \mathbf{b}_3)$ is always greater or equal than $J_{min}(\mathbf{w}_1, \mathbf{b}_1)$.

□

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List of Figure Captions

1. Block diagram used for MMSE channel shortening.
2. Channel capacity for different MMSE TEQ design methods as a function of the CSA loop number, $T = 16$. From left to right: unit-energy constraint; unit-norm constraints $\mathbf{b}^{\mathrm{T}}\mathbf{b} = 1$, $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 1$ and unit-tap constraints $\mathbf{e}_1^{\mathrm{T}}\mathbf{b} = 1$, $\mathbf{e}_1^{\mathrm{T}}\mathbf{w} = 1$. As a reference the optimal SSNR and the minimum ISI TEQ design methods are also depicted.
3. Convergence of the iterative MMSE method with unit-energy constraint for the eight CSA loops as a function of the number of iterations, $T = 16$ and optimal delay.

List of Footnotes

1. $\mathbf{x}_k$ indicates a vector of $\nu + 1$ consecutive transmitted samples, while $\mathbf{y}_k$ determines a vector of T received samples.
2. Instead of computing $\mathbf{b}^{(i)} = \mathbf{B}_2\mathbf{w}^{(i-1)}$ and $\mathbf{w}^{(i)} = \mathbf{B}_1\mathbf{b}^{(i)}$ in every iteration, it is cheaper to solve the linear sets of equations: $\mathbf{R}_{xx}\mathbf{b}^{(i)} = \mathbf{R}_{yx}^{\mathrm{T}}\mathbf{w}^{(i-1)}$ and $\mathbf{R}_{yy}\mathbf{w}^{(i)} = \mathbf{R}_{yx}\mathbf{b}^{(i)}$.
3. The contrived case of an algebraic multiplicity larger than one is not considered here.

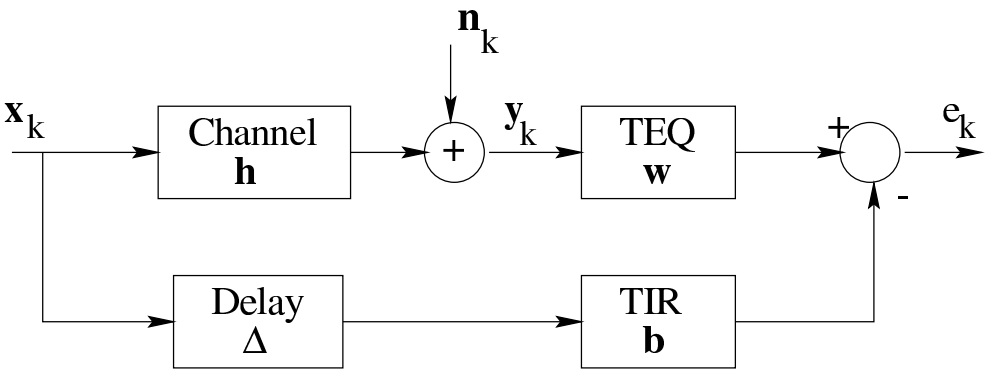

Figure 1: Block diagram used for MMSE channel shortening.

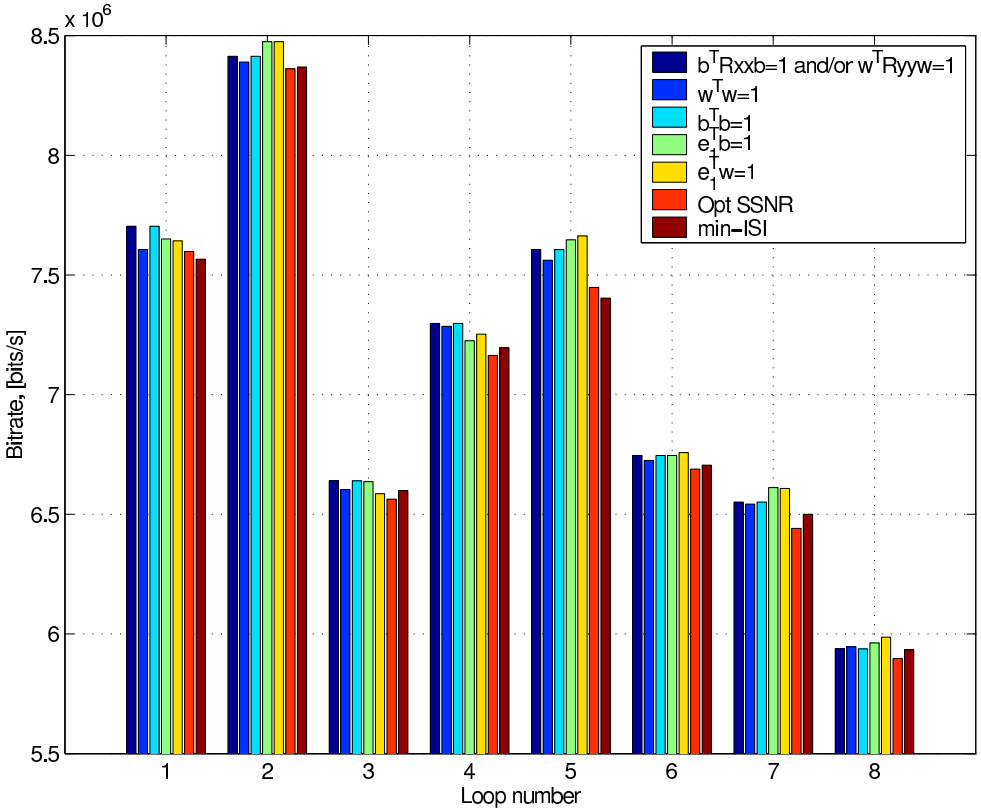

Figure 2: Channel capacity for different MMSE TEQ design methods as a function of the CSA loop number, $T = 16$. From left to right: unit-energy constraint; unit-norm constraints $\mathbf{b}^{\mathrm{T}}\mathbf{b} = 1$, $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 1$ and unit-tap constraints $\mathbf{e}_1^{\mathrm{T}}\mathbf{b} = 1$, $\mathbf{e}_1^{\mathrm{T}}\mathbf{w} = 1$. As a reference the optimal SSNR and the minimum ISI TEQ design methods are also depicted.

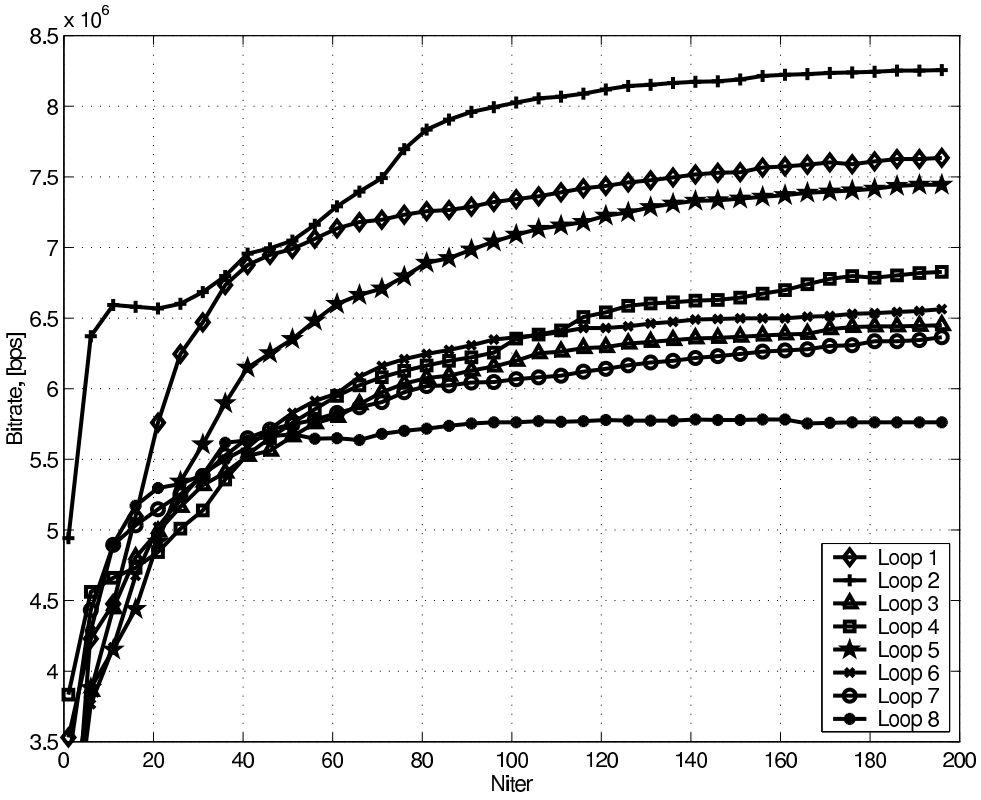

Figure 3: Convergence of the iterative MMSE method with unit-energy constraint for the eight CSA loops as a function of the number of iterations, $T = 16$ and optimal delay.