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A canonical form for regular pencils of matrices

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Abstract: We construct a new canonical form for regular pencils of matrices, which is built with so-called generalized companion blocks. It is shown that every regular pencil is strictly equivalent to such a canonical form.

Keywords: System theory, Regular pencils of matrices, Canonical forms, Generalized companion form, Strict equivalence.

1. Introduction

In this paper, a canonical form for regular pencils of matrices is presented, which is different from that given in Gantmacher [2]. This canonical form is characterized by: (1) the block diagonal structure, (2) the diagonal blocks in so-called generalized companion form. It is shown that every regular pencil can be reduced to such a form by strict equivalence transformations.

As usual, we assume throughout the paper that the entries of all matrices belong to a field F.

2. Main result

First recall that two pencils $Es - A$ and $E_1 s - A_1$, each of dimension $n \times n$, are said to be *strictly equivalent* if there exist constant square nonsingular matrices M and N such that

$$M(Es - A)N = E_1 s - A_1. \tag{1}$$

A pencil $Es - A$ is said to be regular if the determinant $\det(Es - A)$ is not identically zero (i.e. not zero for all s) [2].

Now we state the main theorem.

Theorem. *Every regular pencil $Es - A$ is strictly equivalent to a canonical quasi-diagonal form*

$$Cs - L = \operatorname{diag}\{C_1 s - L_1, C_2 s - L_2, \ldots, C_k s - L_k\} \tag{2}$$

where each of the k blocks $C_i s - L_i$ assumes the form

$$C_i s - L_i = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_0 \end{bmatrix} s - \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_p \\ 1 & 0 & & 0 & -a_{p-1} \\ \vdots & & & \vdots & \\ 0 & 0 & \ldots & 1 & -a_1 \end{bmatrix}. \tag{3}$$

We shall call (2) with (3) the canonical form for regular pencils of matrices. Equation (3), in particular, is called the generalized companion form. It is interesting to note that the distinct feature of (3), compared with the normal companion form, is that a_0 can be zero. Since $\det(C_i s - L_i) = a_0 s^p + a_1 s^{p-1} + \cdots + a_p$, this clearly implies that the characteristic polynomial of (3) can have a lower degree than its dimension.

We now give the proof of the theorem by applying a sequence of strict equivalence operations.

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Proof of the theorem. Since $Es - A$ is a regular pencil, we can find a scalar c such that $\det(Ec - A) \neq 0$. Using this c, the pencil can be shifted to $E(s-c) - A_1$, where $A_1 = A - Ec$ and $\det A_1 \neq 0$. Multiplying the shifted pencil on the left by A_1^{-1}, we get

$$A_1^{-1}E(s-c) - I \tag{4}$$

where I denotes the identity matrix. This symbol will be used throughout to denote the identity of appropriate size if no confusion can arise. Then by recalling the first normal form of a matrix (Gantmacher [3]), a similarity transformation can bring (4) to

$$\operatorname{diag}\{P_1, P_2, \ldots, P_k\}(s-c) - I \tag{5}$$

where each P_i $(i = 1, \ldots, k)$ assumes the form

$$P_i = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_r \\ 1 & 0 & & 0 & -a_{r-1} \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -a_1 \end{bmatrix}. \tag{6}$$

(In order to avoid excessive use of subscripts, we denote the dimension of P_i by r instead of r_i.)

It is quite obvious that $\det P_i = 0$ if and only if $a_r = 0$. Therefore, if $a_r \neq 0$, P_i is nonsingular. Multiplied by P_i^{-1}, the corresponding block in (5) becomes $I(s-c) - P_i^{-1} = Is - (Ic + P_i^{-1})$. It can thus be brought to the first normal form by the similarity transformation, which is precisely of the form (2) with (3). In the second case, if $a_r = 0$, the corresponding block in (5) is

$$P_i(s-c) - I = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & & 0 & -a_{r-1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -a_1 \end{bmatrix} s - \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ c & 1 & & 0 & b_{r-1} \\ \vdots & & & & \vdots \\ 0 & 0 & & 1 & b_2 \\ 0 & 0 & \ldots & c & b_1 \end{bmatrix}. \tag{7}$$

We shall convert it into (11) using a set of strict equivalence transformations. The parameters b_i, e_i, f_i, g_i, p_i, q_i are used to indicate possible nonzero values. By annihilating $a_1, \ldots, a_{r-1}$ using column operations, (7) becomes

$$\begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} s - \begin{bmatrix} 1 & 0 & \ldots & 0 & e_r \\ c & 1 & & 0 & e_{r-1} \\ \vdots & & & & \vdots \\ 0 & 0 & & 1 & e_2 \\ 0 & 0 & \ldots & c & e_1 \end{bmatrix}. \tag{8}$$

Multiplying the first row by (-1) and moving it to the last row, (8) assumes the form

$$\begin{bmatrix} s-c & -1 & \ldots & 0 & f_r \\ 0 & s-c & & 0 & f_{r-1} \\ \vdots & & & & \vdots \\ 0 & 0 & & -1 & f_3 \\ 0 & 0 & & s-c & f_2 \\ 1 & 0 & \ldots & 0 & f_1 \end{bmatrix} = \begin{bmatrix} Is - B & f \\ e_1^{\mathrm{T}} & f_1 \end{bmatrix}, \tag{9}$$

where the superscript T denotes the transpose and

$$B=\begin{bmatrix} c & 1 & \dots & 0 \\ 0 & c & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & c \end{bmatrix}, \quad f^{\mathrm{T}}=[f_r \quad f_{r-1} \quad \dots \quad f_2], \quad e_1^{\mathrm{T}}=[1 \quad 0 \quad \dots \quad 0].$$

For any finite s,

$$\operatorname{rank}\begin{bmatrix} Is-B \\ e_1^{\mathrm{T}} \end{bmatrix}=r-1.$$

Therefore, B and e_1^{T} are an observable pair, we can thus find a transformation T such that (see, for example, Chen [1])

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Is-B \\ e_1^{\mathrm{T}} \end{bmatrix}T=\begin{bmatrix} Is-T^{-1}BT \\ e_1^{\mathrm{T}}T \end{bmatrix}=\begin{bmatrix} s & 0 & \dots & 0 & g_{r-1} \\ -1 & s & & 0 & g_{r-2} \\ \vdots & & & & \vdots \\ 0 & 0 & & -1 & s+g_1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

This allows to construct transformations which bring (9) into the form

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Is-B & f \\ e_1^{\mathrm{T}} & f_1 \end{bmatrix}\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}=\begin{bmatrix} Is-T^{-1}BT & T^{-1}f \\ e_1^{\mathrm{T}}T & f_1 \end{bmatrix}$$

$$=\begin{bmatrix} s & 0 & \dots & 0 & g_{r-1} & p_r \\ -1 & s & & 0 & g_{r-2} & p_{r-1} \\ \vdots & & & & & \vdots \\ 0 & 0 & & -1 & s+g_1 & p_2 \\ 0 & 0 & \dots & 0 & 1 & p_1 \end{bmatrix}. \tag{10}$$

By annihilating $g_{r-1},\dots,g_1$ with the last row in (10) and by multiplying then the last row by (-1), eventually we arrive at the form

$$\begin{bmatrix} s & 0 & \dots & 0 & 0 & q_r \\ -1 & s & & 0 & 0 & q_{r-1} \\ \vdots & & & & & \vdots \\ 0 & 0 & & -1 & s & q_2 \\ 0 & 0 & \dots & 0 & -1 & q_1 \end{bmatrix} \tag{11}$$

which is clearly strictly equivalent to P_i, and it assumes the form (3) with a_0 being zero.

The theorem is proven.

It may be observed that if E is nonsingular, each C_i $(i=1,\dots,k)$ in (2) can further be brought to the identity matrix of appropriate dimension. Hence the canonical form (2) reduces to the well-known first normal form for the matrix $E^{-1}A$.

Let us compare (2) with the canonical form given in [2],

$$\operatorname{diag}\{N_1, N_2,\dots,N_d, Is-J\} \tag{12}$$

where the first d diagonal blocks in (12) correspond to d infinite elementary divisors of the regular pencil

$Es - A$, and the last diagonal block $Is - J$ is uniquely determined by the finite elementary divisors of the given pencil. We note that unlike (12), in (2) the infinite elementary divisors are not separately represented from the finite ones. Thus, these two kinds of elementary divisors are treated on an equal footing in the canonical form (2).

3. Conclusions

The main result of the letter is a very interesting alternative of the well-known canonical form for the regular pencils of matrices [2, p. 28].

In fact, we also explored in Tan and Vandewalle [4] the relationship among these two canonical forms and the connection between the invariant polynomial and the canonical form (2), thereby providing an alternative proof for the theorem. Moreover, it is shown in [4] that (2) is unique, thus it is justified to call this form 'canonical'. Recent research has shown that this canonical form finds wide applications in system theory and more applications may be expected.

References

[1] C.T. Chen, *Introduction to Linear System Theory* (Holt, Rinehart and Winston, New York, 1970).
[2] F.R. Gantmacher, *The Theory of Matrices*, Vol. 2 (Chelsea, New York, 1960).
[3] F.R. Gantmacher, *The Theory of Matrices*, Vol. 1 (Chelsea, New York, 1960).
[4] S.H. Tan and J. Vandewalle, Generalized invariant polynomials and generalized companion form, ESAT internal report, K.U. Leuven (1985).