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Every nonlinear control system can be obtained by parametric convex programming

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Abstract

In this short note we define parametric convex programming (PCP) in a slightly different manner than it is usually done by extending convexity not only to variables but also to the parameters, and we show that the widely applied model predictive control (MPC) technique is a particular case of PCP. The main result of the note is an answer to the inverse question of PCP: which feedback laws can be generated by PCP? By employing results of convex analysis we provide a constructive proof - yet not computational - that allows us to conclude that every continuous feedback law can be obtained by PCP.

Index Terms

Parametric convex programming (PCP), model predictive control (MPC), continuous feedback laws.

I. Introduction

In parametric programming, an optimization problem is considered where the data are functions of some parameters. Explicit parametric programming techniques systematically subdivide the parameter space into characteristic regions where the optimal value and an optimizer are given as explicit functions of the parameters [1]–[3].

In recent years, a new interest in parametric programming arose from model predictive control (MPC) [3]–[8], a well-known technique in the system theory and optimal control community. MPC is the most successful advanced control technology implemented in industry due to its ability to handle complex systems with hard input and state constraints [9]–[11]. The essence of

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MPC is to determine a control profile that optimizes a cost criterion over a prediction window and then to apply this control profile until new process measurements become available. Then the whole procedure is repeated. Feedback is incorporated by using the measurements to update the optimization problem for the next step.

In this paper we study parametric convex programming (PCP), i.e. optimization problems where the functions defining the cost and the constraints are convex in both variables and parameters. The main result of the paper consists in deriving a constructive proof which shows that any continuous feedback law can be obtained from a PCP problem. Typically, such controllers may arise from MPC-type problems with convex stage costs and constraint sets.

This note is organized as follows. In Section II we define the main ingredients of an MPC scheme and in Section III we introduce PCP and we show that linear MPC can be formulated as a PCP. In Section IV we consider the inverse problem of PCP: for a given continuous feedback law we show that it is possible to find a PCP problem related to it. We conclude with some directions for future research.

II. Model Predictive Control

We consider the discrete time dynamic system

$$x(k+1) = f(x(k), u(k)), \quad k \in \mathbb{N}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The control and state sequence must satisfy $(x(k), u(k)) \in X$ for all $k \geq 0$, where usually X is a compact, convex subset of $\mathbb{R}^{n+m}$. We employ $\mathbf{u} = (u_0, u_1, \cdots, u_{N-1})$ to denote a control sequence over a prediction horizon of length N and $\phi(k; x, \mathbf{u})$ to denote the state solution of (1) at step k when the initial state is x at step 0 and the control $\mathbf{u}$ is applied. By definition $\phi(0; x, \mathbf{u}) := x$. We also define a cost function

$$V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x_i, u_i) + V_f(x_N), \tag{2}$$

where $x_i := \phi(i; x, \mathbf{u})$ (thus $x_0 = x$), $\ell(x_i, u_i)$ is the stage cost associated to state x_i and control u_i, and $V_f(x_N)$ is a terminal cost. For a given initial condition x, the set of feasible input

sequences is defined by:

$$\Pi_N(x) := \{\mathbf{u} \in \mathbb{R}^{Nm} : (x_i, u_i) \in X \; \forall i \in \{0, \cdots, N-1\}, \\ x_N \in X_f\}, \tag{3}$$

where X_f specify some terminal constraints on the state. We denote with X_N the set of initial states for which a feasible input sequence exists, i.e.

$$X_N := \{x \in \mathbb{R}^n : \Pi_N(x) \neq \emptyset\}. \tag{4}$$

The optimal control problem for a given initial state x is formulated as follows:

$$\mathbb{P}_N(x): \qquad V_N^0(x) := \inf_{\mathbf{u} \in \Pi_N(x)} V_N(x, \mathbf{u}). \tag{5}$$

It follows that the domain of V_N^0 is the set X_N. The optimal control problem $\mathbb{P}_N(x)$ yields an optimal control sequence $\mathbf{u}_N^0(x) \in \arg\min_{\mathbf{u} \in \Pi_N(x)} V_N(x, \mathbf{u})$ for all $x \in X_N$:

$$\mathbf{u}_N^0(x) = (u_0^0(x), u_1^0(x), \cdots, u_{N-1}^0(x)). \tag{6}$$

We can obtain an infinite-horizon controller by repeatedly solving the finite-horizon optimal control problem (5) where the current state of the plant is used as an initial state for the optimization. From the computed optimal control sequence only the first control sample $u_0^0(x)$ is implemented and the whole procedure is repeated at the next step when new measurements of the state are available. This is referred to as the receding horizon implementation of the controller and the resulting design method is called *model predictive control.*

Note that the optimization problem (5) depends on a parameter, the state x, appearing in the cost function but also in the constraints. Therefore, we can view (5) as a parametric program:

$$g^0(x) = \inf_u g(x, u) \quad \text{s.t. } (x, u) \in \Gamma, \tag{7}$$

where we can easily identify $u \leftarrow \mathbf{u}$, $g(x, u) \leftarrow V_N(x, \mathbf{u})$ and $\Gamma \leftarrow \{(x, \mathbf{u}) : x \in X_N, \mathbf{u} \in \Pi_N(x)\}$.

III. Parametric convex programming: convexity in both variables and parameters

We now define parametric convex programming:

Definition 3.1: Let Γ be a nonempty subset of $\mathbb{R}^n \times \mathbb{R}^m$ and let $g : \Gamma \to \mathbb{R},$. The parametric optimization problem (7) is called a *Parametric Convex Program* (PCP) if the function g is convex and if the set Γ is convex. □

The reader should note that in our definition of PCP we consider convexity in the combined variable vector (x, u) and not only in u as it is sometimes considered (see e.g. [4]).

First let us remark that linear MPC and sometimes even robust linear MPC can be posed in the framework of PCP, as defined before. Indeed, in this case $f(x, u) = Ax + Bu$, X and X_f are polyhedral sets (i.e. described by linear inequalities), and $\ell(x, u) = x^T Q x + u^T R u, V_\mathrm{f}(x) = x^T P x$, where $Q, R, P \succeq 0$ (i.e. the matrices Q, R and P are positive semi-definite [12]). Then the optimal control problem (5) can be written as follows:

$$V_N^0(x) = \inf_{\mathbf{u}} \ \mathbf{u}^T H \mathbf{u} + 2x^T F \mathbf{u} + x^T G x \qquad (8)$$
$$\text{s.t. } Dx + E\mathbf{u} \leq d,$$

where H, F, G, D, E and d are easily obtained from Q, R, the equality $x_i = A^i x + \sum_{j=0}^{i-1} A^j B u_{i-j-1}$ for all $i \geq 0$, with $x_0 = x$ as denoted before, and from the linear inequalities that define X and X_f. Since the matrix $\begin{bmatrix} H & F^T \\ F & G \end{bmatrix} \succeq 0$ it follows that $V_N(x, \mathbf{u}) = \mathbf{u}^T H \mathbf{u} + 2x^T F \mathbf{u} + x^T G x$ is a convex function in both variables and parameters. Moreover, the set $\Gamma = \{(x, u) : Dx + E\mathbf{u} \leq d\}$ is convex. In conclusion, the MPC feedback law can be obtained in this case as solution of a PCP followed by a projection (i.e. we extract the first m components of the optimizer).

The following theorem, which can be found in [1], provides sufficient conditions under which the optimal value function g^0 and the optimizer $\mu^0(x) \in \arg\min_u \{g(x, u) \quad \text{s.t. } (x, u) \in \Gamma\}$ are "well-behaved":

Theorem 3.1: Suppose that Γ is a compact and convex set and g is a continuous and convex function. Then, the optimal value function g^0 is continuous and convex. If additionally $g(x, \cdot)$ is strictly convex for each fixed x, then we can always select a continuous optimizer μ^0. □

Algorithms for solving PCP problems exactly or approximately can be found for example in [3], [13], [14].

IV. The Inverse Problem of Parametric Convex Programming

The discussion from the previous section suggests to consider the following problem: instead of characterizing the properties of a control law corresponding to a given optimal control problem, one might seek to find a convex optimization problem for which a given control law is optimal. For example, considering the feedback control law $\mu(x) = x\sin(x)$ (its graph is displayed in Figure 1) we want to determine the convex function g and the convex set Γ such that the corresponding optimizer of the PCP (7) is this particular feedback law. In the sequel we provide a constructive proof - yet not computational - that allows us to conclude that μ can be obtained from a PCP.

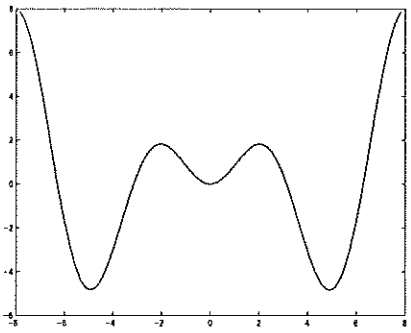

Fig. 1. *The graph of $\mu(x) = x\sin(x)$. Can this function be obtained as solution of a PCP?*

The main result of the paper is stated in the following theorem:

Theorem 4.1: Let $\Omega \subseteq \mathbb{R}^n$ be a closed convex set, $\mu : \Omega \to \mathbb{R}^m$ be a continuous function and $g^0 : \Omega \to \mathbb{R}$ be a continuous strictly convex function. Then we can construct a PCP in the form (7) which has the optimal value function g^0 and as optimizer μ.

Proof: The proof is divided into several lemmata given below. We consider the sets[1]:

$$S := \{(x, \mu(x), t) : x \in \Omega, g^0(x) \leq t\},$$
$$E := \mathbf{conv}(S),$$

and the epigraph of g^0 defined as [12]

[1]Given a set $S \subset \mathbb{R}^n$, $\mathbf{conv}(S)$ denotes the convex hull of S [12]: $\mathbf{conv}(S) = \{\sum_i \alpha_i x_i : \sum_i \alpha_i = 1,\ \alpha_i \geq 0 \text{ and } x_i \in S\ \forall i\}$.

$$\text{epi}g^0 := \{(x,t) : x \in \Omega, g^0(x) \leq t\}.$$

Lemma 4.2: For every $(x,u,t) \in E$, we have $g^0(x) \leq t$.

Proof: Indeed, in view of Caratheodory's Theorem (see e.g. Theorem 2.3 in [12]), there exist nonnegative numbers α_i, t_i and vectors $x_i \in \Omega$ for all $1 \leq i \leq n+m+1$ that satisfy the following conditions:

1) $\sum_{i=1}^{n+m+1} \alpha_i = 1, \ \alpha_i \geq 0 \ \forall i$
2) $(x_i, \mu(x_i), t_i) \in S \ \forall i$
3) $x = \sum_{i=1}^{n+m+1} \alpha_i x_i$
4) $u = \sum_{i=1}^{n+m+1} \alpha_i \mu(x_i)$
5) $t = \sum_{i=1}^{n+m+1} \alpha_i t_i$.

In view of the convexity of g^0, we have:

$$\begin{aligned} t = \sum_{i=1}^{n+m+1} \alpha_i t_i &\geq \sum_{i=1}^{n+m+1} \alpha_i g^0(x_i) \\ &\geq g^0\left(\sum_{i=1}^{n+m+1} \alpha_i x_i\right) = g^0(x). \end{aligned}$$

□

Lemma 4.3: The set E is the epigraph of a convex function.

Proof: It suffices to show that for every $(x,u,t) \in E$, and every $t' \geq t$, the point (x,u,t') belongs to E. This is obvious from Lemma 4.2: it suffices to replace every t_i by $t_i + (t'-t)$ in Caratheodory's decomposition of (x,u,t'). The convexity of E implies the convexity of the corresponding function. □

Lemma 4.4: Let A_M be subsets of $\mathbb{R}^n$ for every nonnegative real number M such that $A_{M_1} \subseteq A_{M_2}$ when $M_1 \leq M_2$. Let $A := \lim_{M\to\infty} A_M = \cup_{M\geq 0} A_M$. Then $\mathbf{conv}(A) = \cup_{M\geq 0}\mathbf{conv}(A_M)$.

Proof: The inclusion $\mathbf{conv}(A) \supseteq \cup_{M\geq 0}\mathbf{conv}(A_M)$ is easy to prove. If $x \in \cup_{M\geq 0}\mathbf{conv}(A_M)$, there exists a number $t \geq 0$ such that $x \in \mathbf{conv}(A_t)$. Hence, by Caratheodory's Theorem, we have some nonnegative numbers $\alpha_1, \cdots, \alpha_{n+1}$ that sum up to 1 and points $x_1, \cdots, x_{n+1} \in A_t$ for which $x = \sum_{i=1}^{n+1} \alpha_i x_i$. But $x_i \in A$, so that $x \in \mathbf{conv}(A)$.

We also use Caratheodory's Theorem to show the reverse inclusion. If $x \in \mathbf{conv}(A)$, there exists $\alpha_1, \cdots, \alpha_{n+1} \in \mathbb{R}_+$ and $x_1, \cdots, x_{n+1} \in A$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$ and $\sum_{i=1}^{n+1} \alpha_i x_i = x$. Each of these points x_i belongs to some A_{M_i} for a $M_i \geq 0$. Since the sets A_M form a growing sequence, we have $x_i \in A_M$, where $M := \max\{M_1, \cdots, M_{n+1}\}$. Hence $x \in \mathbf{conv}(A_M)$, and the inclusion is proved. □

Lemma 4.5: The set E is closed.

Proof: Let us define the sets

$$\begin{aligned} S_M &:= \{(x, \mu(x), t) : \ x \in \Omega, ||x|| \leq M, g^0(x) \leq t\}, \\ E_M &:= \mathbf{conv}(S_M). \end{aligned}$$

According to the previous lemma, we have $E = \cup_{M \geq 0} E_M$. Thus, it suffices to prove that for every $M \in \mathbb{R}$ the set E_M is closed. Let us fix $M \in \mathbb{R}$ such that E_M is nonempty. Observe that, as the intersection of convex sets, the set E_M is also convex. Now, let $\{(x_k, u_k, t_k)\}_{k \geq 0}$ be a sequence of E_M that converges to $z = (x, u, t)$, and let us prove that z belongs to E_M as well. We use Caratheodory's Theorem: there exist nonnegative numbers $\alpha_{i,k}, t_{i,k}$ and vectors $x_{i,k} \in E_M$ for all $1 \leq i \leq n+m+1$ and $k \in \mathbb{N}$ that satisfy the following conditions:

1) $\sum_{i=1}^{n+m+1} \alpha_{i,k} = 1, \quad \alpha_{i,k} \geq 0 \ \ \forall i, k \in \mathbb{N}$
2) $(x_{i,k}, \mu(x_{i,k}), t_{i,k}) \in S_M \ \forall i, k$
3) $x_k = \sum_{i=1}^{n+m+1} \alpha_{i,k} x_{i,k} \ \forall k$
4) $u_k = \sum_{i=1}^{n+m+1} \alpha_{i,k} \mu(x_{i,k}) \ \forall k$
5) $t_k = \sum_{i=1}^{n+m+1} \alpha_{i,k} t_{i,k} \ \forall k.$

Since the vectors $(\alpha_{1,k}, \cdots, \alpha_{n+m+1,k})$ belong to the $n+m+1$-dimensional simplex, which is a compact set, we can assume without loss of generality that they converge to the vector $(\alpha_1, \cdots, \alpha_{n+m+1})$. Similarly, as the sequences $\{x_{1,k}\}_{k \geq 0}, \cdots, \{x_{n,k}\}_{k \geq 0}$ belong to the compact set $\Omega \cap B[0, M]$, where $B[0, M] := \{x : \|x\| \leq M\}$, we can assume that they converge to $v_1, \cdots, v_n \in \Omega \cap B[0, M]$, respectively. Hence, we have $x_k \to \sum_{i=1}^{n+m+1} \alpha_i v_i$. By continuity of μ, we get $\mu(x_{i,k}) \to \mu(v_i)$, implying $u_k \to \sum_{i=1}^{n+m+1} \alpha_i \mu(v_i)$. Since the function g^0 is continuous

and convex, we can write:

$$t \leftarrow t_k = \sum_{i=1}^{n+m+1} \alpha_{i,k} t_{i,k} \geq$$
$$\sum_{i=1}^{n+m+1} \alpha_{i,k} g^0(x_{i,k}) \rightarrow \sum_{i=1}^{n+m+1} \alpha_i g^0(v_i) \geq g^0(x).$$

In conclusion $z \in E_M$. □

Let $g : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ be the function having E as epigraph. In view of the previous lemma, the function g is lower semi-continuous (see [15] for more details). According to our construction g and Γ are defined as follows:

$$\Gamma = \{(x,u) : \exists\ t < \infty \text{ such that } (x,u,t) \in E\}$$
$$g : \Gamma \rightarrow \mathbb{R},\ \ g(x,u) = \inf_t \{t \text{ s.t. } (x,u,t) \in E\}.$$

Lemma 4.6: For every $x \in \Omega$, the convex optimization problem $\inf_u \{g(x,u) \text{ s.t. } (x,u) \in \Gamma\}$ has a unique minimizer, which is $\mu(x)$.

Proof: Let us fix $x \in \Omega$. In view of Lemma 4.2, we know that $g^0(x) \leq \inf\{t : (x,u,t) \in E\}$. Moreover, the point $(x, \mu(x), g^0(x))$ achieves the equality. It remains to show that this is the only point of E doing so. Suppose that there is another point $z := (x, u, g^0(x)) \in E$. Invoking once again Caratheodory's Theorem, the point z can be represented as:

$$u = \sum_{i=1}^{n+m+1} \alpha_i \mu(x_i),\ x = \sum_{i=1}^{n+m+1} \alpha_i x_i, \text{ and}$$
$$g^0(x) = \sum_{i=1}^{n+m+1} \alpha_i t_i,$$

where $t_i \geq g^0(x_i)$. Now,

$$\sum_{i=1}^{n+m+1} \alpha_i g^0(x_i) \geq g^0\left(\sum_{i=1}^{n+m+1} \alpha_i x_i\right) =$$
$$g^0(x) \geq \sum_{i=1}^{n+m+1} \alpha_i g^0(x_i)$$

implies that equality holds throughout. Since the function g^0 is strictly convex, we deduce that $x_i = x$ for all i. □

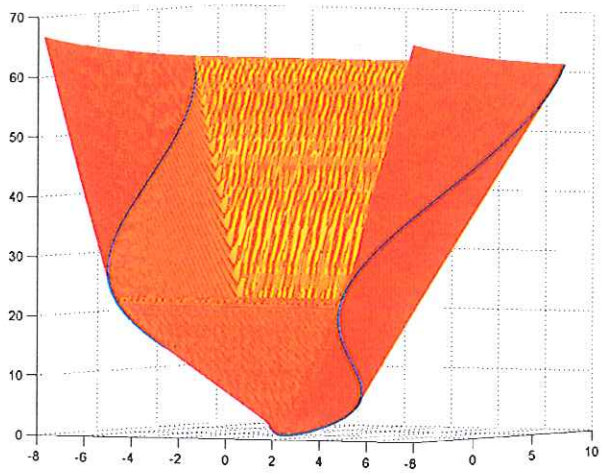

Fig. 2. *The epigraph E of function g for which PCP* (7) *has the optimizer* $\mu(x) = x\sin(x)$.

The procedure is illustrated in Figure 2 which displays the epigraph E of the cost function g corresponding to the feedback law given at the beginning of this section (i.e. $\mu(x) = x\sin(x)$).

From the proof of the previous lemma it follows that we need strict convexity of the optimal value function in order to prove uniqueness of the optimizer. This is not surprising since Theorem 3.1 also requires strict convexity of the cost function in order to be able to select a continuous optimizer.

Note that the construction provided in this section for the convex set Γ and the continuous and convex function g which define the PCP $\inf_u\{g(u,x) \text{ s.t } (x,u) \in \Gamma\}$ is not unique. We have many possibilities to select such a set Γ and function g.

V. Conclusions and Open Questions

In this note we have provided a constructive proof showing that any continuous feedback law can be obtained from solving a parametric convex program. It is well known that continuous nonlinear feedback laws can arise from solving model predictive control type problems with convex stage costs and constraint sets, a special case of parametric convex programming. Our result thus provides a positive answer to the question if every continous state feedback can

be obtained by embedded optimization, as in MPC. We would like to point out, however, that our construction of the parametric convex program is not unique and implies no computational algorithm.

A natural question that can arise from this note would be to particularize our results to piecewise linear controllers: can any continuous piecewise linear feedback law be obtained by parametric linear programming? Should such a construction be possible, it might offer computational advantages for explicit MPC algorithms.

A direction that is worth to be also investigated is the *inverse problem of nonlinear optimal control theory*: for a given system model f and control law μ find out if there exist stage costs ℓ and constraint sets X that give rise to a parametric convex program for which the control law μ is optimal, similar to a related question posed and answered by Kalman for the case of linear control systems [16].

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