Newton-type Alternating Minimization Algorithm for Convex Optimization

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***Abstract*—We propose NAMA (Newton-type Alternating Minimization Algorithm) for solving structured nonsmooth convex optimization problems where the sum of two functions is to be minimized, one being strongly convex and the other composed with a linear mapping. The proposed algorithm is a line-search method over a continuous, real-valued, exact penalty function for the corresponding dual problem, which is computed by evaluating the augmented Lagrangian at the primal points obtained by alternating minimizations. As a consequence, NAMA relies on exactly the same computations as the classical alternating minimization algorithm (AMA), also known as the dual proximal gradient method. Under standard assumptions the proposed algorithm converges with global sublinear and local linear rates, while under mild additional assumptions the asymptotic convergence is superlinear, provided that the search directions are chosen according to quasi-Newton formulas. Due to its simplicity, the proposed method is well suited for embedded applications and large-scale problems. Experiments show that using limited-memory directions in NAMA greatly improves the convergence speed over AMA and its accelerated variant.**

I. Introduction

We consider convex optimization problems of the form

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x)+g(Ax), \tag{P}$$

where f is strongly convex, g is convex and A is a linear mapping. Problems of this form are quite general and appear in various areas of applications, including optimal control [1], system identification [2] and machine learning [3], [4]. For example, whenever g is the indicator function of a convex set C, then (P) models a constrained convex problem: if C is a box, then in particular (P) amounts to minimizing a strongly convex function subject to polyhedral constraints.

A general approach to the solution of (P) is based on the dual proximal gradient method, or forward-backward splitting, also known as alternating minimization algorithm (AMA) [5]. This is the dual application of an algorithm introduced by Lions and Mercier [6] for finding the zero of the sum of two maximal monotone operators, one of which is assumed to be co-coercive. The alternating minimization algorithm is intimately tied to the framework of *augmented Lagrangian* methods, and its global convergence and complexity bounds are well covered in the literature, see [5]: a global convergence rate of order $O(1/\sqrt{k})$ holds for the primal iterates of AMA under very general assumptions, and can be improved to the optimal rate $O(1/k)$ using a simple acceleration technique due to Nesterov, see [7]–[9].

As with all first order methods, the performance of (fast) AMA is severely affected by ill-conditioning of the problem [1]. One way to deal with this issue, which is extensively used in classical smooth, unconstrained optimization, is to precondition the problem using (approximate) second-order information on the cost function, as in (quasi-) Newton methods. However, both (P) and its dual are nonsmooth in general. This motivates considering the concept of *alternating minimization envelope* (AME): this is a real-valued (as opposed to *extended* real-valued) exact merit function for the dual problem, and is precisely the augmented Lagrangian associated with (P) evaluated at the primal points computed by AMA. Under mild assumptions on (P), the AME is continuously differentiable around the set of dual solutions and even strictly twice differentiable there. As a consequence, the AME allows to extend classical, smooth unconstrained optimization algorithms to the solution of the dual problem to (P), which is nonsmooth in general. In this work we propose a dual line-search method, which uses the AME as merit function to compute the stepsizes. The convergence properties of the proposed algorithm greatly improve over AMA when fast-converging directions, computed by means of quasi-Newton formulas, are followed. Furthermore, we show that the AME is equivalent to the *forward-backward envelope* (FBE, see [10]–[12]) associated with the dual problem.

A. Related works

The FBE, as a tool for extending smooth unconstrained algorithms to nonsmooth problems, has first been introduced in [10]: there, two semismooth Newton methods are proposed for minimizing the sum of two convex functions, one of which is smooth and the other having an efficiently computable proximal mapping. This is the classical setting in which the proximal gradient method (and its accelerated variant) can be applied. In [11] the convexity assumption on the smooth term is relaxed, and the authors propose a line-search method with global sublinear rate (in the convex case) and asymptotic superlinear rate when quasi-Newton directions are used: the algorithm relies on descent directions over the FBE which is required to be *everywhere* differentiable. In [13] classical gradient-based line-search methods are considered for minimizing the FBE, see also [14]. In [12] the most general framework, where both summands are allowed to be nonconvex, is

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This work was supported by: KU Leuven internal funding: StG/15/043 Fonds de la Recherche Scientifique — FNRS and the Fonds Wetenschappelijk Onderzoek — Vlaanderen under EOS Project no 30468160 (SeLMA) FWO projects: G086318N; G086518N

taken into account. In this case differentiability of the FBE cannot be assumed: a new algorithm is proposed which computes fast convergent directions with no need for gradient information on the FBE.

A similar approach was used in [15], [16] to analyze and accelerate other splitting algorithms, namely the *Douglas-Rachford splitting* and its dual counterpart ADMM.

B. Contributions and organization of the paper

In the present paper we deal with the case where g in (P) is composed with a linear mapping. In this case, even though g may possess an efficiently computable proximal mapping, $g \circ A$ in general does not. This motivates addressing the dual problem of (P) instead. The contributions and organization of the present work can be summarized as follows.

- We propose the *Newton-type Alternating Minimization Algorithm* (NAMA, Section II, Algorithm 1), a generalization of the alternating minimization algorithm that performs a line-search step over the AME: the proposed algorithm relies on the very same alternating minimization operations of AMA.
- We show that the AME is equivalent to the FBE of the dual problem (Section III). This observation extends a classical result by Rockafellar, relating the Moreau envelope and the augmented Lagrangian, to our setting where an additional strongly convex term is present.
- We show that the proposed method enjoys global sublinear convergence under standard assumptions, and local linear convergence assuming *calmness* of the subdifferentials of the problem terms (Section IV).
- We analyze the first- and second-order properties of the AME, by linking them to generalized second-order properties of the primal functions f and g (Section V).
- We show that the proposed method converges asymptotically superlinearly when the dual problem has a (unique) strong minimum, and the line-search directions are selected so as to satisfy the Dennis-Moré condition, as it is the case when quasi-Newton update formulas are adopted (Section VI). The effectiveness of our approach is demonstrated by numerical simulations on linear MPC problems (Section VII).

Differently from the approaches in [11], [13], [14], NAMA does not require the gradient of the envelope function, therefore no second-order information on the smooth term is needed: this would severely limit its applicability in the present setting where the dual problem is solved. Furthermore, with respect to the approaches of [13], [14], the algorithm presented here possesses strong global convergence properties which are not typical of classical line-search methods. Differently from [12], despite the fact that the selected directions may not be descent directions and the line search is performed on the envelope function, NAMA is a descent method for the dual objective: this allows to simplify the convergence analysis of the method, and to show the global sublinear convergence rate for the dual cost and the primal iterates.

C. Notation

In what follows $\langle \cdot\,, \cdot \rangle$ denotes an inner product over a Euclidean space (whose nature will be clear from the context) and $\|\cdot\| = \sqrt{\langle \cdot\,, \cdot \rangle}$ is the associated norm. For a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$, $\|A\|$ is the operator norm induced by the inner products over $\mathbb{R}^n$ and $\mathbb{R}^m$.

For a set C, we denote by $\mathbf{ri}(C)$ its relative interior, and by $\mathbf{\Pi}_C(x) = \mathbf{argmin}_{y \in C} \|y - x\|$ the projection onto C in the considered norm. We denote the extended real line by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and by $\Gamma_0(\mathbb{R}^n)$ the set of proper, closed, convex functions defined over $\mathbb{R}^n$ with values in $\overline{\mathbb{R}}$. For $h \in \Gamma_0(\mathbb{R}^n)$, its *effective domain* is the set $\mathbf{dom}\, h = \{x \in \mathbb{R}^n \mid h(x) < \infty\}$, and its *Fenchel conjugate* $h^*(y) = \mathbf{sup}_{x \in \mathbb{R}^n}\{\langle x, y\rangle - h(x)\}$ is also proper, closed and convex. Properties of conjugate functions are well described for example in [17]–[20]. Among these we recall the Fenchel-Young inequality [19, Prop. 13.13]

$$\langle x, y\rangle \leq h(x) + h^*(y) \quad \forall x, y, \tag{1}$$

with

$$y \in \partial h(x) \Leftrightarrow \langle x, y\rangle = h(x) + h^*(y) \Leftrightarrow x \in \partial h^*(y), \tag{2}$$

see [17, Thm. 23.5]. For any $\gamma > 0$, the *proximal mapping* associated with h, with stepsize γ, is denoted as

$$\mathbf{prox}_{\gamma h}(x) = \mathbf{argmin}_z \big\{h(z) + (1/2\gamma)\|z - x\|^2\big\}.$$

This satisfies the Moreau identity [19, Thm. 14.3(ii)]

$$y = \mathbf{prox}_{\gamma h}(y) + \gamma\, \mathbf{prox}_{\gamma^{-1} h^*}(\gamma^{-1} y) \quad \forall y. \tag{3}$$

The value function of the problem defining $\mathbf{prox}_{\gamma h}$ is the *Moreau envelope*

$$h^\gamma(x) = \mathbf{min}_z \big\{h(z) + (1/2\gamma)\|z - x\|^2\big\}.$$

An alternative formulation for (P) is

$$\underset{x \in \mathbb{R}^n, z \in \mathbb{R}^m}{\mathbf{minimize}}\ f(x) + g(z) \quad \mathbf{subject\ to}\ Ax = z. \tag{P$'$}$$

Therefore we can define the *augmented Lagrangian* associated with (P), denoted as

$$\mathcal{L}_\gamma(x, z, y) = f(x) + g(z) + \langle y, Ax - z\rangle + \tfrac{\gamma}{2}\|Ax - z\|^2,$$

where $\gamma \geq 0$. We indicate by $\mathcal{L} \equiv \mathcal{L}_0$ the ordinary Lagrangian function.

We follow the terminology of [20] when referring to the concepts of *strict continuity* and *strict differentiability*. We say that a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is *strictly continuous* at $\bar{x}$ if [20, Def. 9.1(b)]

$$\limsup_{\substack{(x,y) \to (\bar{x}, \bar{x}) \\ x \neq y}} \frac{\|F(y) - F(x)\|}{\|y - x\|} < \infty.$$

If F is (Frechét) differentiable, we let $JF : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ denote the *Jacobian* of F. When $m = 1$ we indicate with $\nabla F = JF^\top$ the *gradient* of F and with $\nabla^2 F = J\nabla F^\top$ its *Hessian*, whenever it makes sense. We say that F is *strictly differentiable* at $\bar{x}$ if it satisfies the stronger limit [20, Eq. 9(7)]

$$\lim_{\substack{(x,y) \to (\bar{x}, \bar{x}) \\ x \neq y}} \frac{\|F(y) - F(x) - JF(\bar{x})[y - x]\|}{\|y - x\|} = 0.$$

Some results in the paper are based on generalized second-order properties of extended-real-valued functions.

Definition I.1 ([20, Def. 13.6]). *Function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be* twice epi-differentiable *at x for v, if the second-order difference quotient*

$$\Delta_\tau^2 h(x|v)[d] = \frac{h(x + \tau d) - h(x) - \tau\langle v, d\rangle}{\tau^2/2}$$

epi-converges as $\tau \searrow 0$ *(i.e., its epigraph converges in the sense of Painlevé-Kuratowksi, see [20, Def. 7.1]), the limit being the function* $\mathrm{d}^2 h(x|v)$ *given by*

$$\mathrm{d}^2 h(x|v)[d] = \liminf_{\substack{\tau \searrow 0 \\ d' \to d}} \Delta_\tau^2 h(x|v)[d'].$$

In this case $\mathrm{d}^2 h(x|v)[d]$*, as a function of* d*, is said to be the* second-order epi-derivative *of* h *at* x *for* v*. If* $\Delta_\tau^2 h(\bar{x}|\bar{v})$ *epi-converges as* $\tau \searrow 0$*,* $\bar{x} \to x$ *and* $\bar{v} \to v$*, then* h *is said to be* strictly *twice epi-differentiable.*

Twice epi-differentiability is a mild requirement, and functions with this property are abundant. Refer to [21]–[25] and to [20, §7, §13] for examples and an in-depth account on epi-derivatives, epi-differentiability, and their connections with ordinary differentiability.

II. Background and proposed algorithm

Throughout the paper we will work under the following basic assumption.

Assumption 1. *The following hold for* (P)*:*

(i) (P) *is feasible, i.e.,* $A \operatorname{\mathbf{dom}} f \cap \operatorname{\mathbf{dom}} g \neq \emptyset$*;*

(ii) $f \in \Gamma_0(\mathbb{R}^n)$ *is strongly convex with modulus* $\mu_f > 0$*;*[1]

(iii) $g \in \Gamma_0(\mathbb{R}^m)$*.*

Remark II.1. Assumption 1 guarantees, by strong convexity of f, that a solution to (P) exists and is unique, be it $x_\star$. Assumption 1*(ii)* also implies that f^* is Lipschitz continuously differentiable with constant μ_f^{-1} [20, Th. 12.60]. Assumption 1*(iii)* ensures that g^* is also proper, closed, convex [19, Cor. 13.33], and its Moreau envelope $(g^*)^\gamma$ is strictly continuous [20, Ex. 10.32] with γ^{-1}-Lipschitz gradient

$$\nabla (g^*)^\gamma(y) = \gamma^{-1}\big(y - \mathbf{prox}_{\gamma g^*}(y)\big), \tag{4}$$

as shown in [19, Prop. 12.29]. □

The Fenchel dual problem associated with (P) is

$$\underset{y \in \mathbb{R}^m}{\mathbf{minimize}} \quad \psi(y) = f^*(-A^\top y) + g^*(y). \tag{D}$$

Under Assumption 1 strong duality holds, see [26, Thm. 5.2.1(b)-(c)] and primal-dual solutions $(x_\star, y_\star)$ to (P)-(D) are characterized by the first-order optimality conditions

$$-A^\top y_\star \in \partial f(x_\star) \quad (\Leftrightarrow \ x_\star = \nabla f^*(-A^\top y_\star)) \tag{5a}$$

$$y_\star \in \partial g(A x_\star) \quad (\Leftrightarrow \ A x_\star \in \partial g^*(y_\star)). \tag{5b}$$

A natural way to tackle (P) is to solve (D) by means of forward-backward splitting (or proximal gradient method): starting from an initial dual point $y^0 \in \mathbb{R}^m$, iterate

$$y^{k+1} = T_\gamma(y^k) := \mathbf{prox}_{\gamma g^*}(y^k + \gamma A \nabla f^*(-A^\top y^k)) \tag{6}$$

for some positive stepsize parameter γ. If we define the associated *fixed-point residual*

$$R_\gamma(y) := \gamma^{-1}(y - T_\gamma(y)),$$

then dual optimality can be characterized as follows:

$$y_\star \in Y_\star \ \Leftrightarrow \ y_\star \in \operatorname{\mathbf{fix}} T_\gamma \ \Leftrightarrow \ y_\star \in \operatorname{\mathbf{zer}} R_\gamma \quad \forall \gamma > 0. \tag{7}$$

[1] Function h has convexity modulus $c \geq 0$ if $h - \frac{c}{2}\|\cdot\|^2$ is convex.

Algorithm 1 Newton-type AMA (NAMA)

REQUIRE $y^0 \in \mathbb{R}^m$, $\gamma \in (0, \mu_f/\|A\|^2)$, $\beta \in (0,1)$

INITIALIZE $k = 0$

1: $x^k = \mathbf{argmin}_x \{f(x) + \langle y^k, Ax\rangle\}$
$z^k = \mathbf{argmin}_z \mathcal{L}_\gamma(x^k, z, y^k)$

2: Choose a direction $d^k \in \mathbb{R}^m$

3: Find the largest $\tau_k = \beta^{i_k}, i_k \in \mathbb{N}$, such that

$$\mathcal{L}_\gamma(\tilde{x}^k, \tilde{z}^k, \tilde{y}^k) \geq \mathcal{L}_\gamma(x^k, z^k, y^k), \tag{10}$$

where

$$\tilde{y}^k = y^k + \tau_k d^k + \gamma(1-\tau_k)(Ax^k - z^k)$$
$$\tilde{x}^k = \mathbf{argmin}_x \{f(x) + \langle \tilde{y}^k, Ax\rangle\}$$
$$\tilde{z}^k = \mathbf{argmin}_z \mathcal{L}_\gamma(\tilde{x}^k, z, \tilde{y}^k)$$

4: $y^{k+1} = \tilde{y}^k + \gamma(A\tilde{x}^k - \tilde{z}^k)$, $k = k+1$, go to step 1

Iterations (6) are easily shown to be equivalent to the following scheme, the *alternating minimization algorithm* (AMA)

$$x^k = x(y^k) = \underset{x \in \mathbb{R}^n}{\mathbf{argmin}} \{f(x) + \langle y^k, Ax\rangle\}, \tag{8a}$$

$$z^k = z_\gamma(y^k) = \underset{z \in \mathbb{R}^m}{\mathbf{argmin}} \mathcal{L}_\gamma(x^k, z, y^k), \tag{8b}$$

$$y^{k+1} = y^k + \gamma(Ax^k - z^k). \tag{8c}$$

Note that step (8b) can be equivalently formulated as

$$z^k = \mathbf{prox}_{\gamma^{-1} g}(\gamma^{-1} y^k + Ax(y^k)).$$

Using the notation of (8), T_γ and R_γ can be expressed as

$$T_\gamma(y) = y + \gamma(Ax(y) - z_\gamma(y)) \tag{9a}$$

$$R_\gamma(y) = z_\gamma(y) - Ax(y). \tag{9b}$$

It can be shown that $x^k \to x_\star$ in iterations (8), provided that $\gamma \in (0, 2\mu_f/\|A\|^2)$, see [5, Prop. 3]. Moreover, the dual cost in this case converges sublinearly to the optimum with global rate $O(1/k)$, and the extrapolation techniques introduced by Nesterov [8], [27], [28] allow to obtain accelerated versions of AMA with an optimal global rate $O(1/k^2)$, see [9]: we will here refer to this variant as *fast* AMA.

A. Newton-type alternating minimization algorithm

The convergence speed of (fast) AMA is affected by ill-conditioning of the problem, as it is the case for all first-order methods. To accelerate convergence, we propose Algorithm 1. An overview of the algorithm is as follows:

- Algorithm 1 is composed by the very same operations as AMA: in fact, only alternating minimization steps with respect to x and z are performed.
- Step 3 computes a new dual iterate $\tilde{y}^k$, by performing a line search over the augmented Lagrangian associated with (P) evaluated at the alternating minimization primal points: we will see that this is equivalent to the forward-backward envelope function associated with the dual problem (D).
- The line search is performed using a convex combination of the "nominal" residual direction $\gamma(Ax^k - z^k)$ and an "arbitrary" direction d^k, to be selected so as to ensure fast asymptotic convergence. This novel choice of direction ensures that the line search is feasible at every iteration (*i.e.*, condition (10)

holds for a sufficiently small stepsize) despite the fact that d^k may not be a direction of descent, as we will see.

• Step 4 will allow us to obtain global convergence rates, and it comes at no cost since vectors $\tilde{y}^k, \tilde{x}^k, \tilde{z}^k$ have already been computed in the line-search. In a sense, this step robustifies the algorithmic scheme.

By appropriately choosing d^k, the algorithm is able to greatly improve the convergence of AMA: we will prove that the algorithm converges with superlinear asymptotic rate when Newton-type directions are selected. For this reason we refer to Algorithm 1 as *Newton-type Alternating Minimization Algorithm (NAMA)*.

Remark II.2 (AMA as special case)**.** If in Algorithm 1 one sets $d^k = 0$ for all k, then one can trivially select $\tau_k = 1$. In this case, $(\tilde{y}^k, \tilde{x}^k, \tilde{z}^k) = (y^k, x^k, z^k)$ and Algorithm 1 reduces to AMA, cf. (8). □

Remark II.3 (General equality constrained problems)**.** For any proper, closed, convex $h : \mathbb{R}^r \to \overline{\mathbb{R}}$, $b \in \mathbb{R}^m$ and linear mapping $B : \mathbb{R}^r \to \mathbb{R}^m$, a problem of the form

$$\underset{x\in\mathbb{R}^n, w\in\mathbb{R}^r}{\textbf{minimize}} \; f(x)+h(w) \quad \textbf{subject to} \; Ax+Bw = b \quad (\mathrm{P}'')$$

can be rewritten as (P) by letting

$$g(z) = (Bh)(b-z) = \inf_{w\in\mathbb{R}^r} \{h(w) \mid Bw = b - z\}. \quad (11)$$

Function (Bh) is the *image of h under B*, see [17, Thm. 5.7] and discussion thereafter. If we further assume $\mathbf{ri}(\mathbf{dom}\, h^*) \cap \mathbf{range}(B^\top) \neq \emptyset$, then (Bh) is proper, closed, convex, see [17, Thm. 16.3], therefore g in (11) satisfies Assumption 1*(iii)* (if h is piecewise linear-quadratic then it is sufficient to assume $\mathbf{dom}\, h^* \cap \mathbf{range}(B^\top) \neq \emptyset$, see [20, Cor. 11.33(b)]). In this case steps (8b) and (8c) of AMA become

$$w^k = \underset{w\in\mathbb{R}^r}{\mathbf{argmin}} \left\{g(w) + \langle y^k, Bw\rangle + \tfrac{\gamma}{2}\|Ax^k + Bw - b\|^2\right\}$$
$$y^{k+1} = y^k + \gamma(Ax^k + Bw^k - b).$$

Similar modifications allow to adapt NAMA to this more general setting: in light of these observations, what follows readily applies to problems of the form (P''). □

B. Quasi-Newton directions

There is freedom in selecting d^k in Algorithm 1. To accelerate convergence of the iterates, one possible choice is to employ Newton-type directions for the system of nonlinear equations $R_\gamma(y) = 0$ characterizing dual optimal points, cf. (7). Specifically, in Algorithm 1 one can set

$$d^k = B_k^{-1}(Ax^k - z^k), \quad (12)$$

for a sequence of nonsingular matrices $(B_k)_{k\in\mathbb{N}}$ approximating in some sense the Jacobian JR_γ at the limit point of the dual iterates $(y^k)_{k\in\mathbb{N}}$. In quasi-Newton methods, starting from an initial nonsingular matrix B_0, the sequence of matrices $(B_k)_{k\in\mathbb{N}}$ is determined by low-rank *updates* that satisfy the secant condition: in Algorithm 1 fast asymptotic convergence can be proved if

$$B_{k+1}p^k = q^k \quad \text{with} \quad \begin{cases} p^k = \tilde{y}^k - y^k, \\ q^k = (\tilde{z}^k - A\tilde{x}^k) - (z^k - Ax^k), \end{cases}$$

as will be discussed in Section VI. Note that all quantities required to compute the vectors p^k, q^k are available as by-product of the iterations.

In [29] the modified Broyden update is proposed, that prescribes rank-one updates of the form

$$\textit{Broyden} \qquad B_{k+1} = B_k + \theta_k \frac{(q^k - B_kp^k)(p^k)^\top}{\|p^k\|^2}. \quad (13)$$

Here, $(\theta_k)_{k\in\mathbb{N}} \subset [0, 2]$ is a sequence used to ensure that all terms in $(B_k)_{k\in\mathbb{N}}$ are nonsingular, so that (12) is well defined. The original Broyden method [30] is obtained with $\theta_k \equiv 1$.

Probably the most popular quasi-Newton scheme is BFGS, which prescribes the following rank-two updates

$$\textit{BFGS} \qquad B_{k+1} = B_k + \frac{q^k(q^k)^\top}{\langle q^k, p^k\rangle} - \frac{B_kp^k(B_kp^k)^\top}{\langle p^k, B_kp^k\rangle}. \quad (14)$$

Note that in this case matrices B_k are symmetric, and in fact the fast asymptotic properties of BFGS are guaranteed only if the Jacobian JR_γ is symmetric [31] at the problem solution. This is not the case in our setting (cf. Example V.3) although we have observed that (14) often outperforms other non-symmetric updates such as (13) in practice.

Using the Sherman-Morrison-Woodbury identity in (13) and (14) allows to directly store and update $H_k = B_k^{-1}$, so that d^k can be computed without inverting matrices or solving linear systems.

Ultimately, instead of storing and operating on dense $m \times m$ matrices, *limited-memory* variants of quasi-Newton schemes keep in memory only a few (usually 3 to 30) most recent pairs (p^k, q^k) implicitly representing the approximate inverse Jacobian. Their employment considerably reduces storage and computations over the full-memory counterparts, and as such they are the methods of choice for large-scale problems. The most popular limited-memory method is probably L-BFGS, which is based on the update (14), but efficiently computes matrix-vector products with the approximate inverse Jacobian using a *two-loop recursion* procedure [32]–[34].

III. Alternating minimization envelope

The fundamental tool enabling fast convergence of Algorithm 1 is the *alternating minimization envelope* function associated with (P). This is precisely the (negative) augmented Lagrangian function, evaluated at the primal points given by the alternating minimization steps.

Definition III.1 (Alternating minimization envelope)**.** *The* alternating minimization envelope *(AME) for* (P)*, with parameter $\gamma > 0$, is the function (cf.* (8a)-(8b)*)*

$$\psi_\gamma(y) = -\mathcal{L}_\gamma(x(y), z_\gamma(y), y).$$

The first observation that we make relates the alternating minimization envelope in Definition III.1 with the concept of *forward-backward envelope*.

Theorem III.2. *Function ψ_γ is the* forward-backward envelope *(cf. [11, Def. 2.1]) associated with the dual problem* (D)*:*

$$\psi_\gamma(y) = f^*(-A^\top y) + g^*(T_\gamma(y)) + \tfrac{\gamma}{2}\|Ax(y) - z_\gamma(y)\|^2 + \gamma\langle Ax(y), z_\gamma(y) - Ax(y)\rangle. \quad (15)$$

Proof. The optimality conditions for $x(y)$ and $z_\gamma(y)$ are

$$\partial f(x(y)) \ni -A^\top y, \tag{16a}$$
$$\partial g(z_\gamma(y)) \ni T_\gamma(y) = y + \gamma(Ax(y) - z_\gamma(y)). \tag{16b}$$

From these, using (2), we obtain

$$f(x(y)) + f^*(-A^\top y) = -\langle Ax(y), y\rangle \tag{17a}$$
$$g(z_\gamma(y)) + g^*(T_\gamma(y)) = \langle z_\gamma(y), T_\gamma(y)\rangle \tag{17b}$$

Summing (17) and rearranging the terms we get (15). □

An alternative expression for ψ_γ in terms of the Moreau envelope of g^* is as follows, see [10]:

$$\psi_\gamma(y) = f^*(-A^\top y) - \tfrac{\gamma}{2}\|Ax(y)\|^2 + (g^*)^\gamma(y + \gamma Ax(y)). \tag{18}$$

The AME enjoys several favorable properties, some of which we now summarize. For any $\gamma > 0$, ψ_γ is (strictly) continuous over $\mathbb{R}^m$, whereas if γ is small enough then the problem of minimizing ψ_γ is equivalent to solving (D). These properties are listed in the next result.

Theorem III.3. *For any $\gamma > 0$, ψ_γ is a strictly continuous function on $\mathbb{R}^m$ satisfying*

(i) $\psi_\gamma(y) \le \psi(y) + \frac{\gamma}{2}\|Ax(y) - z_\gamma(y)\|^2$,

(ii) $\psi_\gamma(y) \ge \psi(T_\gamma(y)) + \frac{\gamma}{2}\left(1 - \frac{\gamma\|A\|^2}{\mu_f}\right)\|Ax(y) - z_\gamma(y)\|^2$,

for any $y \in \mathbb{R}^m$. In particular, if $\gamma < \mu_f/\|A\|^2$, then the following also holds

(iii) $\inf \psi_\gamma = \inf \psi$ *and* $\operatorname{argmin} \psi_\gamma = \operatorname{argmin} \psi$.

Proof. Strict continuity of ψ_γ follows immediately by the expression (18).

♠ III.3*(i)*: Follows by Lem. A.1 using $w = y$.

♠ III.3*(ii)*: Due to strong convexity of f, f^* has $1/\mu_f$-Lipschitz gradient, and consequently

$$\begin{aligned} f^*(-A^\top T_\gamma(y)) &\le f^*(-A^\top y) - \langle Ax(y), T_\gamma(y) - y\rangle \\ &\quad + \tfrac{1}{2\mu_f}\|A^\top(T_\gamma(y) - y)\|^2 \\ &= f^*(-A^\top y) - \gamma\langle Ax(y), Ax(y) - z_\gamma(y)\rangle \\ &\quad + \tfrac{\gamma^2}{2\mu_f}\|A^\top(Ax(y) - z_\gamma(y))\|^2. \end{aligned} \tag{19}$$

Combining (15) with (19):

$$\begin{aligned} \psi_\gamma(y) &\ge \psi(T_\gamma(y)) - \tfrac{\gamma^2}{2\mu_f}\|A^\top(Ax(y) - z_\gamma(y))\|^2 \\ &\quad + \tfrac{\gamma}{2}\|Ax(y) - z_\gamma(y)\|^2 \\ &\ge \psi(T_\gamma(y)) + \tfrac{\gamma}{2}\left(1 - \tfrac{\gamma\|A\|^2}{\mu_f}\right)\|Ax(y) - z_\gamma(y)\|^2. \end{aligned}$$

♠ III.3*(iii)*: Easily follows combining III.3*(i)* and III.3*(ii)* with $y = y_\star \in Y_\star$, in light of the dual optimality condition (7). □

A. Analogy with the dual Moreau envelope

Theorem III.2 highlights a clear connection between the augmented Lagrangian, the forward-backward envelope and the alternating minimization algorithm. This closely resembles the one, first noticed by Rockafellar [35], [36], relating the augmented Lagrangian, the Moreau envelope and the *method of multipliers* (also known as *augmented Lagrangian method*) by Hestenes and Powell [37], [38]. Consider the general linear equality constrained convex problem

$$\begin{aligned} &\underset{z\in\mathbb{R}^k}{\textbf{minimize}} && g(z) \\ &\textbf{subject to} && Bz = b, \end{aligned} \tag{20}$$

where $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ is proper, closed, convex, $B \in \mathbb{R}^{m\times k}$ and $b \in \mathbb{R}^m$. When applied to the dual of (20), namely

$$\underset{y\in\mathbb{R}^m}{\textbf{minimize}} \quad \omega(y) = g^*(-B^\top y) + \langle b, y\rangle,$$

the proximal minimization algorithm [39, §5.2] is equivalent to the following augmented Lagrangian method

$$\begin{aligned} z^k &= \underset{z\in\mathbb{R}^n}{\textbf{argmin}}\left\{g(z) + \langle y^k, Bz - b\rangle + \tfrac{\gamma}{2}\|Bz - b\|^2\right\} \\ y^{k+1} &= y^k + \gamma(Bz^k - b). \end{aligned}$$

If $\textbf{range}(B^\top) \cap \textbf{ri}(\textbf{dom}\, g^*) \neq \emptyset$ one can show, with a similar proof to that of Theorem III.2, that the Moreau envelope of ω satisfies

$$\begin{aligned} \omega^\gamma(y^k) &= -g(z^k) - \langle y^k, Bz^k - b\rangle - \tfrac{\gamma}{2}\|Bz^k - b\|^2 \\ &= -\mathcal{L}_\gamma(z^k, y^k). \end{aligned}$$

Therefore the forward-backward and Moreau envelope functions have the same nice interpretation in terms of augmented Lagrangian, when they are applied to the dual of equality constrained convex problems: in a sense, Theorem III.2 extends and generalizes the classical result on the dual Moreau envelope, by allowing for an additional variable x and a strongly convex term f in the problem.

IV. Convergence

We now turn our attention to the global convergence properties of Algorithm 1. In light of Remark II.2, the results in this section directly apply to AMA, which is a special case of NAMA.

Remark IV.1 (Termination of line search)**.** The line-search step 3 is well defined regardless of the choice of d^k: at any iteration k, condition (10) holds for i_k sufficiently large. To see this, suppose that $\|Ax^k - z^k\| > 0$ (otherwise (x^k, y^k) is a primal-dual solution). Then, since $\gamma < \mu_f/\|A\|^2$, Theorem III.3 implies that

$$\psi_\gamma(T_\gamma(y^k)) < \psi_\gamma(y^k). \tag{21}$$

Since $\tilde{y}^k \to T_\gamma(y^k)$ as $\tau_k \to 0$ and ψ_γ is continuous, then necessarily $\psi_\gamma(\tilde{y}^k) \le \psi_\gamma(y^k)$ for τ_k sufficiently small. □

Remark IV.2 (Bounded iteration complexity)**.** In the best case where $\tau_k = 1$ is accepted in step 3, exactly two alternating minimizations are performed at iteration k. In practice, one can also impose a lower bound $\tau_{\min} > 0$ for τ_k: when $\tau_k < \tau_{\min}$ then the ordinary AMA update $y^{k+1} = y^k + \gamma(Ax^k - z^k)$ is executed and the algorithm proceeds to the next iteration. This strategy results in a bounded iteration complexity for NAMA, and does not affect the convergence results of this and later sections. □

Theorem III.3 ensures that the following chain of inequalities, which will be fundamental for convergence results, holds in Algorithm 1:

$$\psi(y^{k+1}) \le \psi_\gamma(\tilde{y}^k) \tag{22a}$$
$$\le \psi_\gamma(y^k) \tag{22b}$$
$$\le \psi(y^k) - \tfrac{\gamma}{2}\|Ax^k - z^k\|^2. \tag{22c}$$

In particular, Algorithm 1 is a descent method for ψ.

We now prove that the iterates of (1) converge to the dual optimal cost and to the primal solution. Moreover, global convergence rates are provided.

Theorem IV.3 (Global convergence)**.** *In Algorithm 1:*

(i) $x^k \to x_\star$, $z^k \to Ax_\star$, *and all cluster points of* $(y^k)_{k\in\mathbb{N}}$ *are dual optimal, i.e., they belong to* $Y_\star$*;*

(ii) if $0 \in \mathbf{int}(\mathbf{dom}\, g - A\,\mathbf{dom}\, f)$ *then* $\psi(y^k) \searrow \mathbf{inf}\,\psi$ *with global rate* $O(1/k)$*, and* $x^k \to x_\star$ *with global rate* $O(1/\sqrt{k})$*;*

(iii) if f *and* g *are piecewise linear-quadratic then* $\psi(y^k) \searrow \mathbf{inf}\,\psi$ *with global Q-linear rate, and* $x^k \to x_\star$ *with global R-linear rate.*

Proof.

♠ IV.3*(i)*: By (22c), for all $i \geq 0$ we have

$$\psi(y^{i+1}) \leq \psi(y^i) - \tfrac{\gamma}{2}\|Ax^i - z^i\|^2.$$

By summing up the inequality for $i = 1, \ldots, k$ we obtain

$$\mathbf{inf}\,\psi \leq \psi(y^{k+1}) \leq \psi(y^1) - \frac{\gamma}{2}\sum_{i=1}^{k}\|Ax^i - z^i\|^2$$

(the sum starts from $i = 1$ since y^0 may be dual infeasible). In particular (cf. (9)) $R_\gamma(y^k) = z^k - Ax^k \to 0$, and since R_γ is continuous, necessarily all cluster points of $(y^k)_{k\in\mathbb{N}}$ are optimal. Moreover, it follows from Lem. A.2 that the sequence $(x^k)_{k\in\mathbb{N}}$ is bounded. Let $K \subseteq \mathbb{N}$ and $\bar{x}$ be such that $(x^k)_{k\in K} \to \bar{x}$; then, since $Ax^k - z^k \to 0$ we also have that $(z^k)_{k\in K} \to A\bar{x}$. By multiplying (16b) on the left by $A^\top$ and summing (16a) we obtain $\gamma A^\top(Ax_k - z_k) \in \partial f(x_k) + A^\top \partial g(z_k)$. By letting $K \ni k \to \infty$, from outer semicontinuity of the subdifferential we obtain that

$$0 \in \partial f(\bar{x}) + A^\top \partial g(A\bar{x}) \subseteq \partial(f + g \circ A)(\bar{x})$$

where the last inclusion follows from [17, Thm.s 23.8 and 23.9]. Thus, $\bar{x}$ is optimal, and being $x_\star$ the unique primal optimal (due to strong convexity), necessarily $\bar{x} = x_\star$. From the arbitrarity of the cluster point we conclude that $x^k \to x_\star$ and $z^k \to Ax_\star$.

♠ IV.3*(ii)*: The assumed condition is equivalent to $Y_\star$ being nonempty and compact, see [26, Thm. 5.2.1], which implies that ψ has bounded level sets [20, Prop. 3.23]. The proof proceeds similarly to that of [8, Thm. 4]. Let $D > 0$ be such that $\mathbf{dist}(y, Y_\star) < D$ for all points $y \in \{y \in \mathbb{R}^m \mid \psi(y) \leq \psi(y^0)\}$. From [11, Prop. 2.5] we know that $\psi_\gamma \leq \psi^\gamma$ (the Moreau envelope of ψ). Therefore,

$$\psi(y^{k+1}) \overset{(22b)}{\leq} \psi_\gamma(y^k) \leq \psi^\gamma(y^k) = \min_{w\in\mathbb{R}^m}\left\{\psi(w) + \tfrac{1}{2\gamma}\|w - y^k\|^2\right\}$$

and in particular, for $y_\star \in \mathbf{argmin}\,\psi$,

$$\begin{aligned}\psi(y^{k+1}) &\leq \min_{\alpha\in[0,1]}\left\{\psi(\alpha y_\star + (1-\alpha)y^k) + \tfrac{\alpha^2}{2\gamma}\|y^k - y_\star\|^2\right\}\\ &\leq \min_{\alpha\in[0,1]}\left\{\psi(y^k) - \alpha(\psi(y^k) - \mathbf{inf}\,\psi) + \tfrac{D^2}{2\gamma}\alpha^2\right\}\end{aligned}$$

where in last inequality we used convexity of ψ. In case $\psi(y^0) - \mathbf{inf}\,\psi \geq D^2/\gamma$, then the optimal solution of the latter problem for $k = 0$ is $\alpha = 1$, and $\psi(y^1) - \mathbf{inf}\,\psi \leq {}^{D^2}/_{2\gamma}$. Otherwise, the optimal solution is

$$\alpha = \tfrac{\gamma}{D^2}(\psi(y^k) - \mathbf{inf}\,\psi) \leq \tfrac{\gamma}{D^2}(\psi(y^0) - \mathbf{inf}\,\psi) \leq 1$$

and we obtain

$$\psi(y^{k+1}) \leq \psi(y^k) - \tfrac{\gamma}{2D^2}(\psi(y^k) - \mathbf{inf}\,\psi)^2.$$

By letting $\lambda_k = \frac{1}{\psi(y^k) - \mathbf{inf}\,\psi}$ the last inequality becomes

$$\lambda_{k+1}^{-1} \leq \lambda_k^{-1} - \tfrac{\gamma}{2D^2}\lambda_{k+1}^{-2}.$$

By multiplying both sides by $\lambda_k\lambda_{k+1}$ and rearranging,

$$\lambda_{k+1} \geq \lambda_k + \tfrac{\gamma}{2D^2}\tfrac{\lambda_{k+1}}{\lambda_k} \geq \lambda_k + \tfrac{\gamma}{2D^2},$$

where the latter inequality follows from the fact that the sequence $(\psi(y^k))_{k\in\mathbb{N}}$ is nonincreasing, as shown in (22). By telescoping the inequality we obtain

$$\lambda_k \geq \lambda_0 + k\tfrac{\gamma}{2D^2} \geq k\tfrac{\gamma}{2D^2},$$

and therefore $\psi(y^k) - \mathbf{inf}\,\psi \leq {}^{2D^2}/_{k\gamma}$. This, together with Lem. A.2, proves IV.3*(ii)*.

♠ IV.3*(iii)*: Since the primal optimum is finite (see Rem. II.1), if f and g are piecewise linear-quadratic then $Y_\star$ is nonempty, see [20, Thm. 11.42, Ex. 11.43]. Using (22) we have that

$$\psi(y^k) - \psi(y^{k+1}) \geq \tfrac{\gamma}{2}\|Ax^k - z^k\|^2. \tag{23}$$

Furthermore, using Lem. A.1 with $w = y_\star^k = \Pi_{Y_\star}y^k$ and $y = y^k$, we obtain

$$\begin{aligned}\psi(y^{k+1}) - \mathbf{inf}\,\psi &\leq \psi_\gamma(y^k) - \mathbf{inf}\,\psi\\ &\leq \langle Ax^k - z^k, y_\star^k - y^k\rangle - \tfrac{\gamma}{2}\|Ax^k - z^k\|^2,\end{aligned}$$

where first inequality is due to (22c). This implies

$$\psi(y^{k+1}) - \mathbf{inf}\,\psi \leq \|Ax^k - z^k\|^2\left(\tfrac{\mathbf{dist}(y^k, Y_\star)}{\|Ax^k - z^k\|} - \tfrac{\gamma}{2}\right)$$

which, by using (23), yields

$$\psi(y^{k+1}) - \mathbf{inf}\,\psi \leq \left(1 - \tfrac{\gamma}{2}\tfrac{\|Ax^k - z^k\|}{\mathbf{dist}(y^k, Y_\star)}\right)(\psi(y^k) - \mathbf{inf}\,\psi). \tag{24}$$

It follows from [20, Thm. 11.14] that f^* and g^* are convex piecewise linear-quadratic in this case, and so is ψ. Therefore by [40, Thm. 2.7] ψ enjoys the following quadratic growth condition: for any $\nu > 0$ there is $\alpha > 0$ such that

$$\tfrac{\alpha}{2}\,\mathbf{dist}^2(y, Y_\star) \leq \psi(y^k) - \mathbf{inf}\,\psi \quad \forall y : \psi(y) - \mathbf{inf}\,\psi \leq \nu,$$

which by [41, Cor. 3.6] is equivalent to the following error bound condition for some $\beta > 0$

$$\mathbf{dist}(y, Y_\star) \leq \beta\|Ax(y) - z_\gamma(y)\| \tag{25}$$

holding for all y such that $\psi(y) - \mathbf{inf}\,\psi \leq \nu$. By using (25) in (24) we obtain global Q-linear convergence of $(\psi(y^k))_{k\in\mathbb{N}}$, and from Lem. A.2 global R-linear convergence of $(x^k)_{k\in\mathbb{N}}$ also follows. □

In general we can prove local linear convergence of Algorithm 1 provided that ∂f and ∂g are *calm*, according to the following definition (see [42, Sec. 3H, Ex. 3H.4]).

Definition IV.4 (Calmness of a mapping)**.** *A multi-valued mapping* $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is said to be* calm *at* $\bar{y} \in \mathbb{R}^m$ *for* $\bar{x} \in F(\bar{y})$ *if there is a neighborhood* U *of* $\bar{x}$ *such that*

$$F(y) \cap U \subseteq F(\bar{y}) + O(\|y - \bar{y}\|), \quad \forall y \in \mathbb{R}^m.$$

We simply say that F *is calm at* $\bar{y} \in \mathbb{R}^m$ *(with no mention of* $\bar{x}$*) if it is calm at* $\bar{y} \in \mathbb{R}^m$ *for all* $\bar{x} \in F(\bar{y})$*.*

Calmness is a very common property of the subdifferential mapping. The subdifferential of all piecewise linear-quadratic functions is calm everywhere, as follows from [42, Prop. 3H.1]. Other examples include the nuclear and spectral norms [43]. Smooth functions, *i.e.*, with Lipschitz gradient, clearly

have calm subdifferential: this includes Moreau envelopes of closed, convex functions, such as the Huber loss for robust estimation, and commonly used loss functions such as the squared Euclidean norm and the logistic loss.

Calmness is equivalent to metric subregularity of the inverse mapping [42, Thm. 3H.3]: from [44, Prop. 6, Prop. 8] we then deduce that the indicator functions of ℓ_1, ℓ_∞ and Euclidean norm balls all have calm subdifferentials.

The following result holds. Its proof is analogous to the one of [41, Thm. 4.2], although our assumption of calmness is equivalent to metric subregularity of ∂f^* and ∂g^*, which is implied by the firm convexity assumed in [41].

Theorem IV.5 (Local linear convergence)**.** *Suppose that the following hold for* (P)*:*

(i) $0 \in \mathbf{int}(\mathbf{dom}\, g - A\,\mathbf{dom}\, f)$ *(nonempty, compact $Y_\star$);*

(ii) $0 \in \mathbf{ri}\,\partial(f + g \circ A)(x_\star)$ *(strict complementarity).*

Suppose also that ∂f is calm at $x_\star$ and ∂g is calm at $Ax_\star$. Then in Algorithm 1 eventually $\psi(y^k) \to \mathbf{inf}\,\psi$ with Q-linear rate and $x^k \to x_\star$ with R-linear rate.

Proof. As discussed in the proof of Thm. IV.3*(iii)*, it suffices to show that an error bound of the form (25) holds for some $\beta, \nu > 0$.

The assumed calmness properties of ∂f and ∂g are equivalent to *metric subregularity* of ∂f^* at $-A^\top y_\star$ for $x_\star$, and of ∂g^* at $y_\star$ for $Ax_\star$, see [42, Thm. 3H.3], for all $y_\star \in Y_\star$. This can be seen, using [45, Thm. 3.3], to be equivalent to the following: there exist $c_{y_\star} > 0$ and a neighborhood $U_{y_\star}$ of $y_\star$ such that for all $y \in U_{y_\star}$

$$\begin{aligned} f^*(-A^\top y) \geq\ & f^*(-A^\top y_\star) + \langle x_\star, A^\top(y_\star - y)\rangle \\ & + \tfrac{c_{y_\star}}{2}\,\mathbf{dist}^2(-A^\top y, (\nabla f^*)^{-1}(x_\star)), \\ g^*(y) \geq\ & g^*(y_\star) + \langle Ax_\star, y - y_\star\rangle \\ & + \tfrac{c_{y_\star}}{2}\,\mathbf{dist}^2(y, (\partial g^*)^{-1}(Ax_\star)). \end{aligned}$$

Since $Y_\star \subset \bigcup_{y_\star \in Y_\star} U_{y_\star}$ and $Y_\star$ is nonempty and compact (due to IV.5*(i)*, see [26, Thm. 5.2.1]), we may select a *finite* subset $W \subset Y_\star$ such that $Y_\star \subset U_{Y_\star} = \bigcup_{y_\star \in W} U_{y_\star}$. Summing the above inequalities for all $y_\star \in W$, and denoting $c = \mathbf{min}\,\{c_{y_\star} \mid y_\star \in W\} > 0$, we obtain

$$\begin{aligned} \psi(y) \geq\ & \mathbf{inf}\,\psi \\ & + \tfrac{c}{2}\big[\mathbf{dist}^2(-A^\top y, \partial f(x_\star)) + \mathbf{dist}^2(y, \partial g(Ax_\star))\big] \end{aligned} \tag{26}$$

for all $y \in U_{Y_\star}$, where we have also used $(\nabla f^*)^{-1} = \partial f$ and $(\partial g^*)^{-1} = \partial g$. Note that IV.5*(i)* implies strict feasibility, therefore from Lem. A.3, and the fact that for any $a, b \in \mathbb{R}$, $a^2 + b^2 \geq 2ab$, we obtain that (26) implies

$$\psi(y) \geq \mathbf{inf}\,\psi + \tfrac{\kappa}{2}\,\mathbf{dist}^2(y, Y_\star), \quad \forall y \in U_{Y_\star},$$

for some $\kappa > 0$, *i.e.*, ψ satisfies the quadratic growth condition, which by [41, Cor. 3.6] is equivalent to the error bound condition (25). This completes the proof. □

Remark IV.6 (Backtracking on γ)**.** In practice, no prior knowledge of the global Lipschitz constant $\|A\|^2/\mu_f$ is required for Algorithm 1: instead of a fixed parameter γ, one can adaptively determine a *sequence* $(\gamma_k)_{k\in\mathbb{N}}$ essentially ensuring that inequalities (21) (which guarantees termination of the line-search step 3) and (22a) (which guarantees descent) hold at every iteration. This is done as follows. Select $\alpha \in (0,1)$ and initialize $\gamma_0 > 0$. At iteration k, let $\bar{y}^k = y^k + \gamma_k(Ax^k - z^k)$ and $\bar{x}^k = x(\bar{y}^k)$, and if

$$f(x^k) > f(\bar{x}^k) - \langle A^\top \bar{y}^k, x^k - \bar{x}^k\rangle + \tfrac{\alpha\gamma}{2}\|Ax^k - z^k\|^2,$$

then $\gamma_k \leftarrow \gamma_k/2$ and restart the iteration. Similarly if

$$f(\tilde{x}^k) > f(x^{k+1}) - \langle A^\top y^{k+1}, \tilde{x}^k - x^{k+1}\rangle + \tfrac{\alpha\gamma}{2}\|A\tilde{x}^k - \tilde{z}^k\|^2.$$

As soon as $\gamma_k \leq \alpha\mu_f/\|A\|^2$, the two inequalities above will never hold. As a consequence, γ_k will be decreased only a finite number of times and will be constant starting from some iteration $\bar{k}$. The inequalities above are obtained by imposing the usual quadratic upper bound on $f^* \circ (-A^\top)$, due to smoothness, and applying the conjugate subgradient theorem (2) in light of (16a). This procedure of adaptively adjusting γ_k is analogous to what is done in practice in (fast) AMA, see [9, Rem. 3.4] and [7, §3, §4], and does not affect the validity of Thm.s IV.3 and IV.5. □

V. First- and second-order properties

Algorithm 1 is a line-search method for the unconstrained minimization of ψ_γ which, by Theorem III.3*(iii)*, is equivalent to solving (D). To enable fast convergence of the iterates, we can apply ideas from smooth unconstrained optimization in selecting the sequence $(d^k)_{k\in\mathbb{N}}$ of directions. To this end, differentiability of ψ_γ around dual solutions $y_\star$ is a desirable property. We will now see that this is implied by generalized second-order properties of f around $x_\star$, which are introduced in the following assumption. Analogous assumptions on g further ensure that ψ_γ is (strictly) twice differentiable at $y_\star$. The interested reader is referred to [20] for an extensive discussion on (second-order) epi-differentiability.

Assumption 2. *The following hold with respect to a primal-dual solution $(x_\star, y_\star)$ to* (P)*-*(D)*:*

(i) f is strictly twice epi-differentiable at all $x \in \mathbf{dom}\, f$ close enough to $x_\star$, and in particular the second-order epi-derivative at $x_\star$ for $-A^\top y_\star$ is, for $w \in \mathbb{R}^n$,

$$\mathrm{d}^2 f(x_\star|-A^\top y_\star)[w] = \langle H_f w, w\rangle + \delta_{S_f}(w), \tag{27}$$

where S_f is a linear subspace of $\mathbb{R}^n$ and $H_f \in \mathbb{R}^{n\times n}$;

(ii) g is (strictly) twice epi-differentiable at $Ax_\star$ for $y_\star$, with

$$\mathrm{d}^2 g(Ax_\star|y_\star)[w] = \langle H_g w, w\rangle + \delta_{S_g}(w), \tag{28}$$

for all $w \in \mathbb{R}^m$, where S_g is a linear subspace of $\mathbb{R}^m$ and $H_g \in \mathbb{R}^{m\times m}$.

When the stronger condition in parenthesis holds we will say that the assumptions are strictly *satisfied.*

Without loss of generality, we consider H_f and H_g symmetric and positive semidefinite, satisfying $\mathbf{range}(H_f) = S_f$, $\mathbf{null}(H_f) = S_f^\perp$, $\mathbf{range}(H_g) \subseteq S_g$ and $\mathbf{null}(H_g) \supseteq S_g^\perp$.

The requirements on H_f and H_g can indeed be made without loss of generality: matrix $H_f' = \frac{1}{2}\,\mathbf{\Pi}_{S_f}(H_f + H_f^\top)\,\mathbf{\Pi}_{S_f}$ has the desired properties and satisfies (27) provided H_f does, and similarly for H_g. In particular, it holds that

$$H_f = \mathbf{\Pi}_{S_f} H_f\, \mathbf{\Pi}_{S_f} \quad \text{and} \quad H_g = \mathbf{\Pi}_{S_g} H_g\, \mathbf{\Pi}_{S_g}. \tag{29}$$

Theorem V.1 (Differentiability of ψ_γ)**.** *Suppose that Assumption 2(i) holds for a primal-dual solution $(x_\star, y_\star)$. Then ψ_γ is of class $\mathcal{C}^1$ around $y_\star$, with*

$$\nabla\psi_\gamma(y) = Q_\gamma(y)R_\gamma(y)$$

where $Q_\gamma(y) = I - \gamma A\nabla^2 f^*(-A^\top y)A^\top$.

Proof. From Lem. A.4 it follows that $\hat{f} = f^* \circ (-A^\top)$ is of class $\mathcal{C}^2$ around $y_\star$. The claim now easily follows from the chain rule of differentiation applied to (18), by using (4). □

Twice differentiability of ψ_γ at a dual solution $y_\star$ is very important: when Newton-type directions are used, this implies that eventually unit stepsize will be accepted and fast asymptotic convergence will take place. In other words, unlike standard nonsmooth merit functions for constrained optimization, ψ_γ does not prevent the acceptance of unit stepsize.

Theorem V.2 (Twice differentiability of ψ_γ)**.** *Suppose that Assumption 2 (strictly) holds with respect to a primal-dual solution $(x_\star, y_\star)$. Then,*

(i) R_γ is (strictly) differentiable at $y_\star$ with Jacobian

$$JR_\gamma(y_\star) = \gamma^{-1}[I - P_\gamma(y_\star)Q_\gamma(y_\star)]; \tag{30}$$

here, Q_γ is as in Theorem V.1 and

$$\begin{aligned} P_\gamma(y_\star) &= J\,\mathbf{prox}_{\gamma g^*}\left(y_\star + \gamma A\nabla f^*(-A^\top y_\star)\right) \\ &= \mathbf{\Pi}_{\bar{S}}\left(I + \gamma H_g^\dagger\right)^{-1}\mathbf{\Pi}_{\bar{S}} \end{aligned} \tag{31}$$

with $\bar{S} = {S_g}^\perp + \mathbf{range}(H_g)$;

(ii) ψ_γ is (strictly) twice differentiable at $y_\star$ with symmetric Hessian

$$\nabla^2\psi_\gamma(y_\star) = \gamma^{-1}Q_\gamma(y_\star)\big[I - P_\gamma(y_\star)Q_\gamma(y_\star)\big]. \tag{32}$$

Proof. Let $\hat{f} = f^* \circ (-A^\top)$ and $L_{\hat{f}} = \mu_f/\|A\|^2$. We know from [25, Thms. 3.8, 4.1] and [20, Thm. 13.21] that $\mathbf{prox}_{\gamma g^*}$ is (strictly) differentiable at $y_\star - \gamma\nabla\hat{f}(y_\star)$ if and only if g (strictly) satisfies Assumption 2*(ii)*; in fact, by (5) we know that $Ax_\star = -\nabla\hat{f}(y_\star)$. Moreover, due to Lem. A.4, $\hat{f} \in \mathcal{C}^2$ in a neighborhood of $y_\star$ and in particular $\nabla\hat{f}$ is strictly differentiable at $y_\star$. The formula for $JR_\gamma(y_\star)$ follows from (4) and the chain rule of differentiation.

We now prove the claimed expression for $P_\gamma(y_\star)$. We may invoke Lem. A.5 and apply [20, Ex. 13.45] to the *tilted* function $g + \langle\nabla\hat{f}(y_\star), \cdot\,\rangle$ which tells us that for all $d \in \mathbb{R}^m$

$$\begin{aligned} &P_\gamma(y_\star)d \\ &= \mathbf{prox}_{(\gamma/2)\mathrm{d}^2 g^*(y_\star|Ax_\star)}(d) \\ &= \mathop{\mathbf{argmin}}_{d'\in\bar{S}}\left\{\tfrac{1}{2}\langle d', H_g^\dagger d'\rangle + \tfrac{1}{2\gamma}\|d' - d\|^2\right\} \\ &= \mathbf{\Pi}_{\bar{S}}\mathop{\mathbf{argmin}}_{d'\in\mathbb{R}^n}\left\{\tfrac{1}{2}\langle \mathbf{\Pi}_{\bar{S}}\,d', H_g^\dagger\,\mathbf{\Pi}_{\bar{S}}\,d'\rangle + \tfrac{1}{2\gamma}\|\mathbf{\Pi}_{\bar{S}}\,d' - d\|^2\right\} \\ &= \mathbf{\Pi}_{\bar{S}}\left(\mathbf{\Pi}_{\bar{S}}[I + \gamma H_g^\dagger]\,\mathbf{\Pi}_{\bar{S}}\right)^\dagger\mathbf{\Pi}_{\bar{S}}\,d \end{aligned}$$

where † indicates the pseudo-inverse. Observe now that, since $\mathbf{range}\,H_g^\dagger = \mathbf{range}\,H_g \subseteq \bar{S}$, we have

$$\mathbf{\Pi}_{\bar{S}}[I + \gamma H_g^\dagger]\,\mathbf{\Pi}_{\bar{S}} = AB \quad \text{for} \quad A = I + \gamma H_g^\dagger \text{ and } B = \mathbf{\Pi}_{\bar{S}}\,.$$

Moreover,

$$\begin{aligned} \mathbf{range}(A^\top AB) &\subseteq \mathbf{range}\,B, \\ \mathbf{range}(B^\top BA) &\subseteq \mathbb{R}^n = \mathbf{range}(A), \end{aligned}$$

therefore we can apply [46, Facts 6.4.12 (i)-(ii) and 6.1.6 (xxxii)] to see that $\left(\mathbf{\Pi}_{\bar{S}}[I + \gamma H_g^\dagger]\,\mathbf{\Pi}_{\bar{S}}\right)^\dagger = \mathbf{\Pi}_{\bar{S}}[I + \gamma H_g^\dagger]^{-1}$, yielding (31).

Since $R_\gamma(y_\star) = 0$, from [11, Lem. 6.2] it follows that $\nabla\psi_\gamma = Q_\gamma R_\gamma$ is (strictly) differentiable at $y_\star$ provided that Q_γ is (strictly) continuous at $y_\star$ and R_γ is (strictly) differentiable at $y_\star$. A simple application of the chain rule of differentiation concludes the proof of V.2*(ii)*. □

To better understand the requirements of Assumption 2, let us consider the following simple but significant example: when f is $\mathcal{C}^2$ and $g\circ A$ models linear inequality constraints, Assumption 2 is implied by strict complementarity.

Example V.3 ($\mathcal{C}^2$ functions subject to polyhedral constraints)**.** Consider problems of the form

$$\mathop{\mathbf{minimize}}_{x\in\mathbb{R}^n} \quad f(x) + \delta_C(Ax),$$

where $g = \delta_C$ is the indicator of $C = \{z \in \mathbb{R}^m \mid z \le b\}$, $b \in \mathbb{R}^m$, and $f \in \mathcal{C}^2$. In this case Assumption 2*(i)* holds with $H_f = \nabla^2 f(x_\star)$, $S_f = \mathbb{R}^n$ (therefore $\mathbf{\Pi}_{S_f}$ is the identity mapping), see [20, Ex. 13.8]. Regarding Assumption 2*(ii)*, one can use [20, Ex. 13.17] to see that

$$\mathrm{d}^2 g(Ax_\star|y_\star)[w] = \delta_{K(Ax_\star, y_\star)}(w),$$

where K is the critical cone. Denoting by $T_C(y)$ the tangent cone of set C at $y \in C$, and by $J = \{i \mid (Ax_\star)_i = b_i\}$ the set of active constraints at the solution $x_\star$, the critical cone is given by

$$\begin{aligned} K(Ax_\star, y_\star) &= \{w \in T_C(Ax_\star) \mid \langle y_\star, w\rangle = 0\} \\ &= \{w \mid \langle y_\star, w\rangle = 0, w_i \le 0\ \forall i \in J\}. \end{aligned}$$

For $K(Ax_\star, y_\star)$ to be a subspace, necessarily $(y_\star)_i > 0$ for all $i \in J$, *i.e.*, strict complementarity must hold at the primal-dual solution $(x_\star, y_\star)$. In this case, Assumption 2*(ii)* holds with $H_g = 0$ and

$$S_g = K(Ax_\star, y_\star) = \{w \mid w_i = 0\ \forall i \in J\}.$$

We may assume that $J = \{1, \dots, k\}$ without loss of generality, *i.e.*, the first k constraints are the active ones, and let $\bar{J} = \{1, \dots, m\} \setminus J$. Note that $\nabla^2 f^*(-A^\top y_\star) = \nabla^2 f(x_\star)^{-1}$ due to strong convexity of f, see [20, Ex. 11.9]. By partitioning the inverse Hessian and constraint matrix as

$$\nabla^2 f(x_\star)^{-1} = \begin{bmatrix} H_{JJ} & H_{J\bar{J}} \\ H_{\bar{J}J} & H_{\bar{J}\bar{J}} \end{bmatrix}, \quad A = \begin{bmatrix} A_J \\ A_{\bar{J}} \end{bmatrix},$$

and using the notation of Theorem V.2*(i)* we obtain

$$P_\gamma(y_\star) = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \quad JR_\gamma(y_\star) = \begin{bmatrix} A_J H_{JJ} A_J^\top & A_J H_{J\bar{J}} A_{\bar{J}}^\top \\ 0 & \frac{1}{\gamma} I_{m-k} \end{bmatrix},$$

as it follows by elementary computations. □

Finally, we can relate strong minimality of ψ and ψ_γ to nonsingularity of the Jacobian of R_γ and to the generalized second-order properties of f and g as follows.

Theorem V.4 (Conditions for strong minimality)**.** *If Assumption 2 holds for a primal-dual solution $(x_\star, y_\star)$, then for all $\gamma < \mu_f/\|A\|^2$ the following are equivalent:*

(a) $y_\star$ is a strong minimum for ψ,[2]

[2]We say that $y_\star$ is a strong local minimum for h if for some $\alpha > 0$, $\alpha\|y - y_\star\|^2 \le h(y) - h(y_\star)$ for all y sufficiently close to $y_\star$.

(b) $\nabla^2\psi_\gamma(y_\star)$ *is nonsingular (in fact, positive definite);*

(c) $JR_\gamma(y_\star)$ *is nonsingular (in fact, similar to a symmetric and positive definite matrix);*

(d) $y_\star$ *is a strong minimum for* ψ_γ.

Proof.

♠ V.4*(b)* ⇔ V.4*(c)*: Let $P = P_\gamma(y_\star)$ and $Q = Q_\gamma(y_\star)$ for brevity. Notice first that, due to Thm. III.3*(iii)*, $y_\star$ minimizes ψ_γ and therefore $\nabla^2\psi_\gamma(y_\star) \succeq 0$. Moreover, since Q is symmetric and positive definite,

$$JR_\gamma(y_\star) = \gamma^{-1}(I - PQ) \sim Q^{-1/2}\nabla^2\psi_\gamma(y_\star)Q^{-1/2}$$

the latter matrix being symmetric and positive semidefinite, where $\sim$ denotes the similitude relation.

♠ V.4*(b)* ⇔ V.4*(d)*: Trivial since $\nabla^2\psi_\gamma(y_\star)$ exists.

♠ V.4*(d)* ⇔ V.4*(a)*: The right implication is trivial since $\psi_\gamma \le \psi$ and $\psi_\gamma(y_\star) = \psi(y_\star)$ as it follows from Thm. III.3. Suppose now that there exist $c, \varepsilon > 0$ such that $\psi(y) - \psi(y_\star) \ge \frac{c}{2}\|y - y_\star\|^2$ for all $y \in \mathbf{B}(y_\star;\varepsilon)$. Since g^* is convex, it follows that $\mathbf{prox}_{\gamma g^*}$ is 1-Lipschitz continuous; combined with the fact that ∇f^* is $\frac{1}{\mu_f}$-Lipschitz continuous, we obtain that the alternating minimization operator T_γ is Lipschitz continuous with modulus $\|A\|^2/\mu_f$. Let $\varepsilon' = \mu_f/\|A\|^2\varepsilon$; since $T_\gamma(y_\star) = y_\star$, for all $y \in \mathbf{B}(y_\star;\varepsilon')$ necessarily $T_\gamma(y) \in \mathbf{B}(y_\star;\varepsilon)$. Therefore, letting $c' = \mathbf{min}\left\{c, \gamma\left(1 - \frac{\gamma\|A\|^2}{\mu_f}\right)\right\} > 0$, it follows from Thm. III.3*(ii)* that for all $y \in \mathbf{B}(y_\star;\varepsilon')$

$$\begin{aligned}\psi_\gamma(y) - \psi_\star &\ge \psi(T_\gamma(y)) - \psi_\star - \tfrac{\gamma}{2}\left(1 - \tfrac{\gamma\|A\|^2}{\mu_f}\right)\|y - T_\gamma(y)\|^2 \\ &\ge \tfrac{c'}{2}\left(\|T_\gamma(y) - y_\star\|^2 + \|y - T_\gamma(y)\|^2\right) \\ &\ge \tfrac{c'}{4}\|y - y_\star\|^2.\end{aligned}$$

This shows that $y_\star$ is a strong local minimum for ψ_γ. □

In the context of Example V.3, notice that

$$JR_\gamma(y_\star) \text{ is nonsingular } \Leftrightarrow\ A_J H_{JJ} A_J^\top \text{ is nonsingular.}$$

Since $\nabla^2 f(x_\star) \succ 0$ by assumption, then $H_{JJ} \succ 0$ and nonsingularity of the Jacobian is equivalent to A_J being full row rank, *i.e.*, linear independence of the active constraints at $x_\star$ (the LICQ assumption).

VI. Superlinear convergence

The following definition (cf. [47, Eq. (7.5.2)]) gives the fundamental condition, on the sequence $(d^k)_{k\in\mathbb{N}}$ of directions, ensuring superlinear asymptotic convergence of Algorithm 1.

Definition VI.1 (Superlinear directions)**.** *For* $(y^k)_{k\in\mathbb{N}}$ *converging to* $y_\star$*, we say that* $(d^k)_{k\in\mathbb{N}}$ *is* superlinearly convergent *w.r.t.* $(y^k)_{k\in\mathbb{N}}$ *if*

$$\lim_{k\to\infty} \frac{\|y^k + d^k - y_\star\|}{\|y^k - y_\star\|} = 0. \tag{33}$$

When $y_\star$ is a strong minimizer, by [41, Cor. 3.6] the error bound (25) holds for some $\beta, \nu > 0$ and $Y_\star = \{y_\star\}$. This, by Thm. IV.3*(i)*, implies $y^k \to y_\star$. Therefore we have the following result.

Theorem VI.2. *Suppose that* f *and* g *satisfy Assumption 2, and that* (D) *has a (unique) strong minimizer* $y_\star$*. If* (33) *holds in Algorithm 1, then*

(i) the stepsize $\tau_k = 1$ *for all* k *sufficiently large,*

(ii) the cost $\psi(y^k) \to \mathbf{inf}\,\psi$ *Q-superlinearly,*

(iii) the dual iterates $y^k \to y_\star$ *Q-superlinearly,*

(iv) the primal iterates $x^k \to x_\star$ *R-superlinearly.*

Proof. We know from Thm.s V.2*(ii)* and V.4*(b)* that ψ_γ is twice differentiable with symmetric and positive definite Hessian $H_\star = \nabla^2\psi_\gamma(y_\star)$. We can expand ψ_γ around $y_\star$ and obtain

$$\begin{aligned}&\frac{\psi_\gamma(y^k + d^k) - \mathbf{inf}\,\psi}{\psi_\gamma(y^k) - \mathbf{inf}\,\psi} \\ &= \frac{\langle H_\star(y^k + d^k - y_\star), y^k + d^k - y_\star\rangle + o(\|y^k + d^k - y_\star\|^2)}{\langle H_\star(y^k - y_\star), y^k - y_\star\rangle + o(\|y^k - y_\star\|^2)} \\ &\le \frac{\|H_\star\|\left(\frac{\|y^k + d^k - y_\star\|}{\|y^k - y_\star\|}\right)^2 + \left(\frac{o(\|y^k + d^k - y_\star\|)}{\|y^k - y_\star\|}\right)^2}{\lambda_{\mathbf{min}}(H_\star) + \left(\frac{o(\|y^k - y_\star\|)}{\|y^k - y_\star\|}\right)^2}\end{aligned}$$

which vanishes for $k \to \infty$. In particular, eventually $\psi_\gamma(y^k + d^k) \le \psi_\gamma(y^k)$ will always hold, proving VI.2*(i)*. In turn, since eventually $\tilde{y}^k = y^k + \tau_k d^k = y^k + d^k$, using Thm. III.3*(ii)* and (22b) we have

$$\frac{\psi(y^{k+1}) - \mathbf{inf}\,\psi}{\psi(y^k) - \mathbf{inf}\,\psi} \le \frac{\psi_\gamma(\tilde{y}^k) - \mathbf{inf}\,\psi}{\psi_\gamma(y^k) - \mathbf{inf}\,\psi} \to 0,$$

which proves VI.2*(ii)*. Moreover, (33) reads

$$\|\tilde{y}^k - y_\star\|/\|y^k - y_\star\| \to 0. \tag{34}$$

Now, using nonexpansiveness of T_γ (cf. the proof of [19, Thm. 25.8]) one has

$$\|y^{k+1} - y_\star\| = \|T_\gamma(\tilde{y}^k) - T_\gamma(y_\star)\| \le \|\tilde{y}^k - y_\star\|$$

which, with (34), proves VI.2*(iii)*. VI.2*(iv)* follows from VI.2*(ii)* and Lem. A.2. □

When quasi-Newton directions are computed as in (12), superlinear convergence holds provided that the sequence of matrices $(B_k)_{k\in\mathbb{N}}$ satisfies the Dennis-Moré condition given in the following result. Such condition is satisfied for example by the modified Broyden method (13) under an assumption of *calm semidifferentiability* of R_γ at the solution, see [48, Thm. 6.8]. Notice that when R_γ is piecewise affine (PWA), under Assumption 2 this requirement is satisfied (combine Thm. V.2*(i)* with the fact that wherever a piecewise affine function is differentiable it is also locally Lipschitz-continuously differentiable). This is for instance the case of QPs, a frequent formulation in control applications, see §VII-A.

Theorem VI.3 (Dennis-Moré condition)**.** *Suppose that* f *and* g *strictly satisfy Assumption 2, and that* (D) *has a (unique) strong minimizer* $y_\star$*. If* $(d^k)_{k\in\mathbb{N}}$ *is selected according to* (12)*, with*

$$\lim_{k\to\infty} \frac{\|(B_k - JR_\gamma(y_\star))d^k\|}{\|d^k\|} = 0, \tag{35}$$

then $(d^k)_{k\in\mathbb{N}}$ *is superlinearly convergent with respect to* $(y^k)_{k\in\mathbb{N}}$*. In particular, the conclusions of Theorem VI.2 hold.*

Proof. From Thm.s V.2*(i)* and V.4*(c)* we know that R_γ is strictly differentiable, with nonsingular Jacobian $J_\star =$

$JR_\gamma(y_\star)$. Let us denote $r^k = z^k - Ax^k = R_\gamma(y^k)$ for simplicity. By using (12) and (35), and by applying the reverse triangle inequality we obtain

$$0 \leftarrow \frac{\|r^k - J_\star d^k\|}{\|d^k\|} \geq \frac{\|J_\star B_k^{-1} r^k\|}{\|d^k\|} - \frac{\|r^k\|}{\|d^k\|} \geq \alpha - \frac{\|r^k\|}{\|d^k\|},$$

where $\alpha = \sqrt{\lambda_{\min}(J_\star^\top J_\star)} > 0$ since $J_\star$ is nonsingular. Therefore,

$$\liminf_{k\to\infty} {}^{\|r^k\|}/_{\|d^k\|} \geq \alpha$$

and as a consequence $\|d^k\| \leq (2/\alpha)\|r^k\|$ for all k sufficiently large. Since $r^k \to 0$ by Thm. IV.3*(i)*, then $d^k \to 0$. We have

$$0 \leftarrow \frac{r^k - J_\star d^k}{\|d^k\|} = \frac{r^k + J_\star d^k - R_\gamma(y^k + d^k)}{\|d^k\|} + \frac{R_\gamma(y^k + d^k)}{\|d^k\|}.$$

The first summand in the above equation tends to zero because of strict differentiability of R_γ at $y_\star$, therefore

$$R_\gamma(y^k + d^k)/\|d^k\| \to 0.$$

By nonsingularity of $J_\star$ then $\|R_\gamma(y)\| \geq \alpha\|y - y_\star\|$ for all y sufficiently close to $y_\star$, and since $y^k + d^k \to y_\star$ we have

$$\begin{aligned} 0 \leftarrow \frac{R_\gamma(y^k + d^k)}{\|d^k\|} &\geq \frac{\alpha\|y^k + d^k - y_\star\|}{\|d^k\|} \\ &\geq \frac{\alpha\|y^k + d^k - y_\star\|}{\|y + d^k - y_\star\| + \|y^k - y_\star\|}. \end{aligned}$$

This implies $\|y^k + d^k - y_\star\|/\|y^k - y_\star\| \to 0$, *i.e.*, $(d^k)_{k\in\mathbb{N}}$ is superlinearly convergent with respect to $(y^k)_{k\in\mathbb{N}}$. □

VII. Simulations: Linear MPC

We now present numerical results obtained with the proposed algorithm. The code reproducing the results in this section is available online.[3] In NAMA we used $\beta = 0.5$ and $\tau_{\min} = 10^{-3}$ (see Remark IV.2). Furthermore, in all experiments we computed directions $(d^k)_{k\in\mathbb{N}}$ according to the L-BFGS method, with memory 20, which is able to scale with the problem dimension much better then full quasi-Newton update formulas. All experiments were performed using MATLAB 2016b (v9.1.0) on a MacBook Pro running macOS 10.12, with an Intel Core i5 CPU (2.7 GHz) and 8 GB of memory.

We consider finite horizon, discrete time, linear optimal control problems of the form

$$\begin{aligned} \underset{\substack{x_0,\dots,x_N \\ u_0,\dots,u_{N-1}}}{\textbf{minimize}} \quad & \sum_{i=0}^{N-1} \ell_i(x_i, u_i) + \ell_N(x_N) && \text{(36a)} \\ \textbf{subject to} \quad & x_0 = x_{\text{init}}, && \text{(36b)} \\ & x_{i+1} = \Phi_i x_i + \Gamma_i u_i + c_i,\ i = 0, \dots, N-1, && \text{(36c)} \end{aligned}$$

where $x_0, \dots, x_N \in \mathbb{R}^{n_x}$ and $u_0, \dots, u_{N-1} \in \mathbb{R}^{n_u}$, and

$$\begin{aligned} \ell_i(x, u) &= q_i(x, u) + g_i(L_i(x, u)), && \text{(36d)} \\ \ell_N(x) &= q_N(x) + g_N(L_N x). && \text{(36e)} \end{aligned}$$

Here functions q_i are strongly convex (typically quadratic), g_i are proper, closed, convex functions, while L_i are linear mappings, for $i = 0, \dots, N$. For example, for a convex set C, one can set

$$\begin{aligned} g_i(\cdot) &= \delta_C(\cdot) && \text{(hard constraints)} \\ g_i(\cdot) &= \alpha\,\mathbf{dist}_C(\cdot), \quad \alpha > 0, && \text{(soft constraints)} \end{aligned}$$

[3] https://github.com/kul-forbes/NAMA-experiments

Set C here is typically the nonpositive orthant or a box, but can be any other convex set onto which one can efficiently project. When $C = [a_1, b_1] \times \dots \times [a_d, b_d]$ is a d-dimensional box, then one can alternatively model soft constraints as

$$g_i(z) = \textstyle\sum_{j=1}^d \alpha_j \big| z_j - \mathbf{max}\{a_j, \mathbf{min}\{b_j, z_j\}\}\big|. \qquad (37)$$

Problem (36) takes the form (P) by reformulating it as follows (see also [1], [49], [50]). Denote the full sequence of states and inputs as $\bar{x} = (x_0, u_0, x_1, u_1, \dots, x_N)$, and let

$$S(p) = \{\bar{x} \mid x_i = \Phi_i x_i + \Gamma_i u_i, x_0 = p\}$$

be the affine subspace of feasible trajectories of the system having initial state p. Then in (P)

$$\begin{aligned} f(\bar{x}) &= \textstyle\sum_{i=0}^{N-1} q_i(x_i, u_i) + q_N(x_N) + \delta_{S(x_{\text{init}})}(\bar{x}), \\ g(\bar{z}) &= \textstyle\sum_{i=0}^{N} g_i(z_i), \quad A = \mathbf{diag}(L_0, \dots, L_N). \end{aligned}$$

Let us further denote by $\bar{y} = (y_0, \dots, y_N)$ the dual variable associated with the above problem. In this case, in the alternating minimization step 1 of NAMA, the iterate $\bar{x}^k$ is obtained by solving

$$\begin{aligned} \textbf{minimize} \quad & \textstyle\sum_{i=0}^{N-1} q_i(x_i, u_i) + \langle y_i^k, L_i(x_i, u_i)\rangle \\ & + q_N(x_N) + \langle y_N^k, L_N x_N\rangle. \\ \textbf{subject to} \quad & x_{i+1} = \Phi_i x_i + \Gamma_i u_i + c_i,\ i = 0, \dots, N-1. \end{aligned}$$

This is an unconstrained LQR problem whose solution can be efficiently computed with a Riccati-like recursion procedure, in the typical case where $q_0, \dots, q_N$ are quadratic, see [49, Alg.s 3, 4]. The expensive "factor" step only needs to be performed once, before the main loop of the algorithm takes place. At every iteration one needs to perform merely a forward-backward sweep and no matrix inversions are required. Furthermore

$$\begin{aligned} \bar{z}_i^k &= \mathbf{prox}_{\gamma^{-1} g_i}(\gamma^{-1} y_i^k + L_i(x_i^k, u_i^k)), \quad i = 0, \dots, N-1, \\ \bar{z}_N^k &= \mathbf{prox}_{\gamma^{-1} g_N}(\gamma^{-1} y_N^k + L_N(x_N^k)), \end{aligned}$$

which in the case of hard/soft constraints essentially consist of projections onto the constrained sets.

A. Aircraft control

We applied the proposed method to the AFTI-16 aircraft control problem [50], [51] with $n_x = 4$ states and $n_u = 2$ inputs, for a sampling time $T_s = 0.05$ seconds. The objective is to drive the *pitch angle* from 0° to 10°, and then back to 0°. We simulated the system for 4 seconds, at the sampling time $T_s = 0.05$, using $N = 50$ and quadratic costs

$$\begin{aligned} q_i(x, u) &= \tfrac{1}{2}\|x - x_{\text{ref}}\|_Q^2 + \tfrac{1}{2}\|u\|_R^2, \quad i = 0, \dots, N-1, \\ q_N(x) &= \tfrac{1}{2}\|x - x_{\text{ref}}\|_{Q_N}^2, \end{aligned}$$

where $Q = \mathbf{diag}(10^{-4}, 10^2, 10^{-3}, 10^2)$, $Q_N = 100 \cdot Q$ and $R = \mathbf{diag}(10^{-2}, 10^{-2})$. The reference was set $x_{\text{ref}} = (0, 0, 0, 10)$ for the first 2 seconds, and $x_{\text{ref}} = (0, 0, 0, 0)$ for the remaining 2 seconds. Furthermore, we imposed hard box constraints on the inputs, and soft box constraints (37) on the states, with weights 10^6. Since soft constraints can be formulated into a QP, by adding linearly penalized nonnegative slack variables, we also compared against standard QP solvers.

The dual problem has a condition number of 10^8. To improve the convergence of the algorithms we therefore considered scaling the dual variables according to the *Jacobi scaling*,

which consists of a diagonal change of variable (in the dual space) enforcing the (dual) Hessian to have diagonal elements equal to one (see also [50], [52] on the problem of preconditioning fast dual proximal gradient methods). Note that a diagonal change of variable in the dual space simply corresponds to a scaling of the equality constraints, when the problem is equivalently formulated as (P′).

We compared NAMA against GPAD [49], which is equivalent to fast AMA [53] in this context, qpOASES v3.2.0 [54] and the commercial QP solver MOSEK v7.1. We also compared against the cone solvers ECOS v2.0.4 [55], SDPT3 v4.0 [56] and SeDuMi v1.34 [57], all accessed through CVX v2.1 in MATLAB: note that the CPU time for these methods does not include the problem parsing and preprocessing by CVX, but only considers the actual running time of the solvers. The results of the simulations are reported in Table I. As termination criterion for NAMA and GPAD we used $\|R_\gamma(y^k)\|_\infty \leq \epsilon_{\text{tol}} = 10^{-4}$. We also report the (average and maximum) number of x- and z-minimization steps performed by NAMA: due to the structure of f, the x-update is a linear mapping, and consequently we can save its computation during the backtracking line-search. GPAD, in contrast, performs one alternating minimization per iteration.

Apparently, NAMA greatly improves the convergence performance with respect to GPAD. When the problem is prescaled, our method performs favorably also with respect to the other QP and cone solvers considered. One must keep in mind that NAMA was executed using a generic, high-level MATLAB implementation. As computation times become smaller and smaller, overheads due to the runtime environment get more and more relevant in the total CPU time. A tailored, low-level implementation of the same algorithm could significantly decrease the CPU times shown in Table I: this is also reported in [50], where a speedup of more than a factor 20 is observed using C code generation.

B. Oscillating masses

Next, we consider a chain of oscillating masses connected by springs, with both ends attached to walls. The chain is composed of $2K$ bodies of unit mass, the springs have constant 1 and no damping, and the system is controlled through K actuators, each being a force acting on a pair of masses, as depicted in Figure 1. Therefore $n_x = 4K$ (the states are the displacement from the rest position and velocity of each mass) and $n_u = K$. The inputs are constrained in $[-0.5, +0.5]$, while the position and velocity of each mass is constrained in $[-4, +4]$.

The continuous-time system was discretized with a sampling time $T_s = 0.5$. Like in the previous example, we considered quadratic costs with $Q = Q_N = I_{n_x}$, $R = I_{n_u}$ and hard constraints on state and input. Furthermore, we imposed a quadratic terminal constraint

$$\tfrac{1}{2}\langle Px_N, x_N\rangle \leq \delta, \tag{38}$$

where P solves the Riccati equation related to the discrete-time LQR problem. Constraint (38) can be enforced by taking L_N in (36) as the Cholesky factor of P, so that $L_N^\top L_N = P$, and g_N as the indicator of the Euclidean ball of radius $\sqrt{\delta}$.

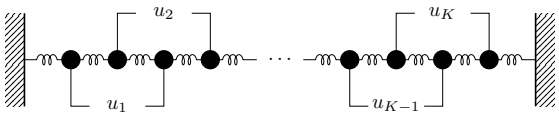

Figure 1. Oscillating masses, schematic representation of the simulated system.

Parameter δ is selected so as to ensure that no constraints are violated in such ellipsoidal set.

We simulated different scenarios, each with a different prediction horizon $N \in \{10, 20, \ldots, 50\}$, with $K = 8, 16$. For each scenario we selected 50 random initial states x_{init} by solving random feasibility problems (*e.g.*, with a cone solver) so as to ensure that a feasible trajectory starting from x_{init} exists. Every algorithm was executed with the same set of initial conditions. The results of this experiment are shown in Figure 2. In addition to fast AMA, we compared NAMA against ECOS, SDPT3 and SeDuMi, all accessed through CVX in MATLAB. NAMA compares favorably with all the other methods in this example, and in particular outperforms fast AMA, both on average and in the worst case.

VIII. Conclusions

In this work we presented NAMA, a line-search method for minimizing the sum of two convex functions, one of which is assumed to be strongly convex, while the other is composed with a linear transformation. The method is an extension of the classical alternating minimization algorithm (AMA), performing an additional line-search step over the *alternating minimization envelope* associated with the problem. By appropriately selecting the line-search directions, for example according to quasi-Newton methods for solving the optimality conditions $R_\gamma(y) = 0$, we have shown that the algorithm converges superlinearly provided that ordinary second-order sufficiency conditions hold for the envelope function at the (unique) dual solution. At the same time, the algorithm possesses the same global sublinear and local linear convergence rates as AMA. Numerical experiments with the proposed method on linear MPC problems suggest that NAMA is able to significantly speed up the convergence of AMA, comparing favorably against its accelerated variant and other state-of-the-art solvers even when limited-memory methods, such as L-BFGS, are used to compute the search directions.

Appendix

Lemma A.1. *Let $y, w \in \mathbb{R}^m$ and $\gamma > 0$. Then,*

$$\begin{aligned}\psi(w) \geq \psi_\gamma(y) + \tfrac{\gamma}{2}\|Ax(y) - z_\gamma(y)\|^2 \\ + \langle z_\gamma(y) - Ax(y), w - y\rangle.\end{aligned} \tag{39}$$

Proof. By (1) we have

$$\begin{aligned}f(x(y)) + f^*(-A^\top w) \geq -\langle Ax(y), w\rangle, \\ g(z_\gamma(y)) + g^*(w) \geq \langle z_\gamma(y), w\rangle.\end{aligned}$$

By summing the two inequalities and using the definition of ψ_γ, after manipulations one obtains the result. □

		Iterations		x-updates		z-updates		CPU time (ms)	
		avg.	max.	avg.	max.	avg.	max.	avg.	max.
GPAD	(no scaling)	6408.2	118.3 k	-	-	-	-	1645.7	23331.9
NAMA (L-BFGS, mem = 20)	(no scaling)	66.0	748	134.2	1527	139.7	1565	36.5	464.6
GPAD	(Jacobi scaling)	104.8	491	-	-	-	-	21.0	96.7
NAMA (L-BFGS, mem = 20)	**(Jacobi scaling)**	**9.7**	**42**	**18.7**	**85**	**18.8**	**88**	**4.9**	**21.3**
qpOASES								2362.7	2573.3
qpOASES	(warm-started)							14.6	286.9
MOSEK								207.4	539.4
ECOS								23.6	37.6
SDPT3								607.7	890.6
SeDuMi								137.2	266.2

Table I

AIRCRAFT CONTROL, PERFORMANCE OF THE ALGORITHMS IN THE CASE OF THE AFTI-16 PROBLEM, FOR $T_s = 50$ MS AND $N = 50$. GPAD AND NAMA WERE STOPPED AS SOON AS $\|R_\gamma(y^k)\|_\infty \leq \epsilon_{\text{TOL}} = 10^{-4}$. SINCE THE PROBLEM IS ILL-CONDITIONED, WE ALSO APPLIED THE METHODS BY PRESCALING THE DUAL PROBLEM. THE NUMBER OF x- AND z- UPDATES OF GPAD EQUALS THE NUMBER OF ITERATIONS. NAMA WAS EXECUTED USING A GENERIC IMPLEMENTATION IN MATLAB, WHILE THE OTHERS QP AND CONE SOLVERS CONSIDERED ARE ALL IMPLEMENTED IN C/C++.

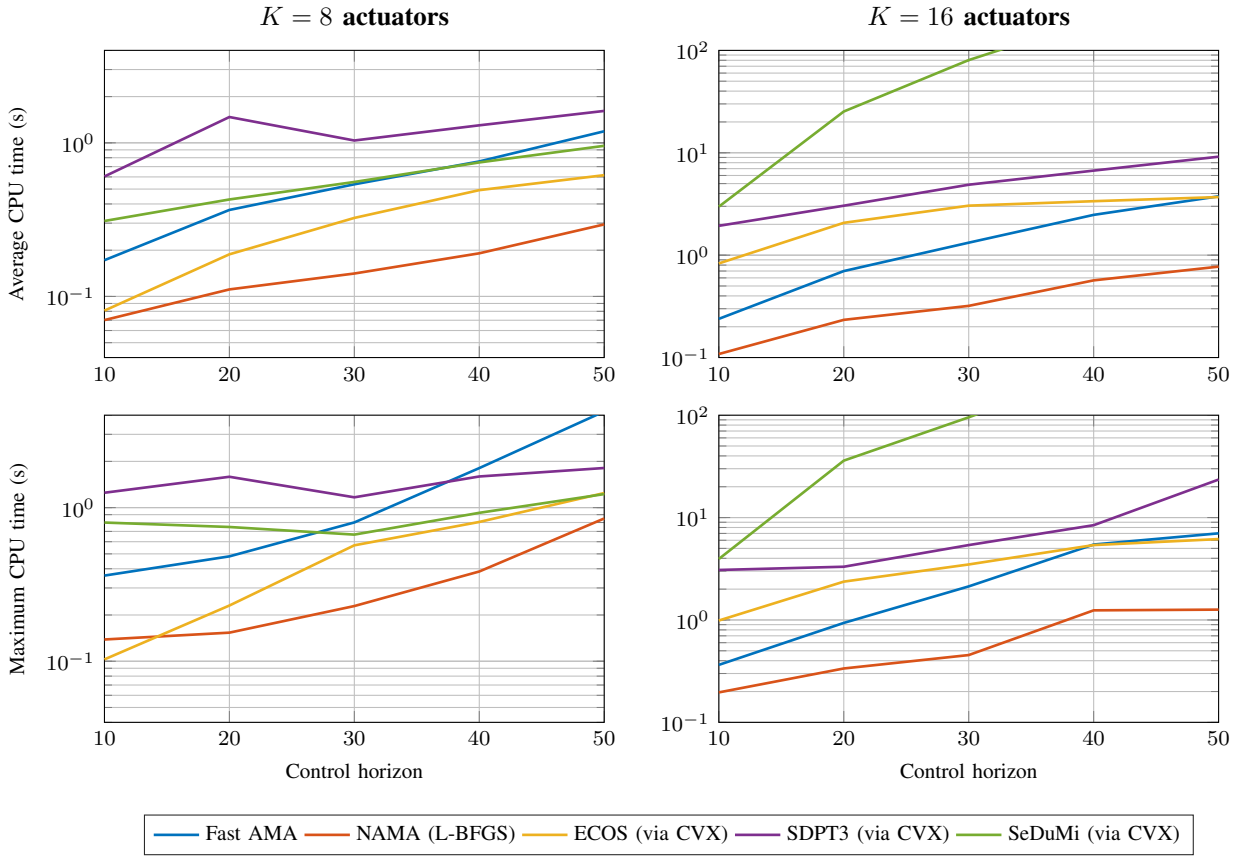

Figure 2. Oscillating masses, average and maximum CPU time (in seconds) for increasing prediction horizon and 50 randomly selected initial states. First column: $K = 8$ actuators. Second column: $K = 16$ actuators. Fast AMA and NAMA were stopped as soon as $\|R_\gamma(y^k)\|_\infty \leq \epsilon_{\text{tol}} = 10^{-4}$.

Lemma A.2. *For all $y \in \mathbb{R}^m$ it holds*

$$\tfrac{\mu_f}{2}\|x(y) - x_\star\|^2 \leq \psi(y) - \inf \psi.$$

Proof. From the optimality condition of the problem defining $x(y)$, one obtains $-A^\top y \in \partial f(x(y))$. Then, by strong convexity of f one gets

$$f(x(y)) - \langle A^\top y, x_\star - x(y)\rangle + \tfrac{\mu_f}{2}\|x(y) - x_\star\|^2 \leq f(x_\star).$$

By using (17a) in the above inequality we obtain

$$\tfrac{\mu_f}{2}\|x(y) - x_\star\|^2 - \langle Ax_\star, y\rangle \leq f(x_\star) + f^*(-A^\top y),$$

By using (1) on g we have instead

$$\langle Ax_\star, y\rangle \leq g(Ax_\star) + g^*(y).$$

By summing the last two inequalities one obtains

$$\tfrac{\mu_f}{2}\|x(y) - x_\star\|^2 \leq f(x_\star) + g(Ax_\star) + \psi(y),$$

and the claimed bound follows by strong duality. □

Lemma A.3. *Suppose that the following hold for (P):*

(i) $A\,\mathbf{ri}(\mathbf{dom}\, f) \cap \mathbf{ri}(\mathbf{dom}\, g) \neq \emptyset$ *(strict feasibility);*

(ii) $0 \in \mathbf{ri}\,\partial(f + g \circ A)(x_\star)$ *(strict complementarity).*

Then for any compact set U there is $\kappa > 0$ such that

$$\mathbf{dist}(y, Y_\star) \leq \kappa\big[\mathbf{dist}(-A^\top y, \partial f(x_\star)) + \mathbf{dist}(y, \partial g(Ax_\star))\big]$$

holds for all $y \in U$.

Proof. From Lem. A.3*(ii)* it follows that

$$\begin{aligned} 0 \in\ & \mathbf{ri}\left[\partial f(x_\star) + A^\top \partial g(Ax_\star)\right] \\ =\ & \mathbf{ri}\,\partial f(x_\star) + A^\top\, \mathbf{ri}\,\partial g(Ax_\star). \end{aligned} \tag{40}$$

In fact, the first inclusion is due to [17, Thm 23.9] in light of Lem. A.3*(i)*, and the equality is due to [17, Thm. 6.6]. Consider $W = \left\{w \mid -A^\top w \in \partial f(x_\star)\right\} \subseteq \mathbb{R}^m$. From (5),

$$Y_\star = W \cap \partial g(Ax_\star).$$

Furthermore, using (40) we obtain

$$\emptyset \neq \left\{w \mid -A^\top w \in \mathbf{ri}\,\partial f(x_\star)\right\} = \mathbf{ri}\, W,$$

where the equality is due to [17, Thm. 6.7], and the fact that $\mathbf{ri}\, W \cap \mathbf{ri}\,\partial g(Ax_\star) \neq \emptyset$. By [58, Cor. 5] then, we conclude that W and $\partial g(Ax_\star)$ are boundedly linearly regular: for any compact set U there is $\alpha > 0$ such that for all $y \in U$

$$\mathbf{dist}(y, Y_\star) \leq \alpha\big[\mathbf{dist}(y, W) + \mathbf{dist}(y, \partial g(Ax_\star))\big]. \tag{41}$$

Similarly, (40) implies with [58, Cor. 5] that the sets $L = \left\{(w, -A^\top w) \mid w \in \mathbb{R}^m\right\}$ and $M = \mathbb{R}^m \times \partial f(x_\star)$ are boundedly linearly regular. Observe that

$$L \cap M = \left\{(w, -A^\top w) \mid -A^\top w \in \partial f(x_\star)\right\}.$$

Therefore, there is $\beta > 0$ such that for all $y \in U$

$$\begin{aligned} \mathbf{dist}(y, W) \leq\ & \mathbf{dist}((y, -A^\top y), L \cap M) \\ \leq\ & \beta[\mathbf{dist}((y, -A^\top y), L) + \mathbf{dist}((y, -A^\top y), M)] \\ =\ & \beta\, \mathbf{dist}(-A^\top y, \partial f(x_\star)), \end{aligned}$$

where the second inequality is due to bounded linear regularity of L and M, while the equality holds since $(y, -A^\top y) \in L$ and $\mathbf{dist}((y, -A^\top y), M) = \mathbf{dist}(-A^\top y, \partial f(x_\star))$ for any y. Using the above inequality in (41) yields the result. □

Lemma A.4 (Twice differentiability of f^*)**.** *Suppose that f satisfies Assumption 2(i) for the primal-dual solution $(x_\star, y_\star)$. Then f^* is of class $\mathcal{C}^2$ around $y_\star$, with*

$$\nabla^2 f^*(y_\star) = H_f^\dagger.$$

Proof. From [20, Thm. 13.21] we know that f^* is twice epi-differentiable at v for $x \in \partial f^*(v)$ iff f is twice epi-differentiable at x for v, with the relation

$$\mathrm{d}^2 f^*(v|x) = \left[\mathrm{d}^2 f(x|v)\right]^*. \tag{42}$$

The cited proof trivially extends to strict twice differentiability, and in fact f^* turns out to be *strictly* twice epi-differentiable at $x_\star$. Since $\mathbf{range}(H_f) + S_f^\perp = \mathbb{R}^n$, by applying (42) to (27) and conjugating $\mathrm{d}^2 f(x_\star|-A^\top y_\star)$ by means of [18, Prop. E.3.2.1] we obtain that function f^* has purely quadratic second epi-derivative (as opposed to generalized quadratic)

$$\mathrm{d}^2 f^*(-A^\top y_\star|x_\star)[w] = \langle(\mathbf{\Pi}_{S_f} H_f \mathbf{\Pi}_{S_f})^\dagger w, w\rangle \overset{(29)}{=} \langle H_f^\dagger w, w\rangle$$

which is everywhere finite in particular. The proof now follows from [25, Cor. 4.7]. □

With similar reasonings, the following result easily follows.

Lemma A.5 (Twice epi-differentiability of g^*)**.** *Suppose that g (strictly) satisfies Assumption 2(ii) for a primal-dual solution $(x_\star, y_\star)$. Then g^* is (strictly) twice epi-differentiable at $y_\star$ for $Ax_\star$. More precisely, letting $\bar{S} = S_g^\perp + \mathbf{range}(H_g)$,*

$$\mathrm{d}^2 g^*(y_\star|Ax_\star) = \left[\mathrm{d}^2 g(Ax_\star|y_\star)\right]^* = \langle H_g^\dagger \cdot, \cdot\rangle + \delta_{\bar{S}}. \tag{43}$$

ACKNOWLEDGMENT

The authors would like to thank Dmitriy Drusvyatskiy for his contribution to the proof of Lemma A.3 in the Appendix.

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