Why chebyshev polynomials are cool

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Abstract

chebyshev polynomials have many uses in engineering. Although their properties are well known, they are little intuitive. The aim of this report is a better understanding of these special polynomials and their behaviour.

1 Definition

chebyshev polynomials of the first kind $T_n(x)$, $x \in [-1...1]$ can be defined in various ways. They are a solution of the Chebyshev differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0,$$

and can be defined recursively:

$$T_n(x) = 2xT_{n-1}(x) - 2T_{n-2}(x). \tag{1}$$

The same recursion formula holds for the chebyshev polynomials of the second kind $U_n(x)$ [1]. The difference between the two sets lies in the initial conditions. More specifically

$$T_0(x) = 1,\ T_1(x) = x$$

Another description is that:

$$T_n(x) = T_n(\cos(\theta)) = \cos(n\theta) \tag{2}$$

They form a complete orthogonal set on the interval $-1 \le x \le 1$ with respect to the weighting function $\frac{1}{\sqrt{1-x^2}}$, i.e.

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)T_n(x)dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n = 1, 2, 3, ... \end{cases} \tag{3}$$

$T_0(x)$ to $T_4(x)$ are shown in fig. 1.

2 Properties

chebyshev polynomials are known to be *equiripple*, making them attractive for e.g. design of filters and array beamformers (*Dolph-Chebyshev* synthesis). These designs are optimum in the sense that they minimise the maximum deviation from the desired behaviour.

Although less known, chebyshev polynomials are also optimum for polynomial interpolation. More specifically, to perform a degree N polynomial approximation in $[-1 \ldots 1]$, interpolation points should be chosen at the roots of $T_N(x)$. This leads to a nonuniform grid which is denser at the edges. The resulting polynomial is a linear combination of $T_0 \ldots T_{N-1}$, for which the coefficients are nothing else than the DCT of the function values on the nonuniform grid [2]! This result is quite remarkable at first sight.

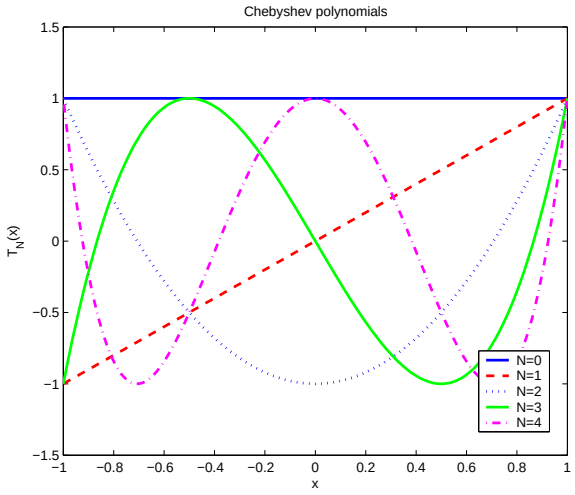

Figure 1: chebyshev polynomials of degree 0 to 4

3 Explanation

Considering the properties of chebyshev polynomials, and looking at fig. 1, they don't have much to do with polynomials, and rather appear to be related to trigonometric functions. Indeed, eq. 2 is crucial in the understanding. We can imagine the abcissa $x = \cos(\theta)$ to be the projection of a point traversing the unit circle, as shown in fig. 2. Also, functions of $\theta \in [0 \dots \pi]$ on the unit

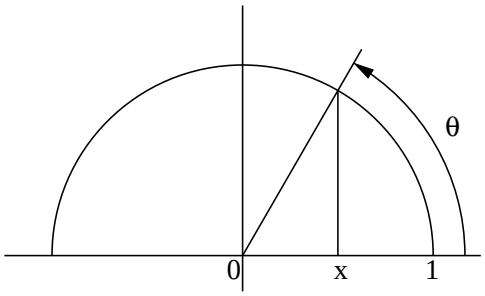

Figure 2: Transformation between x and θ

circle can be projected to the real axis, as in fig. 3. A specific example of this is $\cos(n\theta)$, which will be projected to $T_n(x)$. It is now clear that the two cosines in eq. 2 have a different meaning. The one represents the mere mapping from the unit circle, the other one is a 'real' cosine, which gives rise to the chebyshev polynomials.

The transformation between x and θ also sheds a new light on the orthogonality properties of eq. 3. Taking into account that

$$\frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-\cos(\theta)^2}} = \frac{1}{\sin(\theta)}, \text{ and}$$

$$dx = -\sin(\theta)d\theta,$$

eq. 3 is equivalent to:

$$\int_0^\pi \frac{1}{\sin\theta}\cos(m\theta)\cos(n\theta)\sin(\theta)d\theta = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{\pi}{2} & m = n = 1, 2, 3, \dots \end{cases}$$

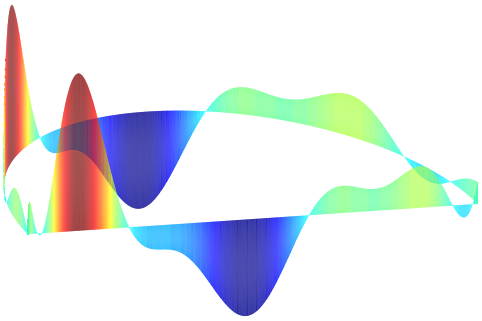

Figure 3: Relation between functions of x and θ

The weighting function disappears, and the result is quite obvious. Furthermore, it becomes clear that if one decomposes any function of θ in terms of cosines, the projected function of x will be decomposed in terms of chebyshev polynomials, see fig. 4. This makes the results from [2] less surprising.

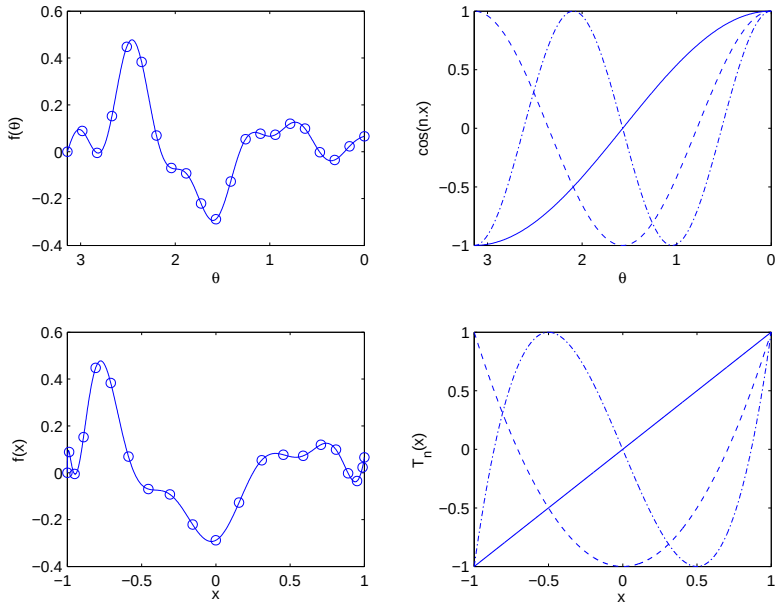

Figure 4: Decomposition of a function of θ in cosines corresponds to a Chebyshev expansion in x

References

[1] J.C. Mason and D.C. Handscomb, *Chebyshev polynomials*, pp. 145–163, Chapman & Hall/CRC, 2003.

[2] Victor-Emil Neogoe, "Chebyshev nonuniform sampling cascaded with the discrete cosine transform for optimum interpolation," *IEEE trans. acoust. speech and sig. proc.*, vol. 38, no. 10, pp. 1812–1815, 1990.