

Tensor-based methods for spectral analysis

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  - The complex exponential model;
  - Standard Matrix approach;



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  - Rank-1 terms
  - n-mode vectors
  - The higher-order SVD
  - Dimensionality reduction



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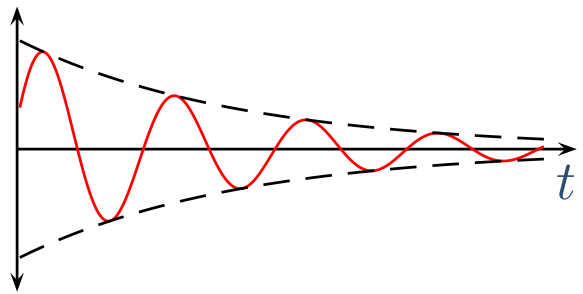
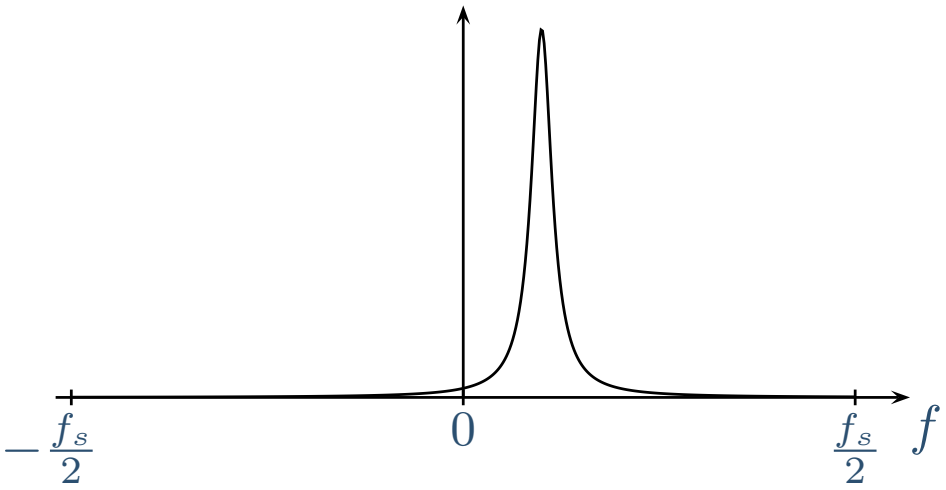
The complex exponential model



The discrete-time model has the following form:

$$x_n = \sum_{k=1}^{K} a_k \exp\{j\varphi_k\} \exp\{(\alpha_k + 2j\pi\nu_k)n.\Delta t\} + e_n, \quad n = 0, \ldots, N-1, \tag{1}$$



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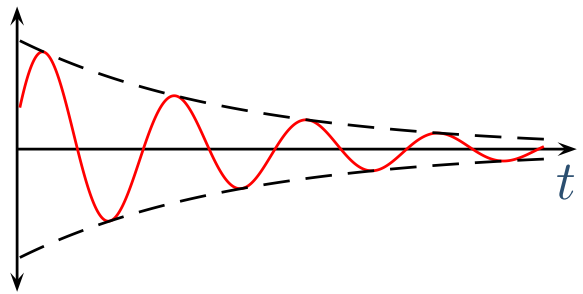

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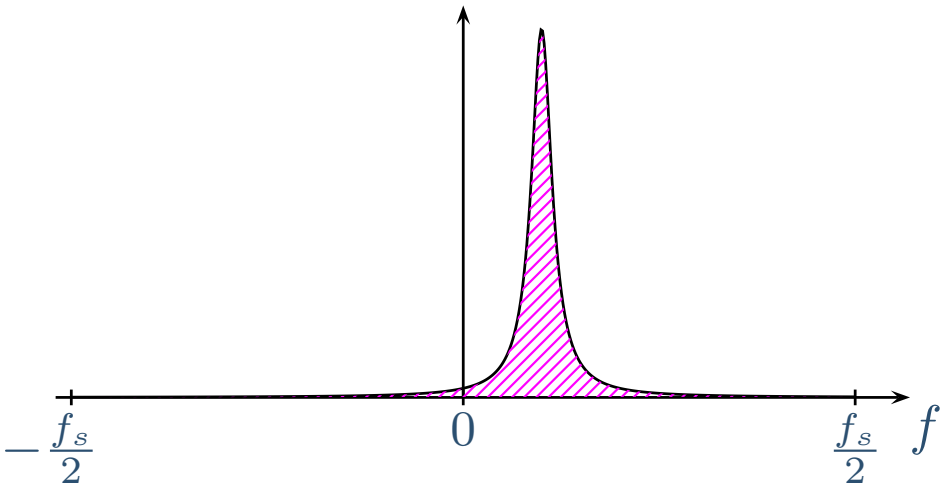
amplitude



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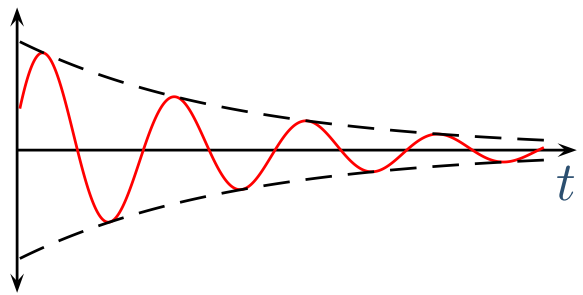

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amplitude

phase

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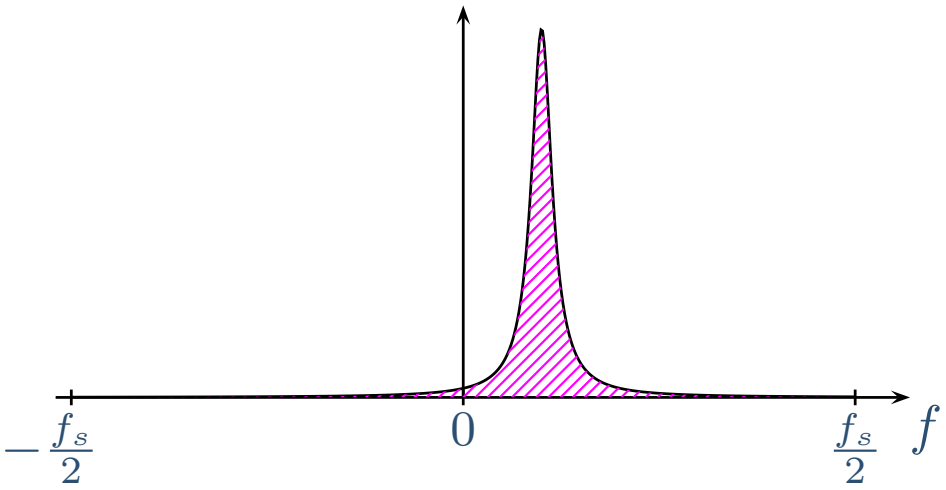
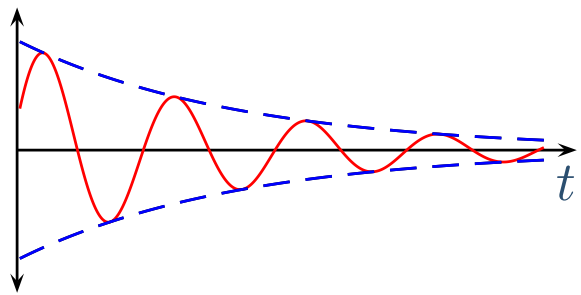




The complex exponential model



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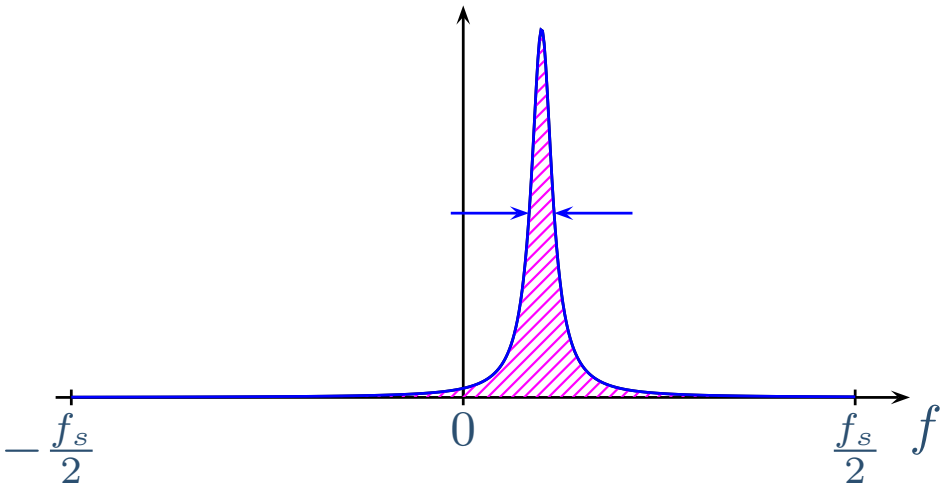
amplitude

phase

damping factor





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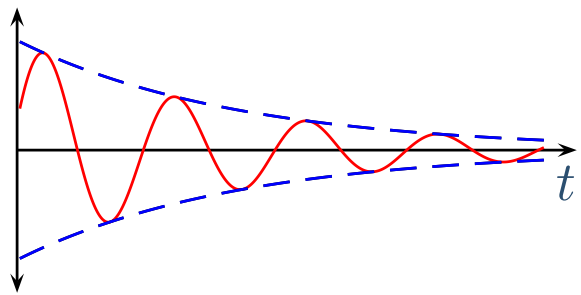

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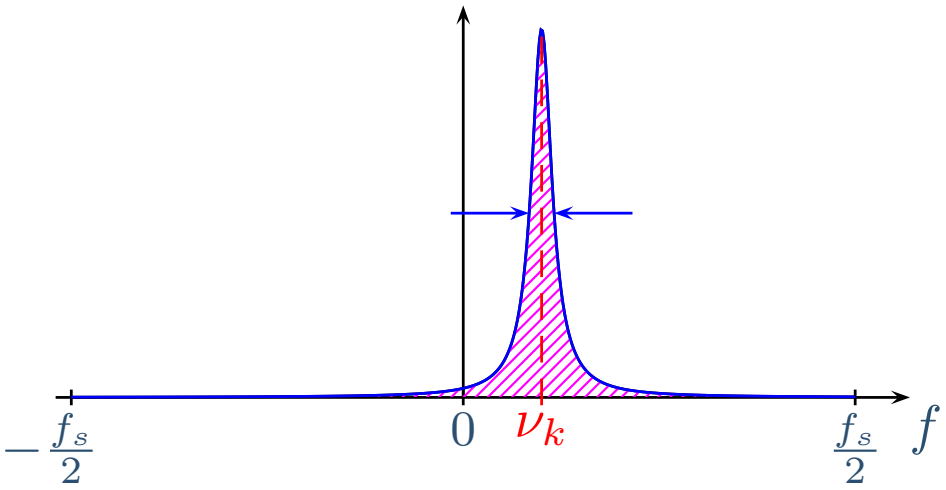
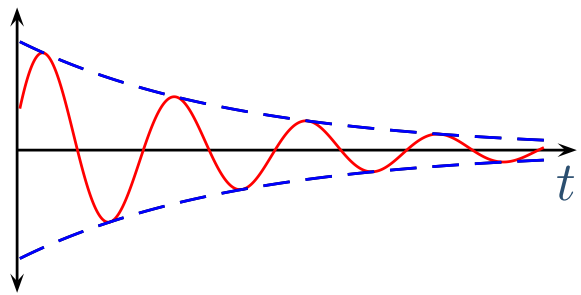
amplitude — phase — damping factor — frequency





The complex exponential model



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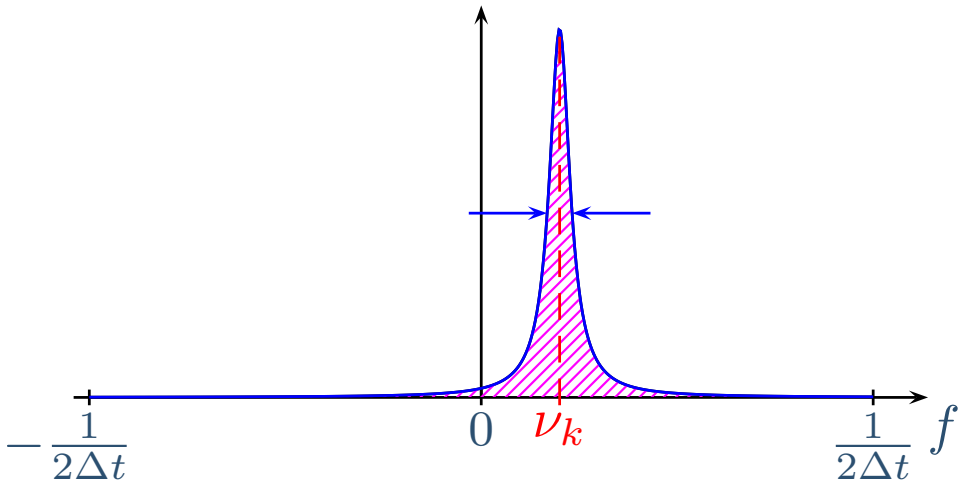
amplitude

phase

damping factor

frequency

sampling time interval



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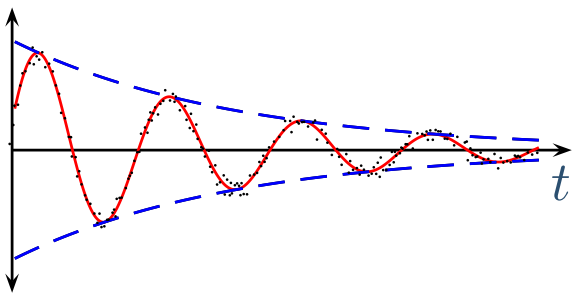

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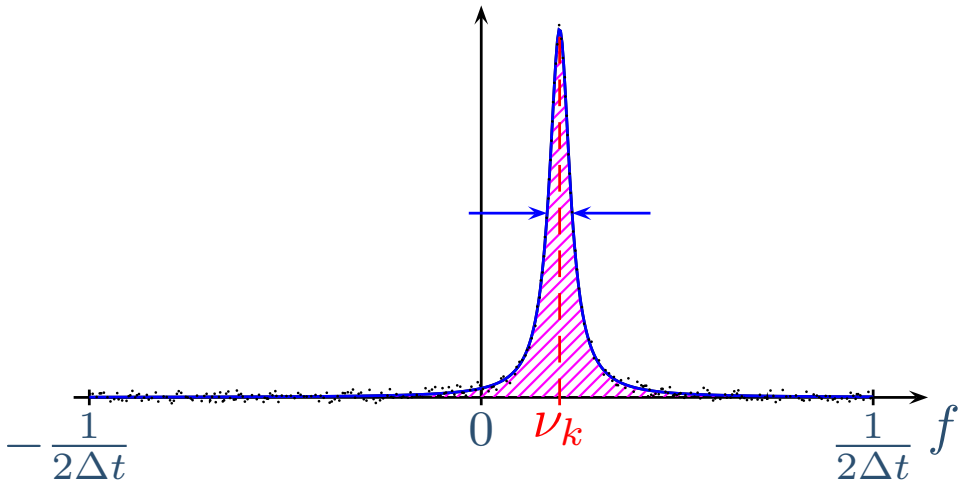
amplitude — phase — damping factor — frequency — WGN — sampling time interval



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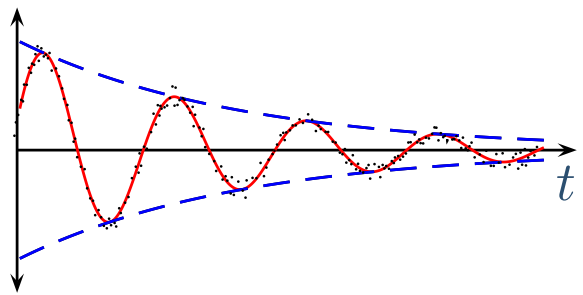

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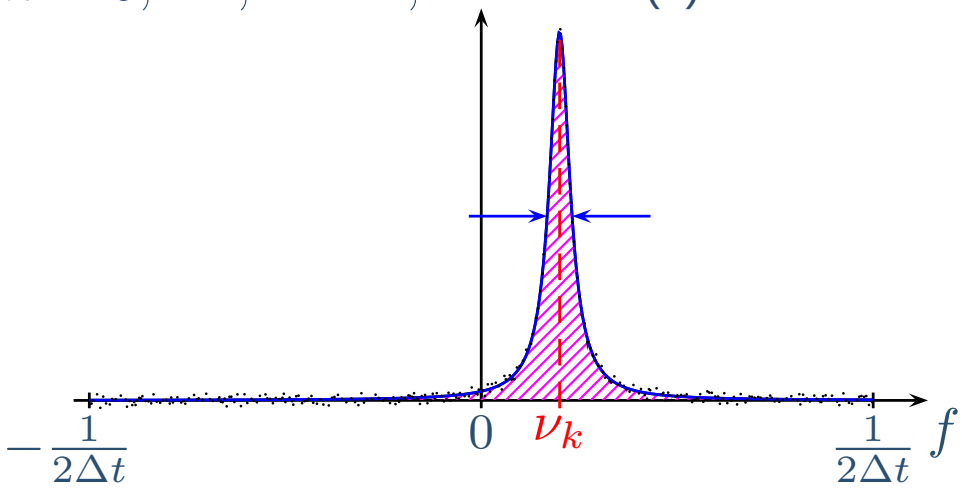
amplitude — phase — damping factor — frequency — WGN — sampling time interval

$$\boxed{x_n = \sum_{k=1}^{K} c_k z_k^n + e_n} \quad n = 0, \ldots, N-1, \qquad (2)$$





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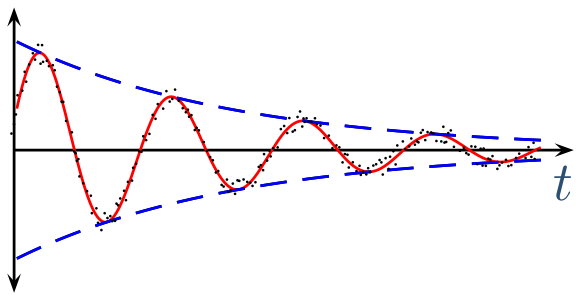

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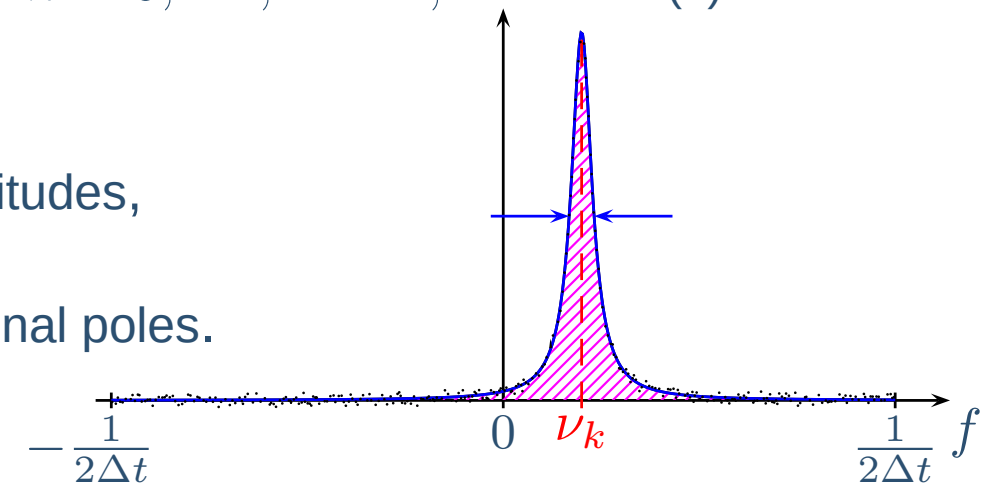
amplitude — phase — damping factor — frequency — WGN — sampling time interval

$$\boxed{x_n = \sum_{k=1}^{K} c_k z_k^n + e_n} \quad n = 0, \ldots, N-1, \qquad (2)$$

- $c_k = a_k \exp\{j\varphi_k\}$: complex amplitudes,
- $z_k = \exp\{(\alpha_k + 2j\pi\nu_k)\Delta t\}$: signal poles.





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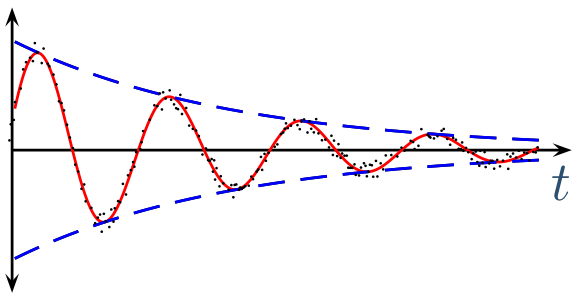

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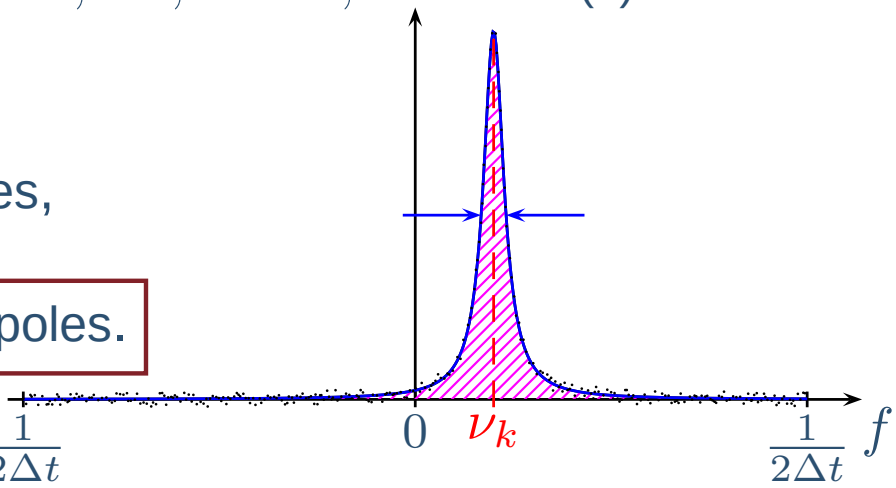
amplitude — phase — damping factor — frequency — WGN — sampling time interval

$$\boxed{x_n = \sum_{k=1}^{K} c_k z_k^n + e_n} \quad n = 0, \ldots, N-1, \qquad (2)$$

- $c_k = a_k \exp\{j\varphi_k\}$: complex amplitudes,
- $\boxed{z_k = \exp\{(\alpha_k + 2j\pi\nu_k)\Delta t\}}$: signal poles.

Assumption: $z_k \neq z_{k'}$, $\forall\, k \neq k'$





Matrix approach

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Starting point for a **subspace representation**

☞ x_n is arranged in a Hankel matrix:

$$\boldsymbol{H} = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{M-1} \\ x_1 & x_2 & \iddots & \cdots & \vdots \\ x_2 & \iddots & \iddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & x_{N-2} \\ x_{L-1} & \cdots & \cdots & x_{N-2} & x_{N-1} \end{pmatrix}$$

Why ?



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Let $x_n = c_1 . z_1^n$

$$\boldsymbol{H} = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{M-1} \\ x_1 & x_2 & \ddots & \cdots & \vdots \\ x_2 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & x_{N-2} \\ x_{L-1} & \cdots & \cdots & x_{N-2} & x_{N-1} \end{pmatrix}$$



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Let $x_n = c_1.z_1^n$

$$\boldsymbol{H} = \begin{pmatrix} c_1.z_1^0 & c_1.z_1^1 & c_1.z_1^2 & \cdots & c_1.z_1^{M-1} \\ c_1.z_1^1 & c_1.z_1^2 & \iddots & \cdots & \vdots \\ c_1.z_1^2 & \iddots & \iddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & c_1.z_1^{N-2} \\ c_1.z_1^{L-1} & \cdots & \cdots & c_1.z^{N-2} & c_1.z^{N-1} \end{pmatrix}$$



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Let $x_n = c_1 . z_1^n$

$$\boldsymbol{H} = c_1 \cdot \begin{pmatrix} z_1^0 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \\ z_1^1 & z_1^2 & \ddots & \cdots & \vdots \\ z_1^2 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & z_1^{N-2} \\ z_1^{L-1} & \cdots & \cdots & z_1^{N-2} & z_1^{N-1} \end{pmatrix}$$



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$\times z_1$



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$\times z_1^2$



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$$\times z_1^{M-1}$$



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Let $x_n = c_1.z_1^n$

$$\boldsymbol{H} = c_1 \cdot \begin{pmatrix} 1 \\ z_1^1 \\ z_1^2 \\ \vdots \\ z_1^{L-1} \end{pmatrix} \begin{pmatrix} 1 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \end{pmatrix}$$

Rank-1 term/matrix

$$\boldsymbol{H} = c_1 \cdot \begin{pmatrix} 1 \\ z_1^1 \\ z_1^2 \\ \vdots \\ z_1^{L-1} \end{pmatrix} \circ \begin{pmatrix} 1 \\ z_1^1 \\ z_1^2 \\ \vdots \\ z_1^{M-1} \end{pmatrix}, \quad (h)_{ij} = c_1.z_1^i \, z_1^j$$

Outer product



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Let $x_n = c_1.z_1^n + c_2.z_2^n$

$$\boldsymbol{H} = c_1 \cdot \begin{pmatrix} 1 \\ z_1^1 \\ z_1^2 \\ \vdots \\ z_1^{L-1} \end{pmatrix} \begin{pmatrix} 1 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \end{pmatrix}$$

$$+ c_2 \cdot \begin{pmatrix} 1 \\ z_2^1 \\ z_2^2 \\ \vdots \\ z_2^{L-1} \end{pmatrix} \begin{pmatrix} 1 & z_2^1 & z_2^2 & \cdots & z_2^{M-1} \end{pmatrix}$$



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Let $x_n = \sum_k c_k z^n$,

$$\boldsymbol{H} = \begin{pmatrix} \sum_k c_k z^0 & \sum_k c_k z_k^1 & \sum_k c_k z_k^2 & \cdots & \sum_k c_k z_k^{M-1} \\ \sum_k c_k z_k^1 & \sum_k c_k z_k^2 & \iddots & \cdots & \vdots \\ \sum_k c_k z_k^2 & \iddots & \iddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \sum_k c_k z_k^{N-2} \\ \sum_k c_k z_k^{L-1} & \cdots & \cdots & \sum_k c_k z_k^{N-2} & \sum_k c_k z_k^{N-1} \end{pmatrix}$$



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$\boldsymbol{H}$ can be factorized as follows (noise-free case):

$$\boldsymbol{H} = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^1 & \cdots & z_K^1 \\ z_1^2 & \cdots & z_K^2 \\ \vdots & \vdots & \vdots \\ z_1^{L-1} & \cdots & z_K^{L-1} \end{pmatrix} \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_K \end{pmatrix} \begin{pmatrix} 1 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & z_K^1 & z_K^2 & \cdots & z_K^{M-1} \end{pmatrix}$$

$$= \boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{\mathrm{T}} \quad (3)$$



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$\Longrightarrow$ *Vandermonde decomposition*



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$$= \underbrace{\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{\mathrm{T}}}_{\text{rank-K matrix}} \quad (3)$$

$\Longrightarrow$ *Vandermonde decomposition*

Due to the structure of the noise-free model x_n, $\boldsymbol{H}$ is rank deficient
The rank equals the number of signal poles (model order)



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Obtaining the subspace of interest

In the noise-free case, this rank deficiency is reflected by the SVD of $\boldsymbol{H}$:

$$\boldsymbol{H} = \begin{pmatrix} | & \cdots & | & \cdots & | \\ U_1 & \cdots & U_K & \cdots & U_L \\ | & \cdots & | & \cdots & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & \lambda_K & & \\ & & & & \\ & 0 & & & 0 \end{pmatrix} \begin{pmatrix} — & V_1^{\mathsf{H}} & — \\ \vdots & \vdots & \vdots \\ — & V_K^{\mathsf{H}} & — \\ \vdots & \vdots & \vdots \\ — & V_M^{\mathsf{H}} & — \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\boldsymbol{U}} & \boldsymbol{U}_0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\Sigma}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{V}}^{\mathsf{H}} \\ \boldsymbol{V}_0^{\mathsf{H}} \end{pmatrix} = \widehat{\boldsymbol{U}} \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{V}}^{\mathsf{H}}$$

- $\left[\widehat{\boldsymbol{U}}\, \boldsymbol{U}_0\right] \in \mathbb{C}^{L \times L}$ is a unitary matrix,
- $\widehat{\boldsymbol{U}} \in \mathbb{C}^{L \times K}$ is a semi-unitary matrix,
- $\|\boldsymbol{U}_k\|_2 = 1, \forall\, k,$
- $\boldsymbol{U}_k \perp \boldsymbol{U}_{k'},\ \forall\, k \neq k'.$



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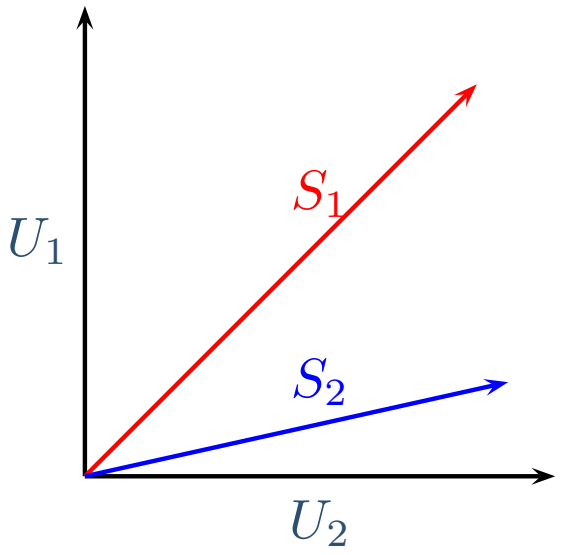
Obtaining the subspace of interest

The column vectors of $\widehat{\boldsymbol{U}}$ span the same subspace as the column vectors of $\boldsymbol{S}$: $\text{span}(\widehat{\boldsymbol{U}}) = \text{span}(\boldsymbol{S})$:



$$\widehat{\boldsymbol{U}} = [U_1 U_2],\ \boldsymbol{S} = [S_1 S_2]$$

$$\Longrightarrow \boxed{\widehat{\boldsymbol{U}} = \boldsymbol{SQ}}$$

$\boldsymbol{Q}$ square nonsingular

☞ In practice x_n is corrupted by additive white (Gaussian) noise



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Estimating the subspace of interest

Let $x_n = \sum_k c_k z^n + e_n$, then we have

$$\boldsymbol{H} = \boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{\mathrm{T}} + \boldsymbol{E}$$

In the presence of noise $\boldsymbol{H}$ is a full rank matrix:

$$\boldsymbol{H} = \left(\begin{array}{c|ccc|c} | & \cdots & | & \cdots & | \\ U_1 & \cdots & U_K & \cdots & U_L \\ | & \cdots & | & \cdots & | \end{array}\right) \left(\begin{array}{ccc|c} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_K & \\ \hline & 0 & & \boldsymbol{\Sigma}_0 \end{array}\right) \left(\begin{array}{c} — \\ \vdots \\ — \\ — \\ \vdots \\ — \end{array}\right.\!\!\left.\begin{array}{c} V_1^{\mathsf{H}} \\ \vdots \\ V_K^{\mathsf{H}} \\ \vdots \\ V_M^{\mathsf{H}} \end{array}\right)$$

$$= \left(\begin{array}{cc} \widehat{\boldsymbol{U}} & \boldsymbol{U}_0 \end{array}\right) \left(\begin{array}{cc} \widehat{\boldsymbol{\Sigma}} & 0 \\ 0 & \boldsymbol{\Sigma}_0 \end{array}\right) \left(\begin{array}{c} \widehat{\boldsymbol{V}}^{\mathsf{H}} \\ \boldsymbol{V}_0^{\mathsf{H}} \end{array}\right) = \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^{\mathsf{H}}$$

$\mathrm{span}(\widehat{\boldsymbol{U}})$ is a good approximation of $\mathrm{span}(\boldsymbol{S})$

$$\Longrightarrow \boxed{\widehat{\boldsymbol{U}} \approx \boldsymbol{S}\boldsymbol{Q}}$$



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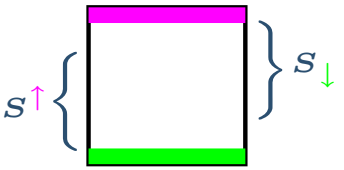
The shift-invariance

The shift-invariance property :

$\boldsymbol{S}^{\uparrow}$ / $\boldsymbol{S}_{\downarrow}$: $\boldsymbol{S}$ with the top / bottom row deleted



$$\left\{ \begin{array}{l} \boldsymbol{S}_{\downarrow}\,\boldsymbol{Z} = \boldsymbol{S}^{\uparrow} \\ \boldsymbol{T}_{\downarrow}\,\boldsymbol{Z} = \boldsymbol{T}^{\uparrow} \end{array} \right. , \quad \boldsymbol{Z} = \mathrm{diag}\left\{ z_1, z_2, \ldots, z_K \right\}$$

Relationship between $\widehat{\boldsymbol{U}}$ and $\boldsymbol{Z}$:

$$\widehat{\boldsymbol{U}} = \boldsymbol{S}\boldsymbol{Q} \Longrightarrow \left\{ \begin{array}{l} \widehat{\boldsymbol{U}}^{\uparrow} = \boldsymbol{S}^{\uparrow}\,\boldsymbol{Q} \\ \widehat{\boldsymbol{U}}_{\downarrow} = \boldsymbol{S}_{\downarrow}\,\boldsymbol{Q} \end{array} \right. \Longrightarrow \widehat{\boldsymbol{U}}^{\uparrow} = \widehat{\boldsymbol{U}}_{\downarrow}\, \underbrace{\boldsymbol{Q}^{-1}\,\boldsymbol{Z}\,\boldsymbol{Q}}_{\widetilde{\boldsymbol{Z}}}$$

$\widetilde{\boldsymbol{Z}}$ has the same eigenvalues as $\boldsymbol{Z}$



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LS-TLS solution

If x_n is corrupted by additive noise, the solution of the (incompatible) set of equation $\widehat{\boldsymbol{U}}^{\uparrow} \approx \widehat{\boldsymbol{U}}_{\downarrow} \widetilde{\boldsymbol{Z}}$ is

- using the least squares (LS) method (HSVD) :

$$\widehat{\widetilde{\boldsymbol{Z}}} = \left(\widehat{\boldsymbol{U}}_{\downarrow}^{\mathsf{H}} \widehat{\boldsymbol{U}}_{\downarrow}\right)^{-1} \widehat{\boldsymbol{U}}_{\downarrow}^{\mathsf{H}} \widehat{\boldsymbol{U}}^{\uparrow}$$

- using the total least squares (TLS) method (HTLS) :

$$[\widehat{\boldsymbol{U}}_{\downarrow} \widehat{\boldsymbol{U}}^{\uparrow}] = \boldsymbol{Y}_{(L-1)\times(L-1)} \boldsymbol{\Gamma} \boldsymbol{W}^{\mathsf{H}}_{2K\times 2K}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} \begin{matrix} K \\ K \end{matrix}$$

$$\Longrightarrow \widehat{\widetilde{\boldsymbol{Z}}} = -\boldsymbol{W}_{12} \boldsymbol{W}_{22}^{-1}$$



Parameter extraction

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The eigenvalues of $\widehat{\widehat{Z}}$ approximate the signal poles

- $\alpha_k = -\frac{1}{\Delta t}\Re\{\ln(z_k)\}, \quad \omega_k = \frac{1}{\Delta t}\Im\{\ln(z_k)\}$

The complex amplitudes c_k are obtained from the LS solution of the system:

- $$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1^1 & z_2^1 & \cdots & z_K^1 \\ z_1^2 & z_2^2 & \cdots & z_K^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_K^{N-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{pmatrix}$$



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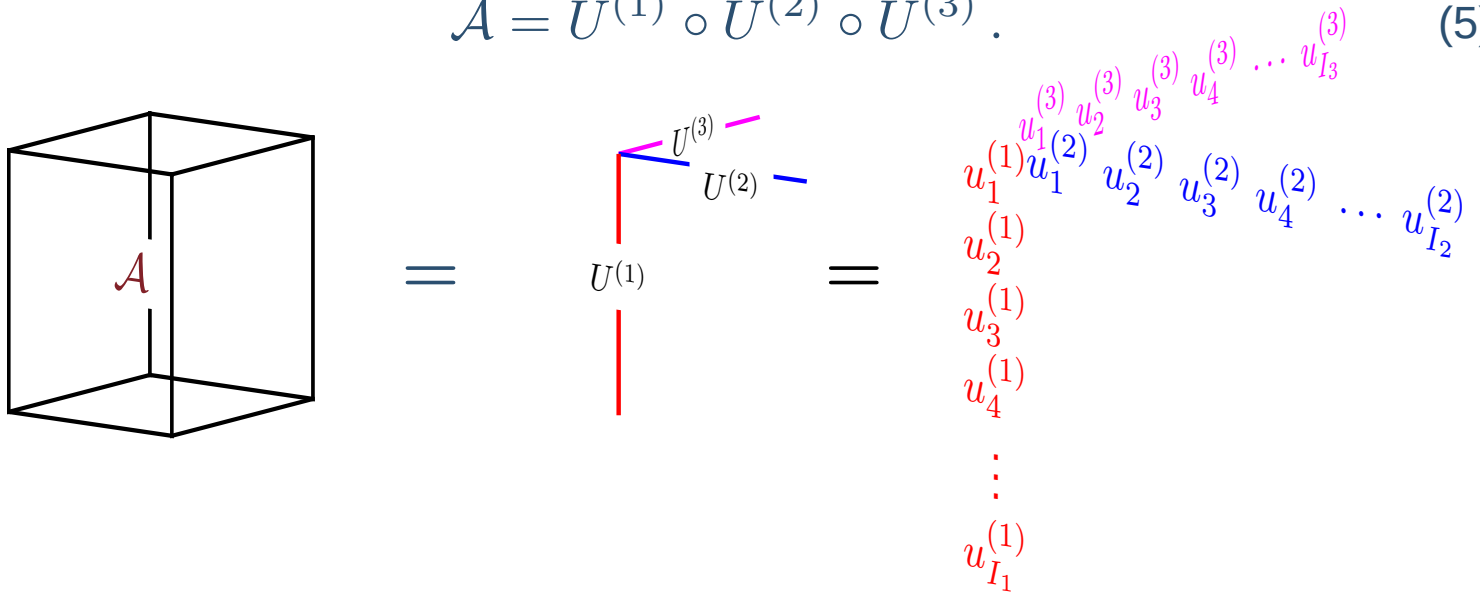
Rank-1 term

In analogy with the matrix case, a third-order tensor $\mathcal{A}$ is called rank-1 when it equals the *outer product* of three vectors $U^{(1)}$, $U^{(2)}$, $U^{(3)}$:

$$a_{i_1 i_2 i_3} = u^{(1)}_{i_1} u^{(2)}_{i_2} u^{(3)}_{i_3} \tag{4}$$

for all index values, which we will denote as:

$$\mathcal{A} = U^{(1)} \circ U^{(2)} \circ U^{(3)} . \tag{5}$$



A tensor that can be written as the sum of K, but not less than K, rank-1 terms is called rank-K



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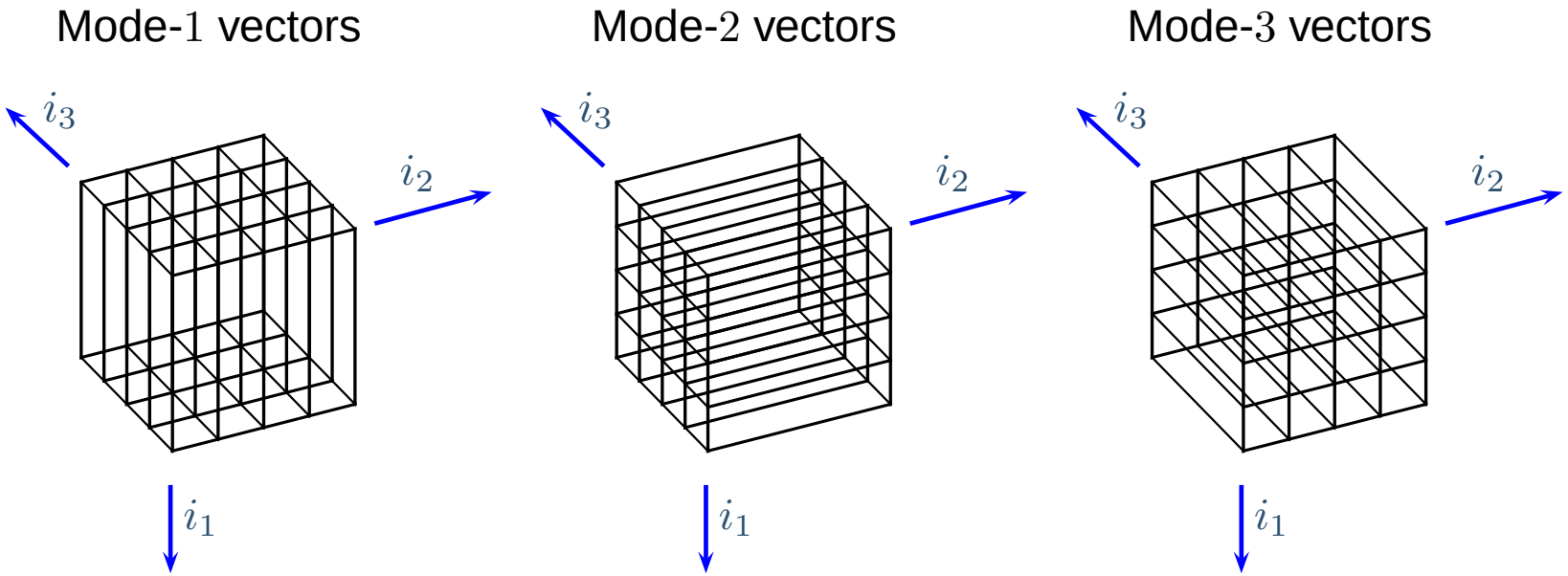
n-mode vectors

- An n-mode vector of a tensor $\mathcal{A}$ is a vector obtained from $\mathcal{A}$ by varying only the nth index.



☞ As such, the n-mode vectors are the generalization of column and row vectors in the matrix case.



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Some remarks

- The subspace spanned by all the n-mode vectors, for a given value of n, is called the n-mode vector space. The dimension of this subspace is called the n-mode rank, R_n.
  ☞ A difference between matrices and tensors is that, for tensors, the different n-mode ranks are not necessarily the same, while for matrices, column and row rank are always the same. For tensors we have the inequalities:

$$\begin{aligned} R_1 &\leq R_2 R_3, \\ R_2 &\leq R_3 R_1, \\ R_3 &\leq R_1 R_2. \end{aligned} \tag{6}$$

- A third-order tensor whose n-mode ranks are equal to R_n $(1 \leq n \leq 3)$ is called a rank-(R_1, R_2, R_3) tensor.



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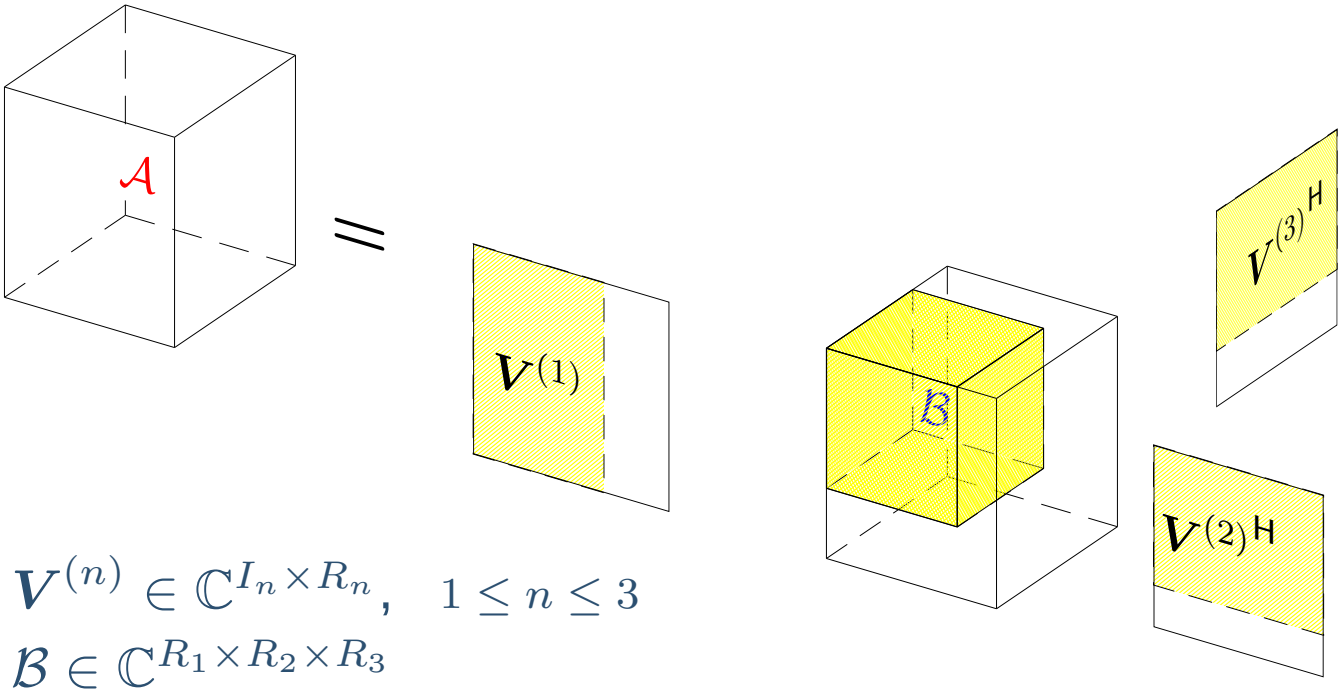
HOSVD of a third order tensor

Every complex $(I_1 \times I_2 \times I_3)$-tensor $\mathcal{A}$ can be written as the product: $\mathcal{A} = \mathcal{B} \times_1 \boldsymbol{V}^{(1)} \times_2 \boldsymbol{V}^{(2)} \times_3 \boldsymbol{V}^{(3)}$,



$\boldsymbol{V}^{(n)} \in \mathbb{C}^{I_n \times R_n}, \quad 1 \leq n \leq 3$

$\mathcal{B} \in \mathbb{C}^{R_1 \times R_2 \times R_3}$

Elementwise notation: $a_{i_1 i_2 i_3} = \sum_{j_1 j_2 j_3} b_{j_1 j_2 j_3} v^{(1)}_{i_1 j_1} v^{(2)}_{i_2 j_2} v^{(3)}_{i_3 j_3}$

☞ If $\mathcal{A}$ is n-mode rank deficient, only the shaded part of the core-tensor contains entries different from zero.



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Dimensionality reduction

Given a complex third-order tensor $\mathcal{H} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$, find a rank-$(R_1, R_2, R_3)$ tensor $\widehat{\mathcal{H}}$ that minimizes the least-squares cost function

$$f(\widehat{\mathcal{H}}) = \left\| \mathcal{H} - \widehat{\mathcal{H}} \right\|^2. \tag{7}$$

Due to the n-rank constraints, $\widehat{\mathcal{H}}$ can be decomposed as :

$$\widehat{\mathcal{H}} = \mathcal{B} \times_1 \boldsymbol{V}^{(1)} \times_2 \boldsymbol{V}^{(2)} \times_3 \boldsymbol{V}^{(3)} \tag{8}$$

in which $\boldsymbol{V}^{(1)} \in \mathbb{C}^{I_1 \times R_1}$, $\boldsymbol{V}^{(2)} \in \mathbb{C}^{I_2 \times R_2}$, $\boldsymbol{V}^{(3)} \in \mathbb{C}^{I_3 \times R_3}$ each have orthonormal columns and $\mathcal{B} \in \mathbb{C}^{R_1 \times R_2 \times R_3}$ is an all-orthogonal tensor.

We use a higher-order orthogonal iteration (HOOI) algorithm to perform the dimensionality reduction. This algorithm is of the Alternating Least Squares (ALS) type.



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Matrix approach

Assume Q simultaneous one-dimensional noiseless time domain signals each consisting of N complex samples:

$$x_n^{(q)} = \sum_{k=1}^{K} a_k^{(q)} \exp\{j\varphi_k^{(q)}\} \exp\{(-\alpha_k + j\omega_k)t_n\} = \sum_{k=1}^{K} c_k^{(q)} z_k^n \quad (9)$$

in which q ($1 \leq q \leq Q$) denotes the channel number. First, the data samples of each channel are arranged into an $L \times M$ Hankel matrix as follows :

$$\boldsymbol{H}_q = \begin{pmatrix} x_0^{(q)} & x_1^{(q)} & x_2^{(q)} & \cdots & x_{M-1}^{(q)} \\ x_1^{(q)} & x_2^{(q)} & \iddots & \cdots & \vdots \\ x_2^{(q)} & \iddots & \iddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & x_{N-2}^{(q)} \\ x_{L-1}^{(q)} & \cdots & \cdots & x_{N-2}^{(q)} & x_{N-1}^{(q)} \end{pmatrix}. \quad (10)$$



Matrix approach

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The Q Hankel matrices are stacked in one block Hankel matrix:

$$\boldsymbol{H} = \left[\boldsymbol{H}_1 \mid \boldsymbol{H}_2 \mid \ldots \mid \boldsymbol{H}_Q \right]. \tag{11}$$

This matrix can be decomposed as follows:

$$\boldsymbol{H} = \boldsymbol{S} \left[\boldsymbol{C}_1\boldsymbol{T}^{\mathsf{T}} \mid \boldsymbol{C}_2\boldsymbol{T}^{\mathsf{T}} \mid \ldots \mid \boldsymbol{C}_Q\boldsymbol{T}^{\mathsf{T}} \right] \tag{12}$$

in which $\boldsymbol{S} \in \mathbb{C}^{L\times K}$, $\boldsymbol{T} \in \mathbb{C}^{K\times M}$ Vandermonde matrices

$$\boldsymbol{S} = \begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_K \\ z_1^2 & \cdots & z_K^2 \\ \vdots & \vdots & \vdots \\ z_1^{(L-1)} & \cdots & z_K^{(L-1)} \end{pmatrix}, \boldsymbol{C}_q = \mathrm{diag}\{c_1^{(q)}, c_2^{(q)}, \ldots, c_K^{(q)}\}.$$



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Matrix approach

The subspace of interest is obtained by SVD:

$$\boldsymbol{H} = \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^{\mathrm{H}}$$

☞ Note that each submatrix has the same row subspace

Does the SVD fully take this structure into account ?

$\Longrightarrow$ NO

Does it exist a tool which allows to exploit this structure ?

$\Longrightarrow$ YES

Let's go for a higher-order approach. . .



Construction of a third-tensor

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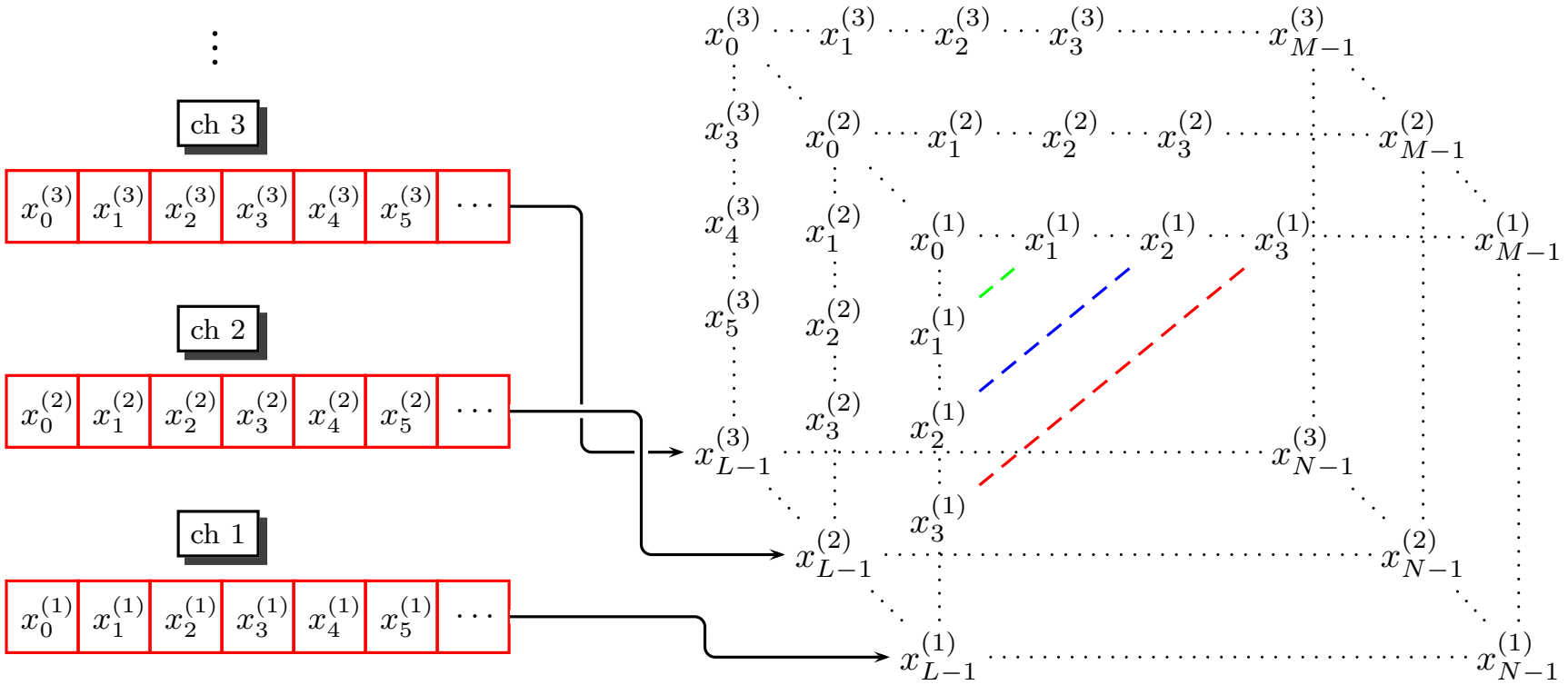



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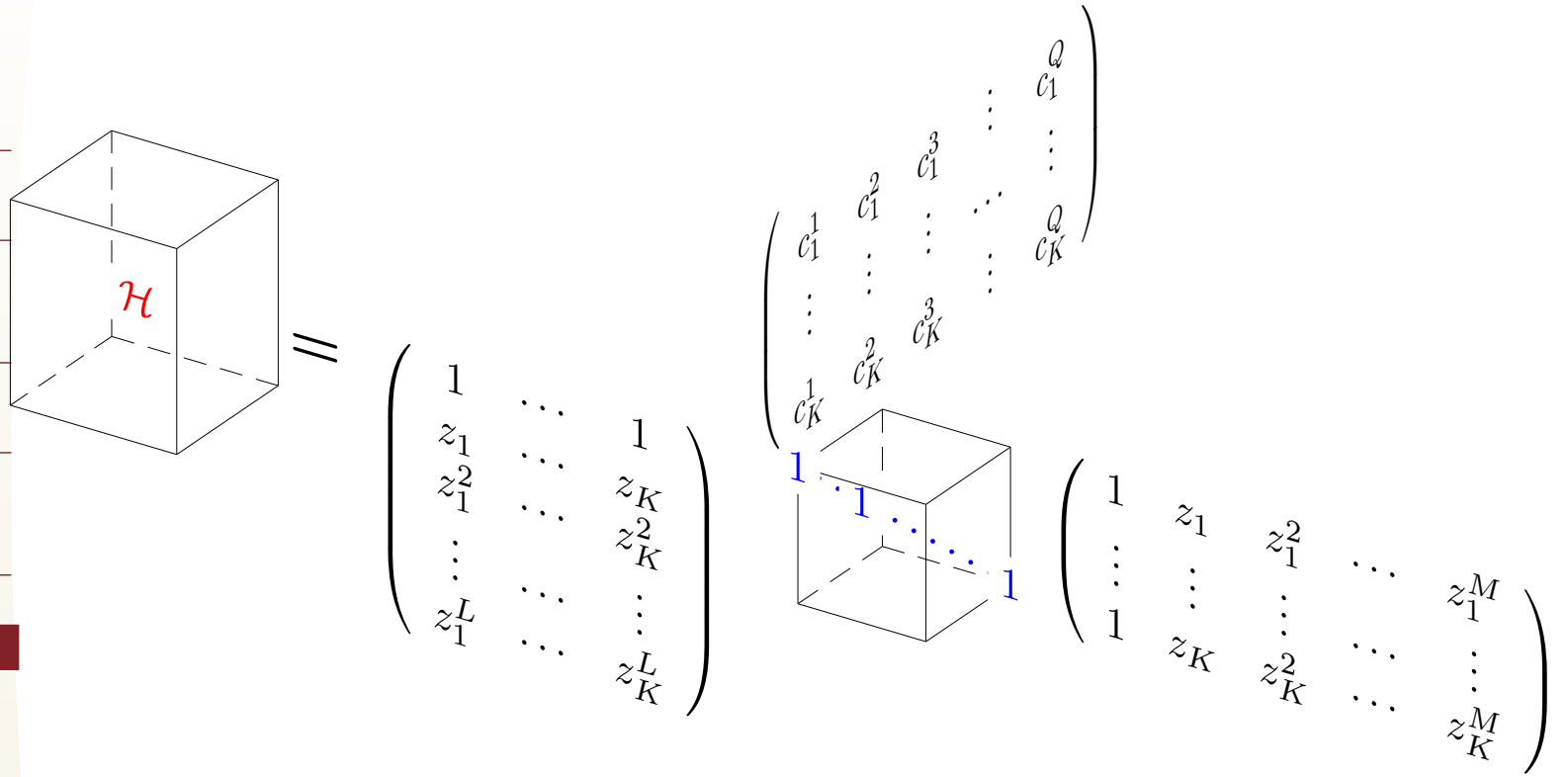

$$\mathcal{H} = \mathcal{I} \times_1 \boldsymbol{S} \times_2 \boldsymbol{T} \times_3 \overline{\boldsymbol{C}}$$

In the noise-free case $\mathcal{H}$ is a rank-K tensor



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Matrix approach in the multi-channel case:

$$x_n^{(q)} \longrightarrow \boldsymbol{H} = \left[\boldsymbol{H_1} \mid \boldsymbol{H_2} \mid \cdots \mid \boldsymbol{H_Q} \right]$$

$$\longrightarrow \boldsymbol{H} \stackrel{SVD}{=} \boldsymbol{U\Sigma V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$

$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\boldsymbol{H} \in \mathbb{C}^{L \times M.Q}$$



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Matrix approach in the multi-channel case:

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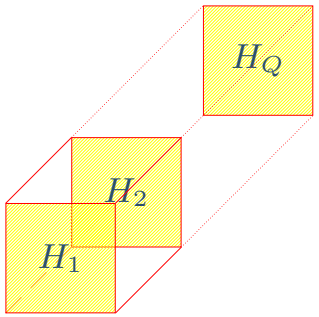
$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\boldsymbol{H} \in \mathbb{C}^{L \times M.Q}$$

Tensor approach in the multi-channel case:

$$x_m^{(q)} \longrightarrow \mathcal{H} =$$



$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\mathcal{H} \in \mathbb{C}^{L \times M \times Q}$$

$\widehat{\mathcal{H}} = \mathcal{B} \times_1 \boldsymbol{V}^{(1)} \times_2 \boldsymbol{V}^{(2)} \times_3 \boldsymbol{V}^{(3)}$ Best rank-(K, K, K) approx. of $\mathcal{H}$



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Matrix approach in the multi-channel case:

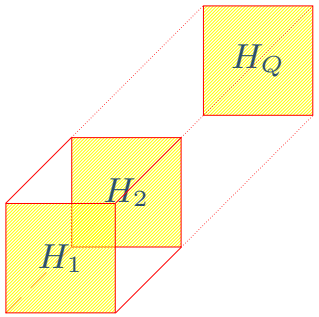
$$x_n^{(q)} \longrightarrow \boldsymbol{H} = \left[\boldsymbol{H_1} \mid \boldsymbol{H_2} \mid \cdots \mid \boldsymbol{H_Q} \right]$$

$$\longrightarrow \boldsymbol{H} \overset{SVD}{=} \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$

$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\boldsymbol{H} \in \mathbb{C}^{L \times M.Q}$$

Tensor approach in the multi-channel case:

$$x_m^{(q)} \longrightarrow \mathcal{H} =$$



$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\mathcal{H} \in \mathbb{C}^{L \times M \times Q}$$

$\widehat{\mathcal{H}} = \mathcal{B} \times_1 \boldsymbol{V}^{(1)} \times_2 \boldsymbol{V}^{(2)} \times_3 \boldsymbol{V}^{(3)}$ Best rank-(K, K, K) approx. of $\mathcal{H}$

Total least squares (TLS) solution:

$$[\widehat{\boldsymbol{U}}_{\downarrow} \widehat{\boldsymbol{U}}^{\uparrow}] = \boldsymbol{Y}_{(L-1)\times(L-1)} \boldsymbol{\Gamma} \boldsymbol{W}^{\mathsf{H}}_{2K \times 2K}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix}$$

$$\Longrightarrow \boxed{\widehat{\widetilde{\boldsymbol{Z}}} = -\boldsymbol{W}_{12}\boldsymbol{W}_{22}^{-1}}$$



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$$
\begin{aligned}
y_n^{(q)} &= x_n^{(q)} + e_n^{(q)} \\
&= c_1^{(q)} \exp\{(-0.01 + 2j\pi 0.2)\, n\} + c_2^{(q)} \exp\{(-0.02 + 2j\pi 0.22)\, n\} \\
&\qquad \text{pole 1} \qquad\qquad\qquad\qquad \text{pole 2} \\
&\quad + e_n^{(q)}, (0 \le n \le N = 24), (1 \le q \le Q = 12), \\
&\quad \text{noise}
\end{aligned}
\tag{13}
$$

- $e_n^{(q)}$ is a different complex circular symmetric WGN sequence for each channel q.
- The complex amplitudes $c_k^{(q)}$ are drawn from a Gaussian distribution with zero-mean and unit-variance for $1 \le k \le K = 2$ and $1 \le q \le Q = 12$.
- Matrix approach: we work with the matrix $\widehat{\boldsymbol{U}}$ containing the 2 dominant left singular vectors of the Hankel matrix $\boldsymbol{H}$.
- Tensor approach, we work with the matrix $\widehat{\boldsymbol{U}}^{(1)}$ in the HOSVD (8) of the best rank-$(2,2,2)$ approximation of $\mathcal{H}$.



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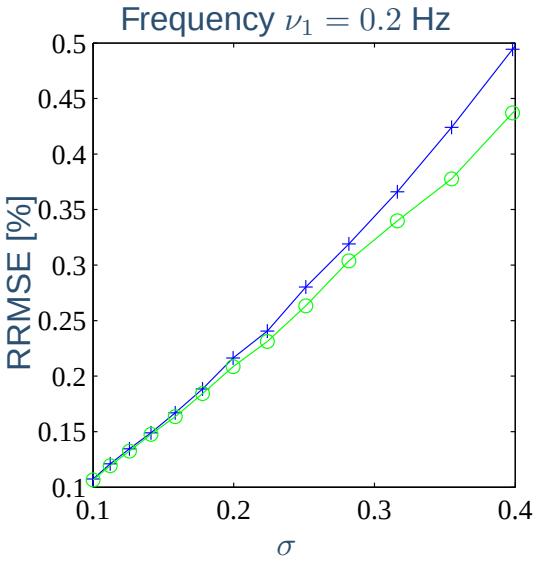

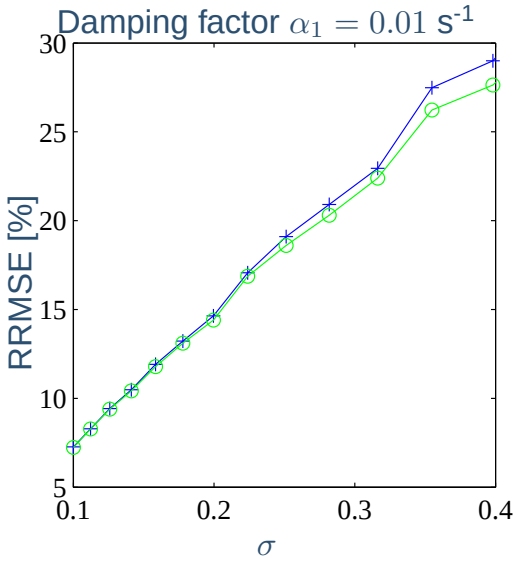

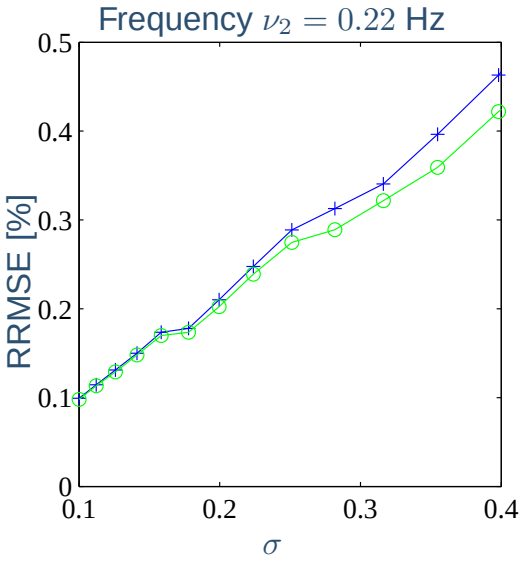

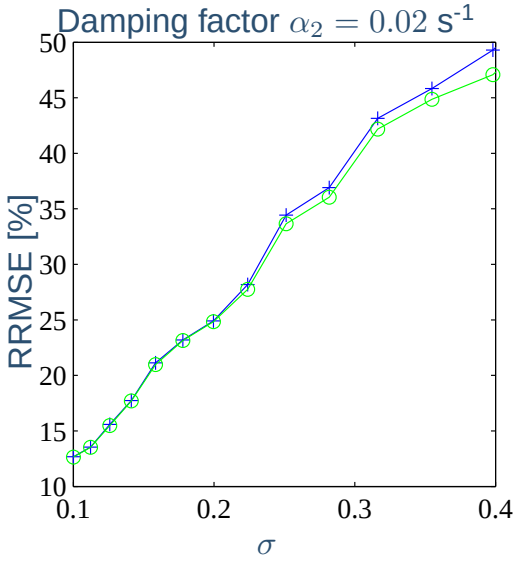



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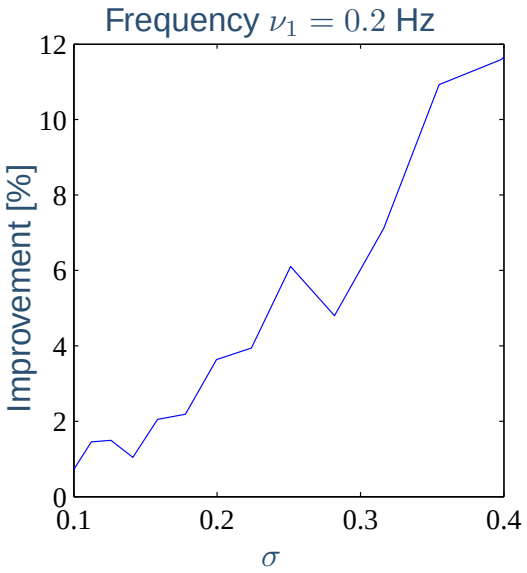

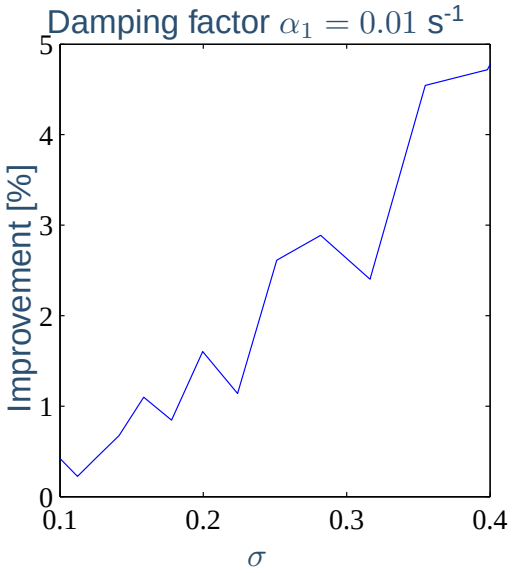

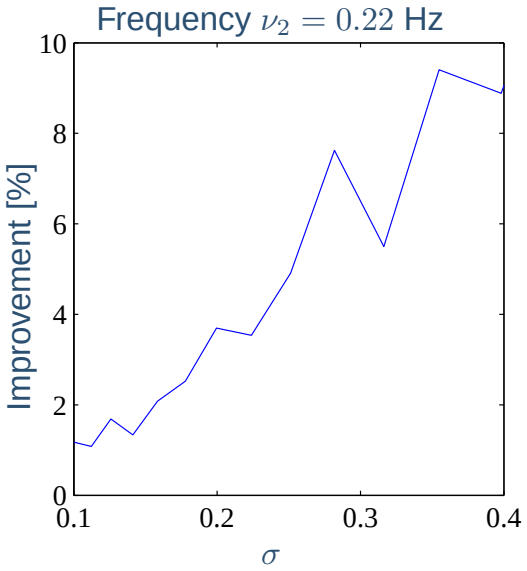

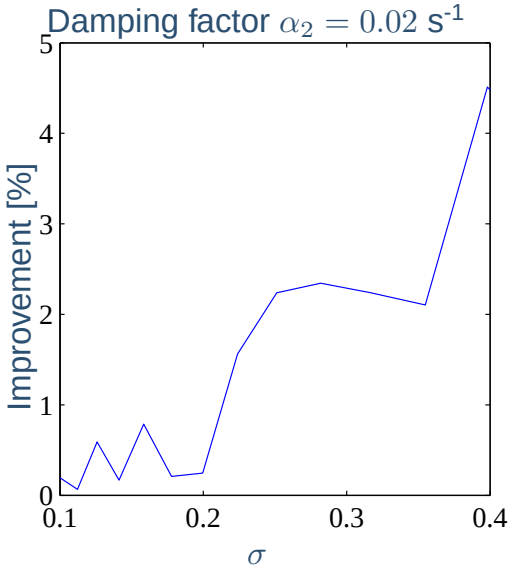



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If the signal poles are close (closely spaced peaks), the Vandermonde vectors are almost dependent. Therefore, in the presence of noise, $\widehat{U}$ might yield a very poor estimate of the column space of S.



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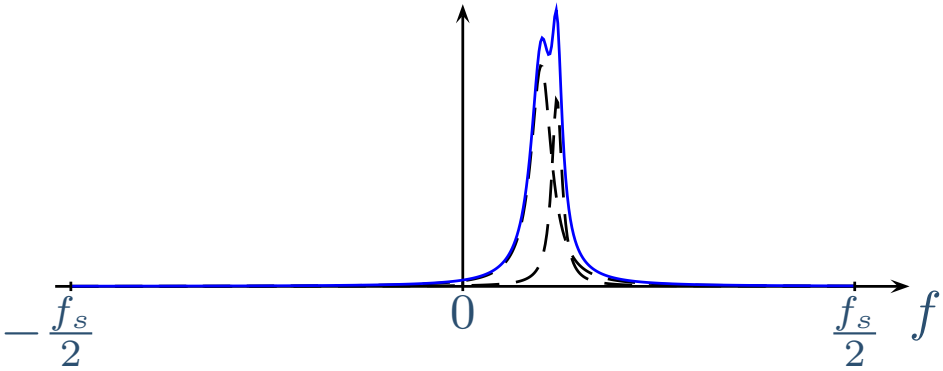

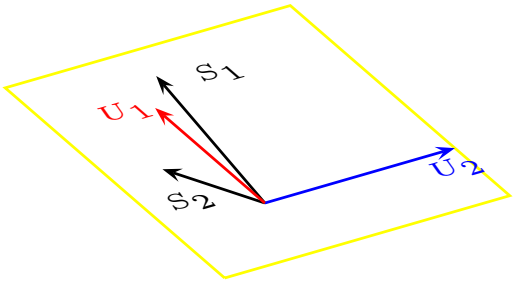

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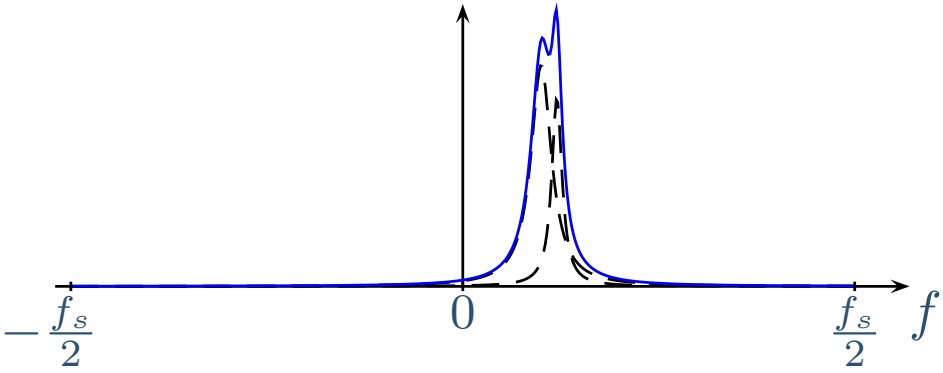
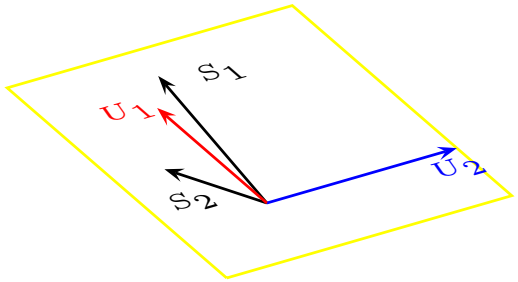
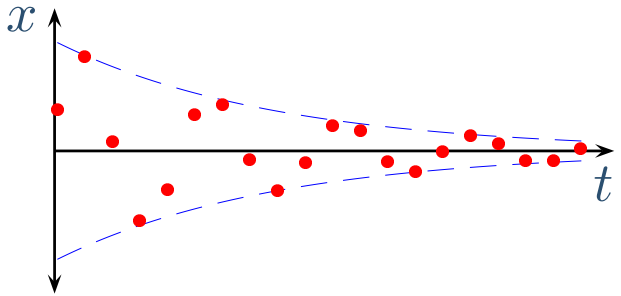




Oversampling yields higher accuracy in the matrix case





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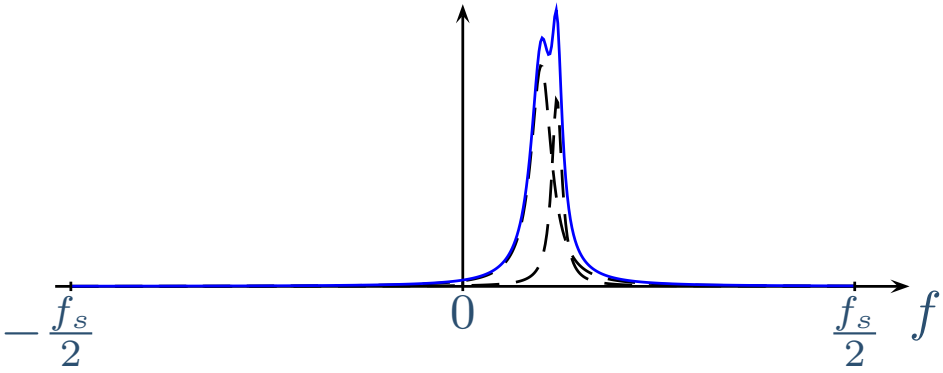

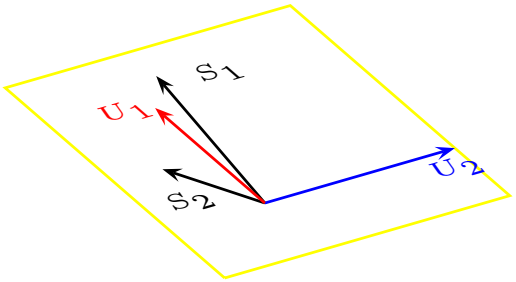

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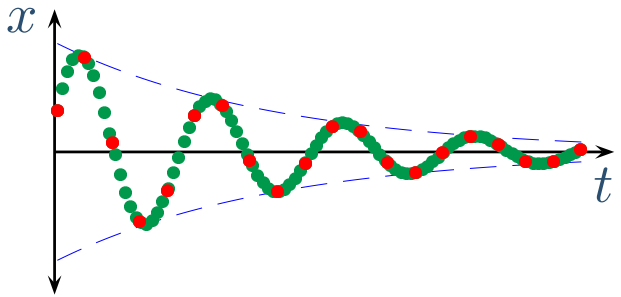



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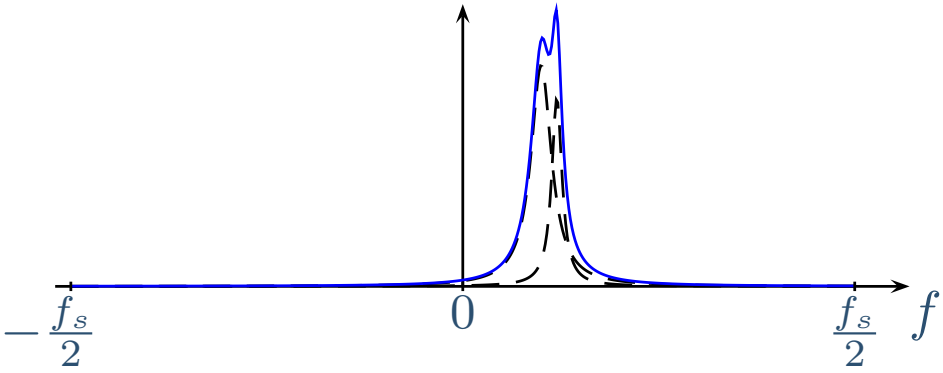

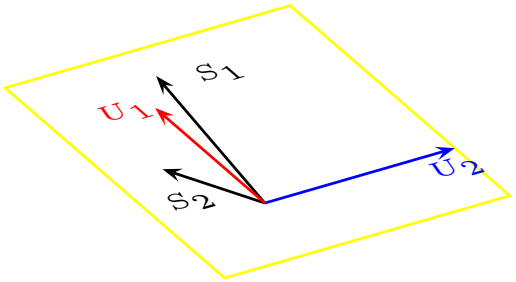

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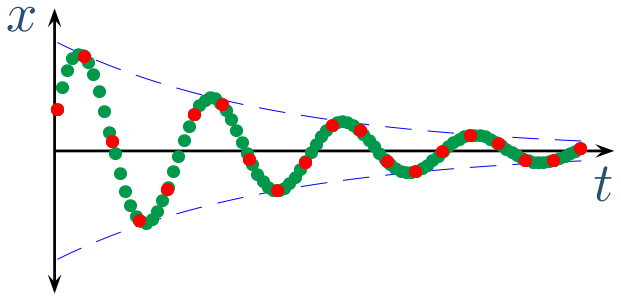

$\Longrightarrow$ The SVD of H becomes computationally expensive



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$$x_n^{(d)} = x_{nD+d} = \sum_{k=1}^{K} c_k z^{nD+d} \quad n=0,\ldots,\tfrac{N}{D}-1,\ d=0,\ldots,D-1\ ,$$



Decimation

$$x_0 \;\; x_1 \;\; \cdots \;\; x_{D-1} \;\; x_D \;\; x_{D+1} \;\; \cdots \;\; x_{2D-1} \;\; x_{2D} \;\; x_{2D+1} \;\; \cdots \;\; x_{N-1}$$

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$$\boldsymbol{S} = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^{D} & \cdots & z_K^{D} \\ z_1^{2D} & \cdots & z_K^{2D} \\ \vdots & \vdots & \vdots \\ z_1^{(L_D-1)D} & \cdots & z_K^{(L_D-1)D} \end{pmatrix}$$

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Low computation cost
good accuracy



More structure

The structure of $\boldsymbol{H}$ is more complex:

$$\boldsymbol{C}_d = \text{diag}\{c_1 \ldots, c_k\}.\,(\text{diag}\{z_1 \ldots, z_k\})^d$$

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The results could be better if this structure could be taken into account.

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The whole structure can be handled using multilinear algebra



Construction of a third-order tensor

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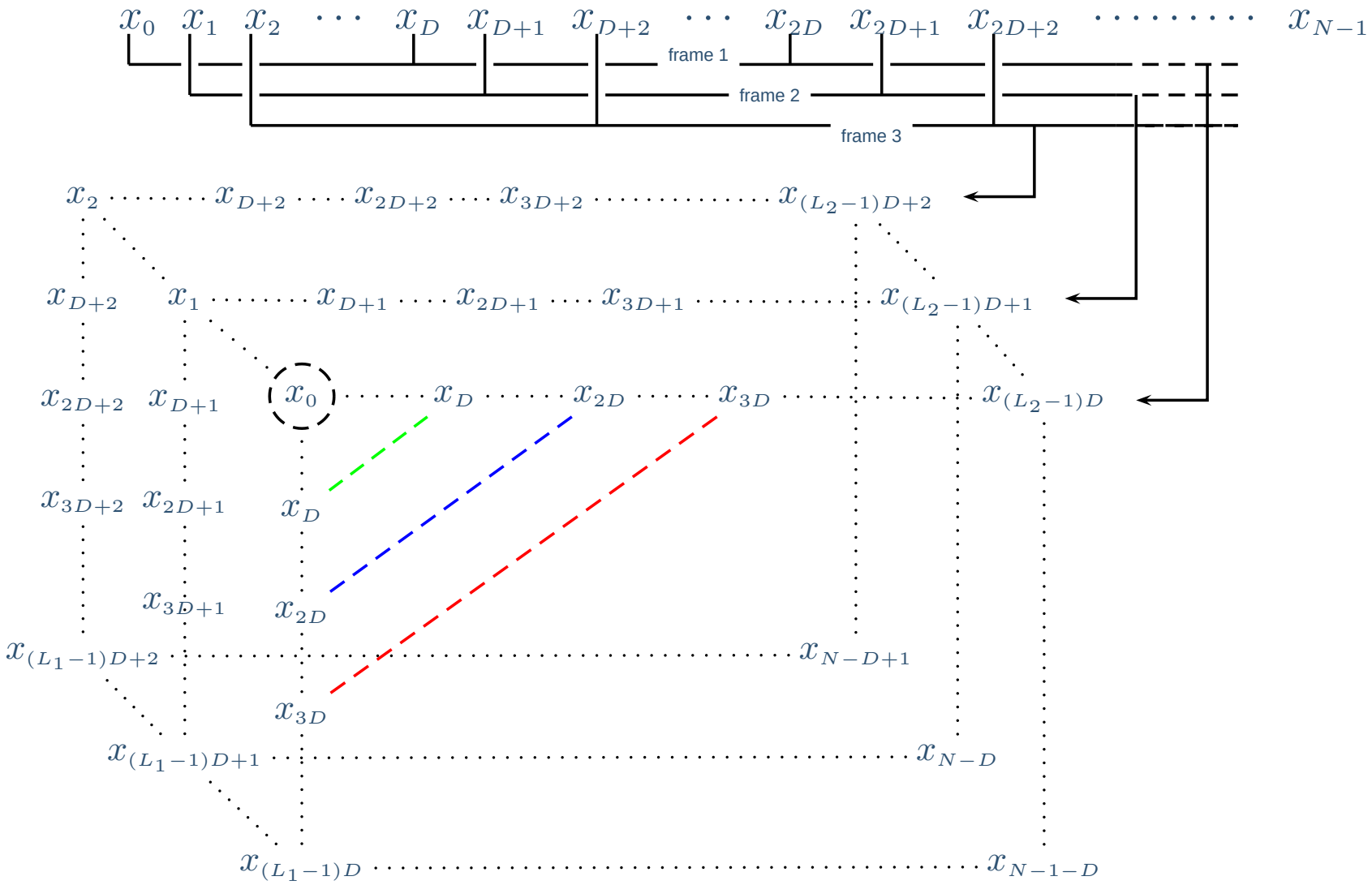



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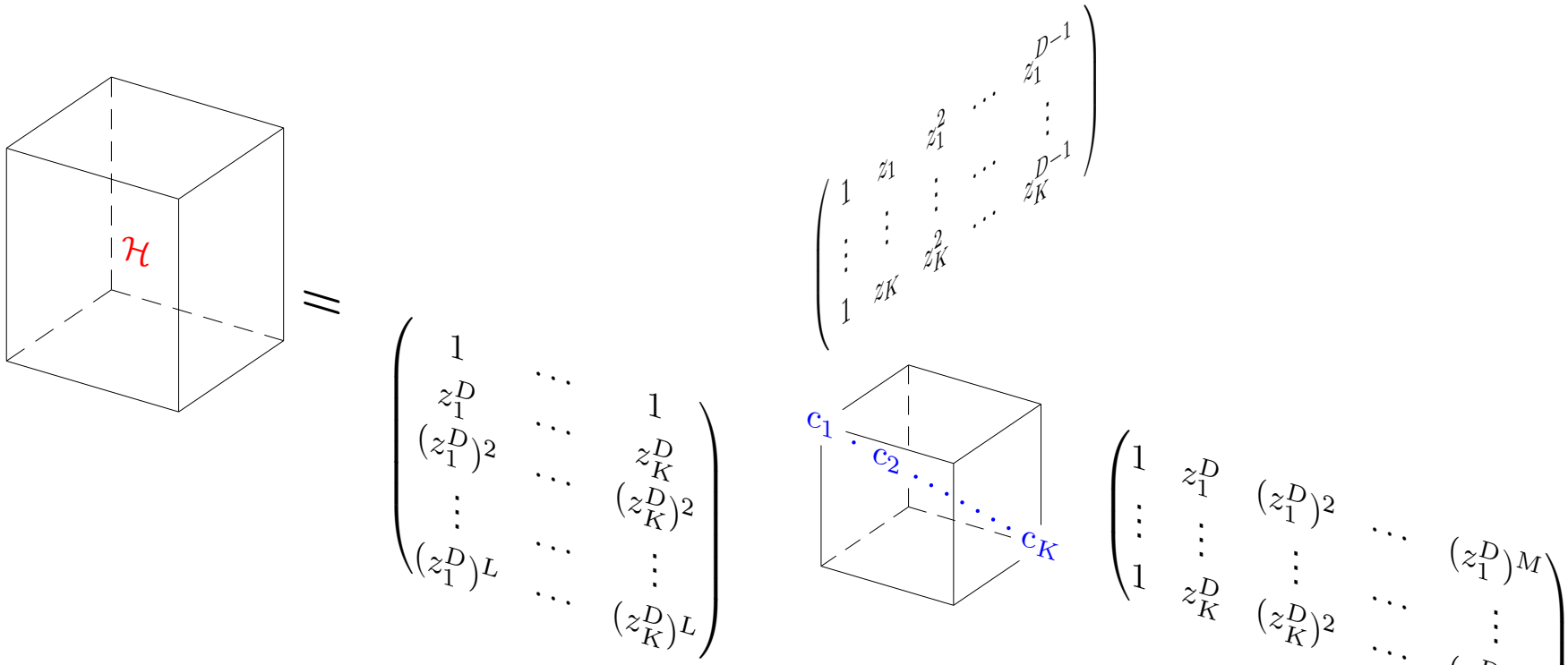

$$\mathcal{H} = \mathcal{C} \times_1 \boldsymbol{S}^{(1)} \times_2 \boldsymbol{S}^{(2)} \times_3 \boldsymbol{S}^{(3)}$$

In the noise-free case $\mathcal{H}$ is a rank-K tensor



Summary

Decimative matrix approach:

$$x_n \longrightarrow x_m^{(d)} \longrightarrow \boldsymbol{H} = \left[\boldsymbol{H_0} \mid \boldsymbol{H_1} \mid \cdots \mid \boldsymbol{H_{D-1}} \right]$$

$$\longrightarrow \boldsymbol{H} \stackrel{SVD}{=} \boldsymbol{U\Sigma V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$

$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\boldsymbol{H} \in \mathbb{C}^{L \times M.D}$$

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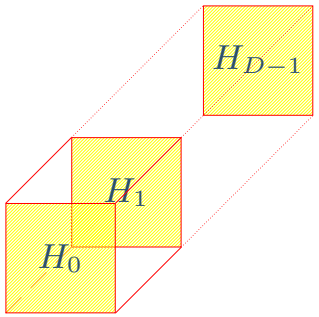
Decimative matrix approach:

$$x_n \longrightarrow x_m^{(d)} \longrightarrow \boldsymbol{H} = \left[\boldsymbol{H_0} \mid \boldsymbol{H_1} \mid \cdots \mid \boldsymbol{H_{D-1}} \right] \qquad \begin{array}{l} \boldsymbol{H_n} \in \mathbb{C}^{L \times M} \\ \boldsymbol{H} \in \mathbb{C}^{L \times M.D} \end{array}$$

$$\longrightarrow \boldsymbol{H} \stackrel{SVD}{=} \boldsymbol{U \Sigma V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$

Decimative tensor approach:

$$x_n \longrightarrow x_m^{(d)} \longrightarrow \mathcal{H} =$$



$$\begin{array}{l} \boldsymbol{H_n} \in \mathbb{C}^{L \times M} \\ \mathcal{H} \in \mathbb{C}^{L \times M \times D} \end{array}$$

$\widehat{\mathcal{H}} = \mathcal{B} \times_1 \boldsymbol{V}^{(1)} \times_2 \boldsymbol{V}^{(2)} \times_3 \boldsymbol{V}^{(3)}$ Best rank-$(K, K, \tilde{K})$ approx. of $\mathcal{H}$ with $\tilde{K} \leq K$



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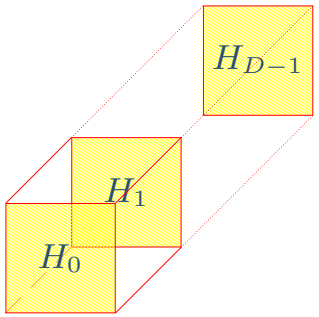

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Total least squares (TLS) solution:

$$[\widehat{\boldsymbol{U}}_{\downarrow} \widehat{\boldsymbol{U}}^{\uparrow}] = \boldsymbol{Y}_{(L-1)\times(L-1)} \boldsymbol{\Gamma} \boldsymbol{W}^{\mathsf{H}}_{2K \times 2K}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix}$$

$$\Longrightarrow \boxed{\widehat{\widetilde{\boldsymbol{Z}}} = -\boldsymbol{W}_{12} \boldsymbol{W}_{22}^{-1}}$$



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Model:

$$x_n = \exp\{(2j\pi 0.2)\Delta t.n\} + \exp\{(2j\pi 0.205)\Delta t.n\}$$

time index: $n = 0, \ldots, N-1$

N=1000

sampling time interval: Δt=0.1

Decimation factor: $D = 10$

Tensor: best rank-$(2,2,2)$ and best rank-$(2,2,1)$ approximation of $\mathcal{H}_{50\times 50\times 10}$

Matrix: best rank-2 approximation of $\boldsymbol{H}_{50\times 500}$



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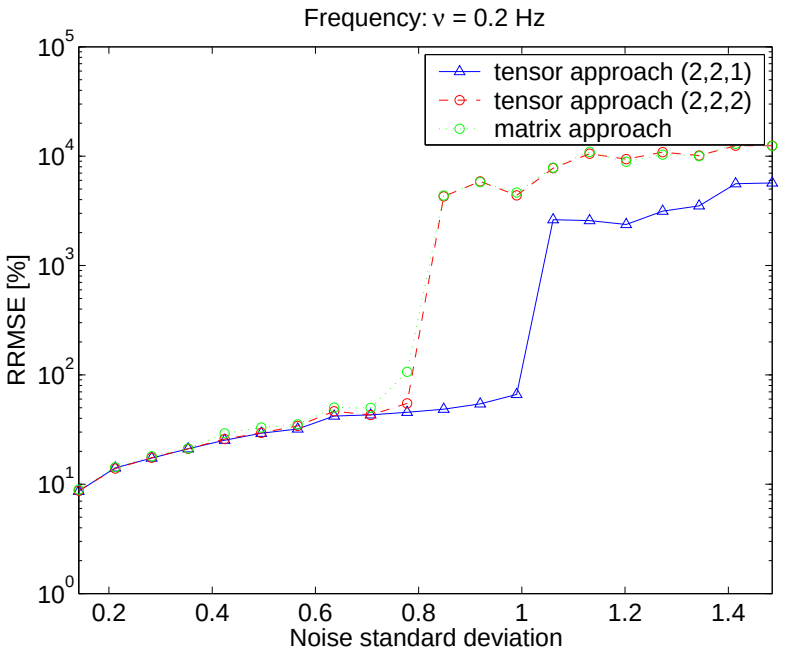

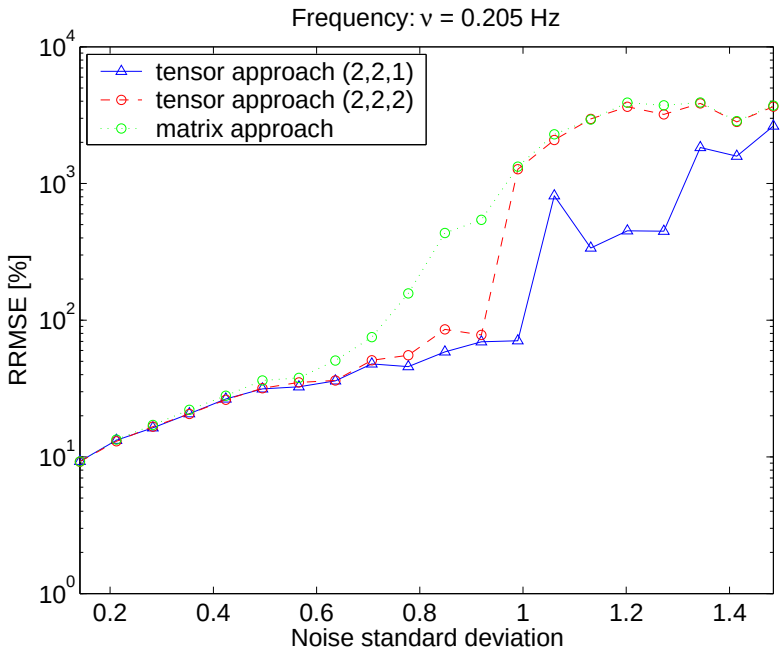



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Parameters

Model:

$$x_n = \exp\{(-0.01 + 2j\pi 0.2)\Delta t.n\} + \exp\{(-0.02 + 2j\pi 0.2)\Delta t.n\}$$

time index: $n = 0, \ldots, N-1$

N=624

sampling time interval: Δt=0.04

Decimation factor: $D = 25$

Tensor: best rank-$(2, 2, 1)$ approximation of $\mathcal{H}_{13\times 13\times 25}$

Matrix: best rank-2 approximation of $\boldsymbol{H}_{13\times 325}$



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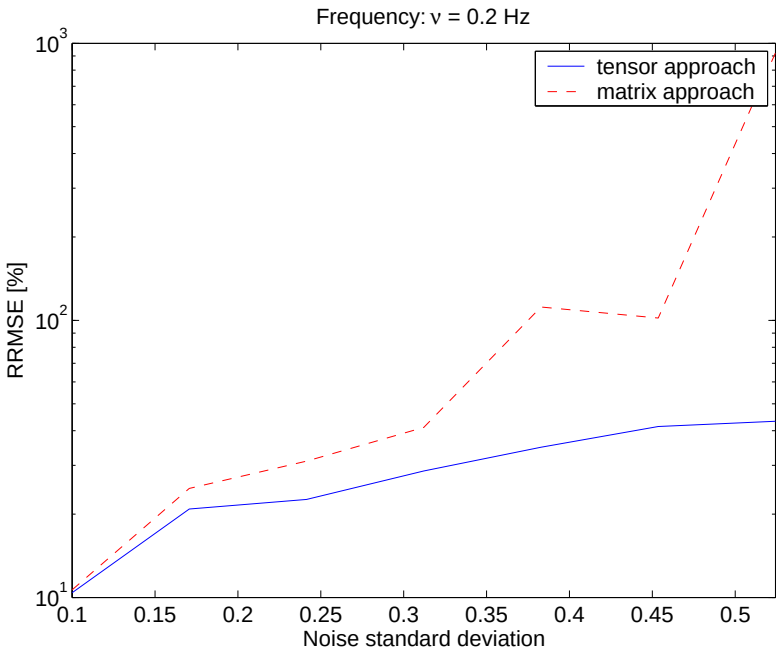

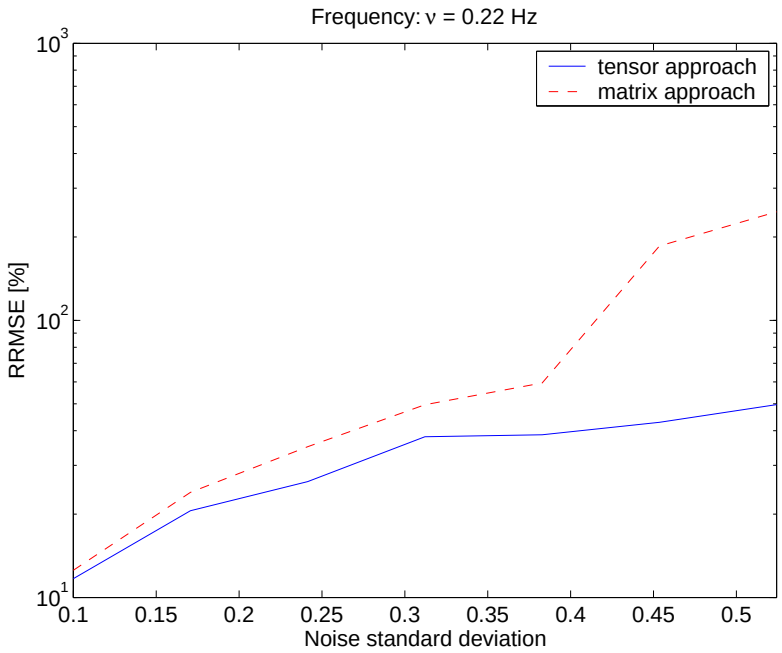



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- We have considered harmonic analysis of:
  - multi-channel signal,
  - single-channel signal whose poles are very close,



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Conclusion

- We have considered harmonic analysis of:
  - multi-channel signal,
  - single-channel signal whose poles are very close,
- These signals lend themselves to subspace representation, but a matrix approach does not take the whole structure into account,



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Conclusion

- We have considered harmonic analysis of:
  - multi-channel signal,
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- A multilinear approach helps to reveal the rest of the structure ($\Longrightarrow$ HOVDMD, rank deficiency...),



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Conclusion

- We have considered harmonic analysis of:
  - multi-channel signal,
  - single-channel signal whose poles are very close,
- These signals lend themselves to subspace representation, but a matrix approach does not take the whole structure into account,
- A multilinear approach helps to reveal the rest of the structure ($\Longrightarrow$ HOVDMD, rank deficiency...),
- We have shown that the dimensionality reduction of the data tensor yields better subspace estimate than the truncated SVD of the data matrix.