Sample Complexity of Data-Driven Stochastic LQR with Multiplicative Uncertainty

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***Abstract*— This paper studies the sample complexity of the stochastic Linear Quadratic Regulator when applied to systems with multiplicative noise. We assume that the covariance of the noise is unknown and estimate it using the sample covariance, which results in suboptimal behaviour. The main contribution of this paper is then to bound the suboptimality of the methodology and prove that it decreases with $1/N$, where N denotes the amount of samples. The methodology easily generalizes to the case where the mean is unknown and to the distributionally robust case studied in a previous work of the authors [1]. The analysis is mostly based on results from matrix function perturbation analysis [2].**

I. INTRODUCTION

The field of learning control has recently seen explosive growth, which can be attributed to the availability of large amounts of data, creating an incentive for controllers that use the available information optimally. A significant amount of this research effort is being directed towards the familiar Linear Quadratic Regulation (LQR) problem where the transition matrices are unknown [3], [4], [5]. Most of these developments however are related to deterministic systems.

Instead this paper takes a different approach, considering systems that intrinsically include the uncertainty in the dynamics through stochastic disturbances. More specifically we study systems with a time-varying multiplicative disturbance. These may cover a wide range of system classes like Linear Parameter Varying (LPV) systems [6], [7] and Linear Difference Inclusions (LDI) [8] or in our case, when the disturbance varies stochastically, systems with multiplicative noise. Such systems have already been studied in the context of learning control by using policy iteration [9] and intrinsically introduce robustness in the controller design [10].

The authors previously developed a control synthesis procedure using the *distributionally robust approach* that guarantees stability with high probability, when the true distribution of the system is not known. This paper is related to that result and provides a methodology to evaluate the performance of the *empirical approach*, where the sample mean and covariance are used to produce a controller making it similar to the *certainty equivanlent approach* for deterministic LQR. Therefore the proofs are similar to the result of Mania et. al. [4], where the sample complexity of this certainty equivalent approach is studied.

The main result is then a suboptimality guarantee for the empirical controller. To produce such a result we make use of Riccati perturbation analysis. This paper is, to the authors' knowledge, the first instance of such a perturbation analysis being applied to discrete time systems with multiplicative noise. A Riccati perturbation bound for continuous time systems was already produced in [11].

The remainder of this paper is then structured as follows. Section II presents the problem statement and the assumptions used throughout the paper. The main result is then presented in Section III in the form of three theorems that show how the uncertainty on the covariance propagates throughout the controller synthesis. The proof of these three components are then given in the following sections. Section IV lists some results that are required for the remainder of the derivations as well as a way of deriving confidence bounds for the sample covariance. Section V then extends upon the results of Konstantinov et. al. [2] to study the perturbed Riccati equation. Section VI uses a result from convex analysis to derive a bound for the perturbation of the controller. Then Section VII proofs the main suboptimality bound, from which a sufficient condition for mean square stability (*m.s.s.*) of the true system under the empirical controller also follows. Finally Section VIII provides a conclusion and suggestions for further work.

A. Notation

Let $\mathbb{R}$ denote the reals, $\mathbb{N}$ the naturals and $\mathbb{N}_+ = \mathbb{N}\backslash\{0\}$. We use $\mathbb{S}^n$ to denote the set of n-by-n symmetric matrices. The set of positive (semi)definite matrices is then written as $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_+$). Then, for $P, Q \in \mathbb{S}^n$, we write $P \succ Q$ ($P \succeq Q$) to signify that $P - Q \in \mathbb{S}^n_{++}$ ($P - Q \in \mathbb{S}^n_+$). We denote by $\otimes$ the Kronecker product, by $A^\dagger$ the pseudoinverse of some matrix A.. We assume that all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω the sample space, $\mathcal{F}$ its associated σ-algebra and $\mathbb{P}$ the probability measure. Let $y : \Omega \to \mathbb{R}^n$ be a random vector defined on $(\Omega, \mathcal{F}, \mathbb{P})$. With some abuse of notation we will write $y \in \mathbb{R}^n$ to state the dimension of this random vector. Let $\mathbb{P}_y$ denote the distribution of y, *i.e.,* $\mathbb{P}_y(A) = \mathbb{P}[y \in A]$, then a trajectory $\{y_i\}_{i=1}^N$ of independent and identically distributed (*i.i.d.*) copies of y is defined by the distribution it induces. That is, for any $A_0, \ldots, A_N \in \mathcal{F}$ we define $\mathbb{P}_y(A_0 \times \cdots \times A_N) := \mathbb{P}[y_0 \in A_0 \wedge \cdots \wedge y_N \in A_N] = \prod_{i=0}^N \mathbb{P}_y(A_i)$. This definition can be extended to infinite trajectories $\{y_i\}_{i\in\mathbb{N}}$ by Kolmogorov's existence theorem

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The work of the first author is supported by: the Research Foundation Flanders (FWO) PhD grant 11E5520N and research projects G086518N and G086318N; Research Council KU Leuven C1 project No. C14/18/068; Fonds de la Recherche Scientifique – FNRS and the Fonds Wetenschappelijk Onderzoek – Vlaanderen under EOS project no 30468160 (SeLMA).

[12]. We will write the expectation operator as $\mathbb{E}$. We denote by $\mathbb{E}[y \mid z]$ the conditional expectation with respect to z. For matrices we will use $\|\cdot\|$ to denote the spectral norm and $\|\cdot\|_F$ to denote the Frobenius norm. For a linear matrix operator $\mathcal{F} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ we similarly use $\|\mathcal{F}\|$ to denote the operator-norm defined as $\|\mathcal{F}\| := \mathbf{max}_{\|X\|\leq 1}\|\mathcal{F}(X)\|$.

II. PROBLEM STATEMENT

In this section, we describe the problem statement and state the main result.

A. LQR for systems with multiplicative noise

This paper considers linear systems with input- and state-multiplicative noise given by:

$$x_{k+1} = A(w_k)x_k + B(w_k)u_k, \tag{1}$$

with $A(w) := A_0 + \sum_{i=1}^{n_w} w^{(i)}A_i$ and $B(w) := B_0 + \sum_{i=1}^{n_w} w^{(i)}B_i$, where at each time k, $x_k \in \mathbb{R}^{n_x}$ denotes the state, $u_k \in \mathbb{R}^{n_u}$ the input and $w_k \in \mathbb{R}^{n_w}$ an *i.i.d.* copy of a square integrable random vector w distributed according to $\mathbb{P}_w$. We use $w^{(i)}$ to denote the i'th element of w. We introduce the following shorthands: $\mathbf{A} := \left[\begin{smallmatrix} A_0^\top & A_1^\top & \dots & A_{n_w}^\top \end{smallmatrix}\right]^\top$, $\mathbf{B} := \left[\begin{smallmatrix} B_0^\top & B_1^\top & \dots & B_{n_w}^\top \end{smallmatrix}\right]^\top$ and define $\Sigma_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \Sigma \end{smallmatrix}\right]$, where we assume that $\mathbb{E}[w] = 0$ and $\mathbb{E}[ww^\top] = \Sigma$. The $\#$ operator, when applied to a matrix, then denotes the block transpose, *i.e.,* $\mathbf{A}^\# := \left[\begin{smallmatrix} A_0 & A_1 & \dots & A_{n_w} \end{smallmatrix}\right]^\top$

The primary goal is to study solutions of the following stochastic LQR problem:

$$\begin{aligned} \underset{u_0,u_1,\dots}{\mathbf{minimize}} \quad & \mathbb{E}\left[\textstyle\sum_{k=0}^{\infty} x_k^\top Q x_k + u_k^\top R u_k\right] \\ \mathbf{subj.\ to} \quad & x_{k+1} = A(w_k)x_k + B(w_k)u_k, \quad k \in \mathbb{N} \\ & x_0 \text{ given} \end{aligned} \tag{2}$$

where we assume that $Q \succ 0$ and $R \succ 0$.[1] The solution of (2) will yield a controller that renders the closed-loop system exponentially mean square stable (*e.m.s.s.*) [1, Definition 1]. Note that for the dynamics in (1) *m.s.s.* is equivalent to *e.m.s.s.* [1, Theorem 2]. Therefore we will say a system is *m.s.s.* throughout the paper, thereby also implying it is *e.m.s.s.*.

The solution of (2) is then described by the following result [1, Proposition 3]:

Proposition II.1 (LQR control synthesis)**.** *Consider a system with dynamics* (1) *and the associated LQR problem* (2)*. Assuming that* (1) *is mean square stabilizable,* i.e., *there exists a K, such that the closed-loop system $x_{k+1} = (A(w_k) + B(w_k)K)x_k$ is* m.s.s.*, then the following statements holds.*

(i) *The optimal solution of* (2) *is given by $K^\star = -(R + \mathcal{G}(P^\star))^{-1}\mathcal{H}(P^\star)$, with $P^\star$ the solution of the following Riccati equation:*

$$\begin{aligned} \mathcal{R}(P^\star, \Sigma_0) := P^\star - Q - \mathcal{F}(P^\star) \\ + \mathcal{H}(P^\star)^\top (R + \mathcal{G}(P^\star))^{-1}\mathcal{H}(P^\star) = 0, \end{aligned} \tag{3}$$

with the linear maps $\mathcal{F}(P)$, $\mathcal{G}(P)$, $\mathcal{H}(P)$ defined in Table I.

[1]This assumption is not strictly necessary, see [13] for some discussion.

(ii) *The controller $K^\star$ renders* (1) m.s.s. *in closed-loop.*

(iii) *The optimal cost is given by* $J_{K^\star}(x_0) = \mathbb{E}[\sum_{k=0}^{\infty} x_k^\top (Q + K^{\star\top} R K^\star) x_k] = x_0^\top P^\star x_0$.

The goal of this paper is then to consider the effect of misestimation of Σ on the closed-loop cost. More specifically we will operate under the following assumption

Assumption II.2. *Let $\hat{\Sigma}_0 = \Sigma_0 + \Delta\Sigma_0$ be some estimator of Σ_0, using N samples of the random vector w. We will assume it satisfies the following:*

$$\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0 \preceq \Delta\Sigma_0 \preceq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0, \tag{4}$$

where $-1 \leq \underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}} \leq 0 \leq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}} = \mathcal{O}\left(1/\sqrt{N}\right)$.

This assumption is valid with high probability when $\hat{\Sigma}_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \hat{\Sigma} \end{smallmatrix}\right]$ where $\hat{\Sigma} = \sum_{i=1}^{N} w_i w_i^\top$ and under some additional assumptions on w, which are stated in Section IV. It is also applicable for the case where the mean is also unknown and estimated as the sample mean. The constants $\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}$ and $\bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}$ depend on N, which is made explicit by using bold symbols.

III. MAIN RESULT

Starting from this assumption we will study the optimal controller produced by applying Theorem II.1 for Σ_0 and $\hat{\Sigma}_0$ which we will denote as K^* (*nominal controller*) and $\hat{K}$ (*empirical controller*) respectively. The goal is then to quantify the difference between $J_{K^\star}(x_0)$ and $J_{\hat{K}}(x_0)$. To do so we study how the perturbation on Σ_0 propagates through the controller synthesis in three stages. The first stage is how the solution of the Riccati equation is perturbed, which is quantified in Theorem III.1. The second stage is the perturbation of the control gain, quantified in Theorem III.2. The final stage is then the suboptimality, quantified in Theorem III.3.

We state these theorems for a system with dynamics (1), with $\mathbb{E}[w] = 0$ and $\mathbb{E}[ww^\top] = \Sigma$ and $K^\star$ the optimal controller and $P^\star$ the solution of (3). Then assume we have some $\hat{\Sigma}_0 = \Sigma_0 + \Delta\Sigma_0$ which satisfies Assumption II.2 and denote by $\hat{K}$ the optimal controller for $\hat{\Sigma}_0$ and $\hat{P}$ the solution of (3). The constants used in the theorems below are listed in Table I.

Theorem III.1 (Riccati Perturbation)**.** *The distance between the solutions of the Riccati equations P^* and $\hat{P}$ for covariances Σ_0 and $\hat{\Sigma}_0$ respectively is bounded as follows:*

$$\|P^\star - \hat{P}\| \leq \frac{\kappa_{\mathcal{L}}^\star - \kappa_{\mathcal{F}}\boldsymbol{\alpha}_{\boldsymbol{\Sigma}} - \sqrt{(\kappa_{\mathcal{L}}^\star - \kappa_{\mathcal{F}}\boldsymbol{\alpha}_{\boldsymbol{\Sigma}})^2 - 4\eta_{\mathcal{F}}\kappa_{\mathcal{R}_0}\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}}}{2\kappa_{\mathcal{R}_0}}, \tag{5}$$

with $\boldsymbol{\alpha}_{\boldsymbol{\Sigma}} = \mathbf{max}\{|\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}|, |\bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}|\}$. This bound holds as long as the following conditions are satisfied:

$$\boldsymbol{\alpha}_{\boldsymbol{\Sigma}} \leq \frac{2\eta_{\mathcal{F}}\kappa_{\mathcal{R}_0} + \kappa_{\mathcal{F}}\kappa_{\mathcal{L}}^\star - 2\sqrt{\eta_{\mathcal{F}}\kappa_{\mathcal{R}_0}(\eta_{\mathcal{F}}\kappa_{\mathcal{R}_0} + \kappa_{\mathcal{F}}\kappa_{\mathcal{L}}^\star)}}{\kappa_{\mathcal{F}}^2}, \tag{6}$$

and $\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}$ sufficiently small such that the right side of (5) *is smaller than $\mu_P^\star = \mathbf{min}\,\sigma(P^\star)$.*

Proof. See Section V for the proof. □

TABLE I

OVERVIEW OF SYSTEM CONSTANTS

System Matrices:	$\mathbf{A}_\star = \mathbf{A} + \mathbf{B}K^\star$	$\mathbf{S} = \mathbf{B}R^{-1}\mathbf{B}^\top$
Spectra:	$\mu_P^\star = \mathbf{min}\,\sigma(P^\star)$	$\mu_R = \mathbf{min}\,\sigma(R)$
Operators	Sensitivities	Offsets
$\mathcal{F} : \mathbb{S}^{n_x} \to \mathbb{S}^{n_x} : P \mapsto \mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A}$	$\kappa_\mathcal{F} = \|\mathcal{F}\|$	$\eta_\mathcal{F} = \|\mathcal{F}(P^\star)\|$
$\mathcal{G} : \mathbb{S}^{n_x} \to \mathbb{S}^{n_u} : P \mapsto \mathbf{B}^\top(\Sigma_0 \otimes P)\mathbf{B}$	$\kappa_\mathcal{G} = \|\mathcal{G}\|$	$\eta_\mathcal{G} = \|\mathcal{G}(P^\star)\|$
$\mathcal{H} : \mathbb{S}^{n_x} \to \mathbb{R}^{n_u \times n_x} : P \mapsto \mathbf{B}^\top(\Sigma_0 \otimes P)\mathbf{A}$	$\kappa_\mathcal{H} = \|\mathbf{A}\|\|\mathbf{B}\|\|\Sigma_0\|$	$\eta_\mathcal{H} = \|\mathbf{A}\|\|\mathbf{B}\|\|\Sigma_0\|\|P^\star\|$
$\mathcal{F}_\star : \mathbb{S}^{n_x} \to \mathbb{S}^{n_x} : P \mapsto \mathbf{A}_\star^\top(\Sigma_0 \otimes P)\mathbf{A}_\star$	$\kappa_\mathcal{F}^\star = \|\mathcal{F}_\star\|$	$\eta_\mathcal{F}^\star = \|\mathcal{F}_\star(P^\star)\|$
$\mathcal{H}_\star : \mathbb{S}^{n_x} \to \mathbb{R}^{n_u \times n_x} : P \mapsto \mathbf{B}^\top(\Sigma_0 \otimes P)\mathbf{A}_\star$	$\kappa_\mathcal{H}^\star = \|\mathbf{A}_\star\|\|\mathbf{B}\|\|\Sigma_0\|$	$\eta_\mathcal{H}^\star = \|\mathbf{A}_\star\|\|\mathbf{B}\|\|\Sigma_0\|\|P^\star\|$
$\mathcal{L}_\star : \mathbb{S}^{n_x} \to \mathbb{S}^{n_x} : P \mapsto P - \mathbf{A}_\star^\top(\Sigma_0 \otimes P)\mathbf{A}_\star$	$\kappa_\mathcal{L}^\star = \|\mathcal{L}_\star^{-1}\|^{-1}$	
$\mathcal{G}^\# : \mathbb{S}^{n_u} \to \mathbb{S}^{n_x} : X \mapsto \mathbf{B}^{\#\top}(\Sigma_0 \otimes X)\mathbf{B}^\#$	$\kappa_\mathcal{G}^\# = \|\mathcal{G}^\#\|$	
$\mathcal{H}^\# : \mathbb{R}^{n_u \times n_x} \to \mathbb{R}^{n_x \times n_u} : X \mapsto \mathbf{B}^{\#\top}(\Sigma_0 \otimes X)\mathbf{A}_\star^\#$	$\kappa_\mathcal{H}^\# = \|\mathbf{A}^\#\|\|\mathbf{B}^\#\|\|\Sigma_0\|$	
$\mathcal{L}_\star^\# : \mathbb{S}^{n_x} \to \mathbb{S}^{n_x} : X \mapsto X - \mathbf{A}_\star^{\#\top}(\Sigma_0 \otimes X)\mathbf{A}_\star^\#$	$\kappa_\mathcal{L}^\# = \|\mathcal{L}_\star^{\#-1}\|^{-1}$	
Other:	$\kappa_{\mathcal{R}_0} = \|\mathbf{A}_\star\|^2\|\mathbf{S}\|\|\Sigma_0\|^2$	$\bar{\eta}_\mathcal{G} = \|\mathcal{G}(P^\star) + R\|$

Theorem III.2 (Controller Perturbation)**.** *The distance between the optimal controllers for Σ_0 and $\hat{\Sigma}_0$ is bounded as follows:*

$$\|K^\star - \hat{K}\| \leq \tfrac{1}{\mu_R}[(1 + \boldsymbol{\alpha}_\mathbf{\Sigma})\boldsymbol{\epsilon}_\boldsymbol{P}\kappa_K + \boldsymbol{\alpha}_\mathbf{\Sigma}\eta_K], \tag{7}$$

with $\boldsymbol{\epsilon}_\boldsymbol{P}$ the right-hand side of (5)*, $\kappa_K = \kappa_\mathcal{G}\|K^\star\| + \kappa_\mathcal{H}$, $\eta_K = \eta_\mathcal{G}\|K^\star\| + \eta_\mathcal{H}$ and $\mu_R = \mathbf{min}\,\sigma(R)$.*

Proof. See Section VI for the proof. □

Theorem III.3 (Suboptimality)**.** *The difference between the optimal cost $J_{K^\star}(x_0)$ and the closed-loop cost achieved when applying $\hat{K}$ to the true system, denoted by $J_{\hat{K}}(x_0)$, is bounded as follows:*

$$\begin{aligned} &J_{\hat{K}}(x_0) - J_{K^\star}(x_0) \\ &\quad\leq \bar{n}\boldsymbol{\epsilon}_\boldsymbol{K}{}^2\bar{\eta}_\mathcal{G}\|x_0\|^2 \tfrac{1}{\kappa_\mathcal{L}^\#}\left[1 + \tfrac{2\kappa_\mathcal{H}^\#\boldsymbol{\epsilon}_\boldsymbol{K} + \kappa_\mathcal{G}^\#\boldsymbol{\epsilon}_\boldsymbol{K}{}^2}{\kappa_\mathcal{L}^\# - 2\kappa_\mathcal{H}^\#\boldsymbol{\epsilon}_\boldsymbol{K} - \kappa_\mathcal{G}^\#\boldsymbol{\epsilon}_\boldsymbol{K}{}^2}\right], \end{aligned} \tag{8}$$

where $\bar{n} = \mathbf{min}(n_x, n_u)$ and $\boldsymbol{\epsilon}_\boldsymbol{K}$ the right-side of (7)*. The bound in* (8) *is valid as long as:*

$$\boldsymbol{\epsilon}_\boldsymbol{K} < \tfrac{\kappa_\mathcal{L}^\#}{\kappa_\mathcal{H}^\# + \sqrt{\kappa_\mathcal{H}^{\#\,2} + \kappa_\mathcal{G}^\#\kappa_\mathcal{L}^\#}}. \tag{9}$$

Proof. See Section VII for the proof. □

The rate of decrease predicted by these theorems is then given in the Corollary below.

Corollary III.4 (Suboptimality bound)**.** *Let $\hat{\Sigma}_0$ satisfy Assumption II.2 and let $K^\star$ denote the nominal controller and $\hat{K}$ the emprical controller. Then*

$$J_{\hat{K}}(x_0) - J_{K^\star}(x_0) = \mathcal{O}\left(1/N\right), \tag{10}$$

assuming that N is sufficiently large.

Proof. The proof is quite straightforward. Note that $\boldsymbol{\alpha}_\mathbf{\Sigma} = \mathcal{O}(1/\sqrt{N})$ by Assumption II.2. Let $\boldsymbol{\epsilon}_\boldsymbol{P}$ denote the right-hand side of (5). Evaluating the limit $\mathbf{lim}_{N\to\infty}\sqrt{N}\boldsymbol{\epsilon}_\boldsymbol{P}$ results in

$$\begin{aligned} &\lim_{N\to\infty}\sqrt{N}\left(\tfrac{\kappa_\mathcal{L}^\star - \kappa_\mathcal{F}\boldsymbol{\alpha}_\mathbf{\Sigma} - \sqrt{(\kappa_\mathcal{L}^\star - \kappa_\mathcal{F}\boldsymbol{\alpha}_\mathbf{\Sigma})^2 - 4\eta_\mathcal{F}\kappa_{\mathcal{R}_0}\boldsymbol{\alpha}_\mathbf{\Sigma}}}{2\kappa_{\mathcal{R}_0}}\right) \\ &= \lim_{N\to\infty}\tfrac{4\eta_\mathcal{F}\kappa_{\mathcal{R}_0}(\sqrt{N}\boldsymbol{\alpha}_\mathbf{\Sigma})}{2\kappa_{\mathcal{R}_0}((\kappa_\mathcal{L}^\star - \kappa_\mathcal{F}\boldsymbol{\alpha}_\mathbf{\Sigma} + \sqrt{(\kappa_\mathcal{L}^\star - \kappa_\mathcal{F}\boldsymbol{\alpha}_\mathbf{\Sigma})^2 - 4\eta_\mathcal{F}\kappa_{\mathcal{R}_0}\boldsymbol{\alpha}_\mathbf{\Sigma}})} \\ &= \tfrac{4\eta_\mathcal{F}\kappa_{\mathcal{R}_0}c}{4\kappa_{\mathcal{R}_0}\kappa_\mathcal{L}^\star}, \end{aligned}$$

where the final equality follows from the fact that $\mathbf{lim}_{N\to\infty}\sqrt{N}\boldsymbol{\alpha}_\mathbf{\Sigma} = c > 0$ (since $\boldsymbol{\alpha}_\mathbf{\Sigma} = \mathcal{O}(1/\sqrt{N})$ by assumption), which implies that the limit of the numerator is $4\eta_\mathcal{F}\kappa_{\mathcal{R}_0}c$. The limit of the denominator meanwhile is $4\kappa_{\mathcal{R}_0}\kappa_\mathcal{L}^\star$, since $\mathbf{lim}_{N\to\infty}\boldsymbol{\alpha}_\mathbf{\Sigma} = 0$. The overall limit being some positive constant then directly implies that $\boldsymbol{\epsilon}_\boldsymbol{P} = \mathcal{O}(1/\sqrt{N})$.

Let $\boldsymbol{\epsilon}_\boldsymbol{K}$ be the right-hand side of (7) then we can see that $\boldsymbol{\epsilon}_\boldsymbol{K} = \mathcal{O}(1/\sqrt{N})$, since it depends linearly on $\boldsymbol{\alpha}_\mathbf{\Sigma}\boldsymbol{\epsilon}_\boldsymbol{P} = \mathcal{O}(1/N)$ and $\boldsymbol{\alpha}_\mathbf{\Sigma}$. Finally (8) implies the required result since the factor in square brackets is $\mathcal{O}(1)$ – which can be seen by noting that the limit for $N \to \infty$ is one – and since $\boldsymbol{\epsilon}_\boldsymbol{K}{}^2 = \mathcal{O}(1/N)$. □

Note that the rate predicted by Corollary III.4 is the same as the one achieved by the certainty equivalent controller for deterministic LQR [4].

IV. PRELIMINARY RESULTS

In this section we provide some results that will be used throughout the remainder of this paper. First we slightly alter a previous result from high-dimensional statistics that results in a condition on $\hat{\Sigma}_0$ as in (4). Second we introduce three lemmas that are related to bounding the operator norms of versions of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{L}$.

A. Concentration inequalities for the sample covariance

When using the sample-covariance $\hat{\Sigma} = \sum_{i=0}^M w_i w_i^\top$, with $\{w_i\}_{i=0}^M$ *i.i.d.* copies of w, we can find a high confidence bound of the parameters $\boldsymbol{\alpha}_\mathbf{\Sigma}$ and $\bar{\boldsymbol{\alpha}}_\mathbf{\Sigma}$ under the following assumptions:

Assumption IV.1. *We assume that (i) w is square integrable, (ii) w_k and w_ℓ are independent for all $k \neq \ell$, (iii) $\mathbb{E}[w] = 0$ (iv) $\mathbb{E}[ww^\top] = \Sigma \succ 0$ and (v) $\Sigma^{-1/2}w \sim \mathbf{subG}_{n_w}(\sigma^2)$ for some $\sigma \geq 1$.*

Here we follow the definition of a sub-Gaussian random vector (denoted by $\mathbf{subG}$) given in [1, Definition 5]. Condition (iv) holds for example for gaussian w ($\sigma = 1$) and for w with bounded support (where σ can be estimated from data [14]). Under these assumptions we can prove a slightly altered version of [1, Theorem 8], which is stated as:

Theorem IV.2. *Let $w \in \mathbb{R}^{n_w}$ be a random vector satisfying Assumption IV.1 and $\hat{\Sigma}$ the sample covariance as defined above. Then with probability at least $1-\beta$,*

$$-\boldsymbol{t}_{\boldsymbol{\Sigma}}\Sigma \preceq \hat{\Sigma} - \Sigma \preceq \boldsymbol{t}_{\boldsymbol{\Sigma}}\Sigma, \tag{11}$$

with $\boldsymbol{t}_{\boldsymbol{\Sigma}} := \frac{\sigma^2}{1-2\epsilon}\left(\sqrt{\frac{32q(\beta,\epsilon,n_w)}{M}} + \frac{2q(\beta,\epsilon,n_w)}{M}\right)$, $\epsilon \in (0, 1/2)$ *chosen freely and* $q(\beta,\epsilon,n_w) := n_w \mathbf{log}(1+1/\epsilon) + \mathbf{log}(2/\beta)$.

Proof. The proof is a specialised version of that of [1, Theorem 8] and combines [15, Lemma A.1.] with the methodology of [14]. The major difference is that no uncertainty on the mean is considered and the difference $\hat{\Sigma} - \Sigma$ is bounded instead of simply finding an upper bound for $\hat{\Sigma}$. □

Note $-\boldsymbol{t}_{\boldsymbol{\Sigma}}\Sigma_0 \preceq \Delta\Sigma_0 \preceq \boldsymbol{t}_{\boldsymbol{\Sigma}}\Sigma_0$ follows directly from (11).

B. Norms of matrix operators

We will consider bounding norms associated with $\mathcal{F}$ and $\mathcal{G}$ in two circumstances. The first being where we have some $\Delta\Sigma_0$ that is constrained by (4). The second being the case where we have some $\|\Delta P\| \leq \epsilon$. To deal with these two cases we will use the lemmas given below.

Lemma IV.3. *Consider the matrices $\mathbf{A} \in \mathbb{R}^{n_x n_w \times p}$, $P \in \mathbb{S}_+^{n_x}$, $\Sigma_0 \in \mathbb{S}_+^{n_w}$ and $\Delta\Sigma_0 \in \mathbb{S}^{n_w}$, where $\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0 \preceq \Delta\Sigma_0 \preceq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0$. Let $\boldsymbol{\alpha}_{\boldsymbol{\Sigma}} = \mathbf{max}\{|\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}|, |\bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}|\}$ We can then state the following bound:*

$$\|\mathbf{A}^\top(\Delta\Sigma_0 \otimes P)\mathbf{A}\| \leq \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}\|\mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A}\|.$$

Proof. From $\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0 \preceq \Delta\Sigma_0 \preceq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0$, due to the fact that the eigenvalues of a kronecker product of two matrices are the products of the eigenvalues of the matrices, we have that $\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0 \otimes P \preceq \Delta\Sigma_0 \otimes P \preceq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\Sigma_0 \otimes P$, implying $\underline{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A} \preceq \mathbf{A}^\top(\Delta\Sigma_0 \otimes P)\mathbf{A} \preceq \bar{\boldsymbol{\alpha}}_{\boldsymbol{\Sigma}}\mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A}$, which implies the required result. □

Lemma IV.4. *Consider the matrices $\mathbf{A} \in \mathbb{R}^{n_x n_w \times p}$, $P \in \mathbb{S}_+^{n_x}$ and $\Sigma \in \mathbb{S}_+^{n_w}$. Suppose $\|P\| \leq \epsilon$ then:*

$$\|\mathbf{A}^\top(\Sigma \otimes P)\mathbf{A}\| \leq \epsilon\|\mathbf{A}^\top(\Sigma \otimes I)\mathbf{A}\|.$$

Proof. Note that $\|P\| \leq \epsilon$ implies $P \preceq \epsilon I$. Therefore we can prove the required result using the same arguments as for the proof of Lemma IV.3. □

Using Lemma IV.4 we can see that $\kappa_{\mathcal{F}} = \|\mathbf{A}^\top(\Sigma_0 \otimes I)\mathbf{A}\|$, since $\|\mathcal{F}\| := \mathbf{max}_{\|P\|\leq 1}\|\mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A}\|$. Analogously we can find $\kappa_{\mathcal{G}}$, $\kappa_{\mathcal{G}}^{\#}$ and $\kappa_{\mathcal{F}}^{\star}$. The lemma is however not applicable to $\kappa_{\mathcal{H}}$, which is why we define it as $\kappa_{\mathcal{H}} = \|\mathbf{A}\|\|\mathbf{B}\|$. The same is true for $\kappa_{\mathcal{H}}^{\star}$ and $\kappa_{\mathcal{H}}^{\#}$. The applicability of Lemma IV.3 is less direct and will be used in Section V and Section VII. To evaluate $\kappa_{\mathcal{L}}^{\star}$ and $\kappa_{\mathcal{L}}^{\#}$ we use Lemma IV.5, which is similar to a result for deterministic dynamics [16]:

Lemma IV.5. *Let $\mathcal{L}_\star(P)$ be an invertible Lyapunov operator as defined in Table I. Then,*

$$\|\mathcal{L}_\star^{-1}\| = \|\mathcal{L}_\star^{-1}(I)\|. \tag{12}$$

Proof. First note that $\|I\| = 1$. Therefore $\|\mathcal{L}_\star^{-1}\| \geq \|\mathcal{L}_\star^{-1}(I)\|$. To prove $\|\mathcal{L}_\star^{-1}\| \leq \|\mathcal{L}_\star^{-1}(I)\|$ note that $x_0^\top\mathcal{L}_\star^{-1}(Q)x_0 = \mathbb{E}[\sum_{k=0}^{\infty} x_k^\top Q x_k]$, with $x_{k+1} = A(w_k)x_k$ [17]. We can write this as $\mathbf{Tr}\,(\sum_{k=0}^{\infty}\mathbb{E}[x_k x_k^\top]Q) = \mathbf{Tr}\,HQ$, where $H = \mathcal{L}_\star^{\#\,-1}(I) \succeq 0$ and apply [18, Proposition 2.1] to show that $I = \mathbf{argmax}_{\|Q\|\leq 1}\mathbf{Tr}\,HQ$. Therefore $\|\mathcal{L}_\star^{-1}\| \leq \|\mathcal{L}_\star^{-1}(I)\|$. □

V. RICCATI PERTURBATION

In this section we study the stochastic Riccati equation with perturbed parameters. The goal is to bound how much such perturbations affect the solutions, thereby proving Theorem III.1. To do so we will use the methodology applied in [2], [19] to the deterministic case. The main proof is stated at the end of the section, for which we state the main component first. This is a reformulation of the perturbed Riccati equation as a fixed-point equation.

More specifically let $P^\star$ be the solution of $\mathcal{R}(P^\star, \Sigma_0) = 0$ and $\Delta\Sigma_0$ selected such that $\Sigma_0 + \Delta\Sigma_0 \in \mathbb{S}_+^{n_w}$. Then a $\Delta P \in \mathbb{S}^{n_x}$ is a solution of $\mathcal{R}(P^\star + \Delta P, \Sigma_0 + \Delta\Sigma_0) = 0$ iff it is a solution to the following fixed-point equation:

$$\Phi(\Delta P) := \mathcal{L}_\star^{-1}(\mathcal{R}_0(\Delta P) + \mathcal{R}_\Delta(\Delta P)) = \Delta P, \tag{13}$$

with $\mathcal{L}_\star$ the Lyapunov operator for the optimal closed-loop system — which is invertible since the closed-loop system is *m.s.s.* [17] — and where

$$\mathcal{R}_0(\Delta P) := \mathcal{R}(P^\star + \Delta P, \Sigma_0) - \mathcal{L}_\star(\Delta P)$$
$$\mathcal{R}_\Delta(\Delta P) := \mathcal{R}(P^\star + \Delta P, \Sigma_0 + \Delta\Sigma_0) - \mathcal{R}(P^\star + \Delta P, \Sigma_0).$$

Using the constants in Table I, Lemma V.1 then describes two essential properties of Φ. The proof is deferred to Appendix A.

Lemma V.1. *Let Φ be defined as in* (13) *and $\mathcal{D} := \{\Delta P \in \mathbb{S}^{n_x} \mid \|\Delta P\| \leq \boldsymbol{\epsilon}_{\boldsymbol{P}} \leq \mu_P^\star, \Delta P + P^\star \succeq 0\}$. For every $\Delta P \in \mathcal{D}_P$*

(i) *the spectral norm of $\Phi(\Delta P)$ is bounded as:*

$$\begin{aligned}\|\Phi(\Delta P)\| &\leq h(\boldsymbol{\epsilon}_{\boldsymbol{P}}, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}) \\ &= \kappa_{\mathcal{L}}^{\star\,-1}(\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}(\eta_{\mathcal{F}} + \kappa_{\mathcal{F}}\boldsymbol{\epsilon}_{\boldsymbol{P}}) + \kappa_{\mathcal{R}_0}\boldsymbol{\epsilon}_{\boldsymbol{P}}{}^2),\end{aligned} \tag{14}$$

(ii) *the matrix $\Phi(\Delta P)$ is symmetric.*

Proof of Theorem III.1. We can now complete proof of Theorem III.1. To do so first note that $h(\boldsymbol{\epsilon}_{\boldsymbol{P}}, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}) = \boldsymbol{\epsilon}_{\boldsymbol{P}}$ is a quadratic equation. It is easy to check that (6) is a necessary and sufficient condition for the existence of a positive solution, which is given by (5). By assumption we then also have that $\boldsymbol{\epsilon}_{\boldsymbol{P}} \leq \mu_P^\star := \mathbf{min}\,\sigma(P^\star)$.

Under these conditions we can verify three properties of the mapping Φ: (i) it preserves symmetry, (ii) $\|\Delta P\| \leq \boldsymbol{\epsilon}_{\boldsymbol{P}}$

implies $\|\Phi(\Delta P)\| \leq \boldsymbol{\epsilon_P}$, (iii) $P^\star + \Delta P \succeq 0$ implies $P^\star + \Phi(\Delta P) \succeq 0$. Property (i) directly follows from Lemma V.1. From (14) we also know that for every $\Delta P \in \mathcal{D}_P$ we have $\|\Phi(\Delta P)\| \leq h(\boldsymbol{\epsilon_P}, \boldsymbol{\alpha_\Sigma}) = \boldsymbol{\epsilon_P}$, which implies property (ii). Since $\|\Phi(\Delta P)\| \leq \boldsymbol{\epsilon_P} \leq \mu_P^\star$ property (iii) holds as well. Therefore $\Phi(\mathcal{D}_P) \subseteq \mathcal{D}_P$ and we can apply the Brouwer Fixed-Point Theorem [20, Corollary 17.56], which proves $\Delta P \in \mathcal{D}_P$ and therefore Theorem III.1. □

VI. CONTROLLER PERTURBATION

In this section we derive a bound on $\|K^\star - \hat{K}\|$, thereby proving Theorem III.2. We state the proof at the end of the section, but first introduce some of the components.

We will use a result from convex optimization, [4, Lemma 1], which we can apply since both the nominal as well as the empirical controllers are optima of the following cost functions:

$$f^*(u) = u^\top R u + (\mathbf{A}x + \mathbf{B}u)^\top (\Sigma \otimes P^\star)(\mathbf{A}x + \mathbf{B}u) \quad (15)$$

$$\hat{f}(u) = u^\top R u + (\mathbf{A}x + \mathbf{B}u)^\top (\Sigma \otimes \hat{P})(\mathbf{A}x + \mathbf{B}u). \quad (16)$$

Both functions are strongly convex with $\mu_R = \mathbf{min}\,\sigma(R)$. We will then need a bound for $\|\nabla f^\star(u) - \nabla \hat{f}(u)\|$:

Lemma VI.1. *Let $f^\star(u)$ and $\hat{f}(u)$ defined respectively as in* (15) *and* (16)*. Then the difference between their gradients is bounded as:*

$$\begin{aligned}\|\nabla f^\star(u) - \nabla \hat{f}(u)\| \leq\ & ((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}\kappa_\mathcal{G} + \boldsymbol{\alpha_\Sigma}\eta_\mathcal{G})\|u\| \\ & + ((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}\kappa_\mathcal{H} + \boldsymbol{\alpha_\Sigma}\eta_\mathcal{H})\|x\|. \end{aligned} \quad (17)$$

Proof. The gradients are given by $\nabla f^\star(u) = (\mathbf{B}^\top(\Sigma_0 \otimes P^\star)\mathbf{B} + R)u + \mathbf{B}^\top(\Sigma_0 \otimes P^\star)\mathbf{A}x$ and $\nabla \hat{f}(u) = (\mathbf{B}^\top(\hat{\Sigma}_0 \otimes \hat{P})\mathbf{B} + R)u + \mathbf{B}^\top(\hat{\Sigma}_0 \otimes \hat{P})\mathbf{A}x$. The difference between the first terms of the gradients can be bounded by using Lemma IV.3 and Lemma IV.4. More specifically we have

$$\begin{aligned}&(\mathbf{B}^\top(\hat{\Sigma}_0 \otimes \hat{P})\mathbf{B} + R)u - (\mathbf{B}^\top(\Sigma_0 \otimes P^\star)\mathbf{B} + R)u \\ &= \mathbf{B}^\top(\Sigma_0 \otimes \Delta P + \Delta\Sigma_0 \otimes (P^\star + \Delta P))\mathbf{B}u.\end{aligned}$$

We can remove the dependency on $\Delta\Sigma_0$ and ΔP by applying Lemma IV.3 and Lemma IV.4 respectively, resulting in:

$$\begin{aligned}&\mathbf{B}^\top(\Sigma_0 \otimes \Delta P + \Delta\Sigma_0 \otimes (P^\star + \Delta P))\mathbf{B} \\ &\preceq \mathbf{B}^\top((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}(\Sigma_0 \otimes I) + \boldsymbol{\alpha_\Sigma}(\Sigma_0 \otimes P^\star))\mathbf{B}.\end{aligned}$$

Since Lemma IV.4 and Lemma IV.3 are not applicable for the difference between the second term of the gradients we instead produce the following bound:

$$\begin{aligned}&\|\mathbf{B}^\top(\hat{\Sigma}_0 \otimes \hat{P})\mathbf{A}x - \mathbf{B}^\top(\Sigma_0 \otimes P^\star)\mathbf{A}x\| \\ &= \|\mathbf{B}^\top((\Sigma_0 \otimes \Delta P) + \Delta\Sigma_0 \otimes (P + \Delta P))\mathbf{A}\|\|x\| \\ &\leq \|\mathbf{B}\|\|\mathbf{A}\|\|\Sigma_0\|\|\Delta P\| \\ &\quad + \|\mathbf{B}\|\|\mathbf{A}\|\|\Delta\Sigma_0\|(\|P^\star\| + \|\Delta P\|))\|x\| \\ &= ((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}\|\mathbf{B}\|\|\mathbf{A}\|\|\Sigma_0\| \\ &\quad + \boldsymbol{\alpha_\Sigma}\|\mathbf{B}\|\|\mathbf{A}\|\|\Sigma_0\|\|P^\star\|)\|x\|,\end{aligned}$$

where we used (4) for $\|\Delta\Sigma_0\| \leq \boldsymbol{\alpha_\Sigma}\|\Sigma_0\|$. Using the definitions of $\kappa_\mathcal{G}$, $\kappa_\mathcal{H}$, $\eta_\mathcal{G}$ and $\eta_\mathcal{H}$ we get (17). □

We are now ready to bound $\|\hat{K} - K^\star\|$.

Proof of Theorem III.2. Let x be any vector with $\|x\| = 1$. Then we have

$$\begin{aligned}\|(\hat{K} - K^\star)x\| &\leq \|\hat{u} - u^\star\| \\ &\leq {\mu_R}^{-1}((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}\kappa_\mathcal{G} + \boldsymbol{\alpha_\Sigma}\eta_\mathcal{G})\|K^\star x\| \\ &\quad + {\mu_R}^{-1}((1+\boldsymbol{\alpha_\Sigma})\boldsymbol{\epsilon_P}\kappa_\mathcal{H} + \boldsymbol{\alpha_\Sigma}\eta_\mathcal{H})\|x\|, \end{aligned} \quad (18)$$

where we used $\hat{K}x = \hat{u}$ and $K^\star x = u^\star$ for the first inequality and we combined [4, Lemma 1] and Lemma VI.1 for the second inequality. Let $x^\star = \mathbf{argmax}_{\|x\|=1}\|(\hat{K} - K^\star)x\|$, for which $\|(\hat{K} - K^\star)x^\star\| = \|\hat{K} - K^\star\|$. Then (18) also holds for $x^\star$. If we also use $\|K^\star x^\star\| \leq \|K^\star\|\|x^\star\| = \|K^\star\|$, then we have proven Theorem III.2. □

VII. SUBOPTIMALITY

This section is dedicated to the proof of the main result of this paper. More specifically we derive a bound for the suboptimality of the empirical controller compared to the nominal one, given in (8) as a part of Theorem III.3. The proof of which is stated at the end of this section.

We first introduce the main component, which is a perturbation bound on a matrix operator, similar to the one derived in Section V. More specifically we will study the adjoint Lyapunov operator $\mathcal{L}^\# : \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_u \times n_x} \to \mathbb{R}^{n_x \times n_x}$ for the closed-loop system $x_{k+1} = (A(w) + B(w)\hat{K})x_k$ given by $\mathcal{L}^\#(X, K) := X - (\mathbf{A} + \mathbf{B}K)^{\#^\top}(\Sigma_0 \otimes X)(\mathbf{A} + \mathbf{B}K)^\#$. In the remainder of this section we will omit the second argument of $\mathcal{L}^\#$ and use a subscript $\star$, when $K^\star$ is implied (*i.e.,* $\mathcal{L}^\#(X, K^\star) = \mathcal{L}_\star^\#(X)$). This corresponds with the definition in Table I.

We can then state the following Lemma

Lemma VII.1. *Let $X_\infty^\star$ and $\hat{X}_\infty = \hat{X}_\infty + \Delta X_\infty$ denote the solution to $\mathcal{L}_\star^\#(X_\infty^\star) = x_0 x_0^\top$ and $\mathcal{L}^\#(\hat{X}_\infty, \hat{K}) = x_0 x_0^\top$ respectively. Then ΔX_∞ is also the solution of the following fixed-point equation:*

$$\begin{aligned}\Delta X_\infty = \Phi_\mathcal{L}(\Delta X_\infty) := {\mathcal{L}_\star^\#}^{-1}\big(& \\ {\mathbf{A}_\star}^{\#^\top}(\Sigma_0 \otimes (X_\infty^\star + \Delta X_\infty))(\mathbf{B}\Delta K)^\# & \\ + (\mathbf{B}\Delta K)^{\#^\top}(\Sigma_0 \otimes (X_\infty^\star + \Delta X_\infty)){\mathbf{A}_\star}^\# & \\ + (\mathbf{B}\Delta K)^{\#^\top}(\Sigma_0 \otimes (X_\infty^\star + \Delta X_\infty))(\mathbf{B}\Delta K)^\#\big), & \end{aligned} \quad (19)$$

with $\Delta K = \hat{K} - K^\star$. The following bounds then hold

$$\begin{aligned}&\|\Phi_\mathcal{L}(\Delta X_\infty)\| \leq g(\|\Delta X_\infty\|) \\ &:= {\kappa_\mathcal{L}^\#}^{-1}(2\kappa_\mathcal{H}^\star\boldsymbol{\epsilon_K} + \kappa_\mathcal{G}^\#{\boldsymbol{\epsilon_K}}^2)(\|X_\infty^\star\| + \|\Delta X_\infty\|), \end{aligned} \quad (20)$$

$$\begin{aligned}&\|\Phi_\mathcal{L}(\Delta X_\infty) - \Phi_\mathcal{L}(\Delta X'_\infty)\| \\ &\leq {\kappa_\mathcal{L}^\#}^{-1}(2\kappa_\mathcal{H}^\star\boldsymbol{\epsilon_K} + \kappa_\mathcal{G}^\#{\boldsymbol{\epsilon_K}}^2)\|\Delta X_\infty - \Delta X'_\infty\|, \end{aligned} \quad (21)$$

with $\kappa_\mathcal{H}^\star$, $\kappa_\mathcal{G}^\#$ and $\kappa_\mathcal{L}^\#$ defined as in Table I.

Proof. We can use basic algebra and $\mathcal{L}_\star^\#(X_\infty^\star) = x_0 x_0^\top$ to rewrite the perturbed Lyapunov equation $\mathcal{L}^\#(X_\infty^\star + \Delta X_\infty, K^\star + \Delta K) = x_0 x_0^\top$ as in (19). The bounds are then derived by noting that $\|(\mathbf{B}\Delta K)^{\#^\top}(\Sigma_0 \otimes X_\infty^\star)(\mathbf{B}\Delta K)^\#\| =$

$\|\mathbf{B}^{\#^\top}(\Sigma_0 \otimes \Delta K X_\infty^\star \Delta K^\top)\mathbf{B}^\#\| \leq \kappa_{\mathcal{G}}^\# \|\Delta K\|^2 \|X_\infty^\star\|$. We also know $\|(\mathbf{B}\Delta K)^{\#^\top}(\Sigma_0 \otimes X_\infty^\star)\mathbf{A}_\star^\#\| \leq \|\mathbf{A}_\star^\#\|\|\mathbf{B}^\#\|\|\Sigma_0\|\|\Delta K\|\|X_\infty^\star\| = \kappa_{\mathcal{H}}^\#\|\Delta K\|\|X_\infty^\star\|$. Using the same tricks for the final term in (19) and the definition of $\kappa_{\mathcal{L}}^\#$ allows us to prove (20). The same tricks also produce (22) where we use $A \otimes B + A \otimes C = A \otimes (B + C)$. □

Similarly to how Lemma V.1 was used to bound the Riccati perturbation, we can bound $\|\Delta X_\infty\|$.

Lemma VII.2. *Suppose* $\kappa_{\mathcal{L}}^{\#^{-1}}(2\kappa_{\mathcal{H}}\|\Delta K\| + \kappa_{\mathcal{G}}^\#\|\Delta K\|^2) < 1$ *then we can bound* $\|\Delta X_\infty\|$ *as*

$$\|\Delta X_\infty\| \leq \frac{2\kappa_{\mathcal{H}}^\#\|\Delta K\| + \kappa_{\mathcal{G}}^\#\|\Delta K\|^2}{\kappa_{\mathcal{L}}^\# - 2\kappa_{\mathcal{H}}^\#\|\Delta K\| - \kappa_{\mathcal{G}}^\#\|\Delta K\|^2}\|X_\infty^\star\| \quad (22)$$

and $\hat{K}$ *renders the true system* m.s.s..

Proof. It is easy to verify that the solution to $\boldsymbol{\epsilon_X} = g(\boldsymbol{\epsilon_X})$ is given by the right-hand side of (22), with g as defined in Lemma VII.1. Let $\mathcal{D}_X = \{\Delta X_\infty \in \mathbb{S}^{n_x} \mid \|\Delta X_\infty\| \leq \boldsymbol{\epsilon_X}, X_\infty^\star + \Delta X_\infty \succeq 0\}$. Then, due to $\kappa_{\mathcal{L}}^{\#^{-1}}(2\kappa_{\mathcal{H}}\|\Delta K\| + \kappa_{\mathcal{G}}^\#\|\Delta K\|^2) < 1$ and Lemma VII.1, the operator $\Phi_{\mathcal{L}}$ is a contraction on $\mathcal{D}_X$. Invoking the Banach Fixed-Point Theorem [20, Theorem 3.48] then guarantees that $\mathcal{D}_X$ contains a fixed-point of $\Phi_{\mathcal{L}}$. Therefore $\hat{K}$ stabilizes the true system since $\|X_\infty^\star + \Delta X_\infty\| \leq \|X_\infty^\star\| + \boldsymbol{\epsilon_X}$ is finite, implying *m.s.s.* and we have $\|\Delta X_\infty\| \leq \boldsymbol{\epsilon_X}$. □

We are now ready to prove the suboptimality bound

Proof of Theorem III.3. We start by using [9, Lemma 3.5], which states:

$$J_{\hat{K}}(x_0) - J_{K^\star}(x_0) = \mathbf{Tr}\left[\Delta K^\top (R + \mathcal{G}(P^\star))\Delta K \hat{X}_\infty\right], \quad (23)$$

where $\hat{K} = K^\star + \Delta K$. We can then write:

$$J_{\hat{K}}(x_0) - J_{K^\star}(x_0) \leq \bar{n}\boldsymbol{\epsilon_K}^2 \bar{\eta}_{\mathcal{G}} \|\hat{X}_\infty\|, \quad (24)$$

with $\bar{n} = \mathbf{max}\{n_x, n_u\}$ and $\bar{\eta}_{\mathcal{G}} = \|\mathcal{G}(P^\star) + R\|$. The value of $\boldsymbol{\epsilon_K}$ is given as the right-hand side of (7) in Theorem III.3, leaving only the derivation of a bound for $\hat{X}_\infty$. Note that, by definition of $\kappa_{\mathcal{L}}^\#$, we have $\|X_\infty^\star\| \leq \kappa_{\mathcal{L}}^{\#^{-1}}\|x_0\|^2$. Hence using Lemma VII.2 — which is applicable due to (9) in Theorem III.3 — we can prove

$$\begin{aligned}\|X_\infty^\star + \hat{X}_\infty\| &\leq \|X_\infty^\star\| + \|\hat{X}_\infty\| \\ &\leq \frac{1}{\kappa_{\mathcal{L}}^\#}\left[1 + \frac{2\kappa_{\mathcal{H}}^\#\|\Delta K\| + \kappa_{\mathcal{G}}^\#\|\Delta K\|^2}{\kappa_{\mathcal{L}}^\# - 2\kappa_{\mathcal{H}}^\#\|\Delta K\| - \kappa_{\mathcal{G}}^\#\|\Delta K\|^2}\right]\|x_0\|^2.\end{aligned}$$

Substituting this into (23) results in (8). □

VIII. CONCLUSIONS AND FUTURE WORKS

This paper studied the sample complexity of LQR applied to systems with multiplicative noise. Overall we provided three types of sample complexities in Theorem III.1-III.3.

The first is given in Theorem III.1, which produces a bound on the amount of samples required to make the resulting problem stabilizable and the Riccati perturbation finite.

The second sample-complexity is the one related to stability, given in Theorem III.3. It gives a bound on the amount of samples required before the produced controller stabilizes the true system.

The final sample-complexity is then related to performance. It is given in Corollary III.4 and states that the suboptimality decreases with $1/N$. This is the same rate as was derived for determinstic certainty equivalent LQR in [4].

In future work, we aim to extend the results to partially observed systems and to the distributionally robust approach, where the stability complexity is absent, since it is satisfied automatically.

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APPENDIX

A. Proof of Lemma V.1

The proof of Lemma V.1 is stated in three steps. First a simplified expression of Φ is derived. Then the bound in (14) is derived. Finally symmetry of $\Phi(\Delta P)$ is verified.

a) Simplification of Φ: The result given in (13) follows from inspection. To prove Lemma V.1 however we require a simplified expression of $\mathcal{R}_0$ and $\mathcal{R}_\Delta$. To derive these we can use the matrix inversion lemma[2], to produce a simplified version of the Riccati equation, similarly to [2] and [4]:

$$\begin{aligned}\mathcal{R}(P^\star,\Sigma_0) &= P^\star - Q \\ &\quad - \mathbf{A}^\top(\Sigma_0\otimes P^\star)(I+\mathbf{S}(\Sigma_0\otimes P^\star))^{-1}\mathbf{A} = 0.\end{aligned} \tag{25}$$

Where we applied the lemma to $\mathcal{R}(P^*,\Sigma_0) = P^* - Q - \mathbf{A}^\top P^\star_\otimes(I - I\mathbf{B}(R+\mathbf{B}^\top P^\star_\otimes\mathbf{B})^{-1}\mathbf{B}^\top P^\star_\otimes)\mathbf{A}$, with $X = I$, $Y = R^{-1}$, $U = \mathbf{B}$ and $V = \mathbf{B}^\top P^\star_\otimes$, with $P^\star_\otimes := \Sigma_0\otimes P^\star$ and $\Delta P_\otimes := \Sigma_0\otimes\Delta P$. Similarly we can verify $\mathbf{A}_\star = (I + \mathbf{S}(\Sigma_0\otimes P^\star))^{-1}\mathbf{A}$. The inverse of $(I+\mathbf{S}(\Sigma_0\otimes P^\star))$ then exists due to the matrix inversion lemma, since $R+\mathbf{B}^\top(\Sigma_0\otimes P^*)\mathbf{B}$ is invertible. We then want to prove the following:

$$\mathcal{R}_0(\Delta P) := \mathbf{A}_\star^\top\Delta P_\otimes(I+\mathbf{S}P^\star_\otimes+\mathbf{S}\Delta P_\otimes)^{-1}\mathbf{S}\Delta P_\otimes\mathbf{A}_\star, \tag{26}$$

To simplify the derivation of (26), let $Y := I + \mathbf{S}P^\star_\otimes$ and $\Delta Y := \mathbf{S}\Delta P_\otimes$. As mentioned for (25) Y is invertible and similarly $Y+\Delta Y$ is invertible due to similar arguments as before. More specifically we assumed $\|\Delta P\| \le \mu_P^\star$, which implies that $P+\Delta P \succeq 0$. Therefore $R+\mathbf{B}^\top(P^\star_\otimes+\Delta P_\otimes)\mathbf{B}$ is invertible, implying that $Y+\Delta Y$ is invertible due to the matrix inversion lemma. Therefore we can derive the following relation:

$$\begin{aligned}&Y^\top(P^\star_\otimes+\Delta P_\otimes)(Y+\Delta Y)^{-1} \\ &\quad= (I+P^\star_\otimes\mathbf{S})(P^\star_\otimes+\Delta P_\otimes)(Y+\Delta Y)^{-1} \\ &\quad= (P^\star_\otimes+\Delta P_\otimes+P^\star_\otimes\mathbf{S}P^\star_\otimes+P^\star_\otimes\mathbf{S}\Delta P_\otimes)(Y+\Delta Y)^{-1} \\ &\quad= (P^\star_\otimes(I+\mathbf{S}P^\star_\otimes+\mathbf{S}\Delta P_\otimes)+\Delta P_\otimes) \\ &\qquad\times(I+\mathbf{S}P^\star_\otimes+\mathbf{S}\Delta P_\otimes)^{-1} \\ &\quad= P^\star_\otimes+\Delta P_\otimes(Y+\Delta Y)^{-1}.\end{aligned} \tag{27}$$

Using this property as well as $\mathbf{A}_\star = Y^{-1}\mathbf{A}$ we can write:

$$\begin{aligned}\mathcal{R}_0(\Delta P) &= \mathcal{R}(P^\star+\Delta P,\Sigma_0) - \mathcal{R}(P^\star,\Sigma_0) - \mathcal{L}_\star(\Delta P) \\ &= -\mathbf{A}^\top(P^\star_\otimes+\Delta P_\otimes)(Y+\Delta Y)^{-1}\mathbf{A} \\ &\quad + \mathbf{A}^\top P^\star_\otimes Y^{-1}\mathbf{A} + \mathbf{A}_\star^\top\Delta P_\otimes\mathbf{A}_\star \\ &= -\mathbf{A}_\star^\top Y^\top(P^\star_\otimes+\Delta P_\otimes)(Y+\Delta Y)^{-1}\mathbf{A} \\ &\quad + \mathbf{A}^\top P^\star_\otimes Y^{-1}\mathbf{A} + \mathbf{A}_\star^\top\Delta P_\otimes\mathbf{A}_\star \\ &\overset{Eq.\ (27)}{=} \overbrace{-\mathbf{A}_\star^\top P^\star_\otimes\mathbf{A} - \mathbf{A}_\star^\top\Delta P_\otimes(Y+\Delta Y)^{-1}\mathbf{A}} \\ &\quad + \mathbf{A}^\top P^\star_\otimes Y^{-1}\mathbf{A} + \mathbf{A}_\star^\top\Delta P_\otimes\mathbf{A}_\star \\ &= \mathbf{A}_\star^\top[-P^\star_\otimes Y - \Delta P_\otimes(Y+\Delta Y)^{-1}Y \\ &\quad + Y^\top P^\star_\otimes + \Delta P_\otimes]\mathbf{A}_\star,\end{aligned}$$

The final equation can be simplified further to (26) by noting that $Y^\top P^\star_\otimes = P^\star_\otimes Y$ and $I-(Y+\Delta Y)^{-1}Y = (Y+\Delta Y)^{-1}(Y+\Delta Y-Y) = (Y+\Delta Y)^{-1}\Delta Y$.

We then derive the a simplified description of $\mathcal{R}_\Delta(\Delta P)$. To do so we again define similar shorthands as we used for the derivation of $\mathcal{R}_0(\Delta P)$, but now in terms of $\hat{P} = P^* + \Delta P$. More specifically we now have $\hat{P}_\otimes = \Sigma_0\otimes\hat{P}$, $\Delta\hat{P}_\otimes = \Delta\Sigma_0\otimes\hat{P}$, $\hat{Y} = I + \mathbf{S}\hat{P}_\otimes$ and $\Delta\hat{Y} = \mathbf{S}\Delta\hat{P}_\otimes$. Note that $\hat{Y}$ and $\hat{Y}+\Delta\hat{Y}$ are invertible from the same arguments as before. The final expression is then:

$$\mathcal{R}_\Delta(\Delta P) := \mathbf{A}^\top\hat{Y}^{-\top}\Delta\hat{P}_\otimes(\hat{Y}+\Delta\hat{Y})^{-1}\mathbf{A}. \tag{28}$$

To derive (28) we first use the matrix inversion lemma:

$$\begin{aligned}(\hat{Y}+\Delta\hat{Y})^{-1} &= \hat{Y}^{-1} - \hat{Y}^{-1}\Delta\hat{Y}(I+\hat{Y}^{-1}\Delta\hat{Y})^{-1}\hat{Y}^{-1} \\ &= \hat{Y}^{-1}(I - \Delta\hat{Y}(\hat{Y}+\Delta\hat{Y})^{-1}).\end{aligned} \tag{29}$$

Therefore we have:

$$\begin{aligned}\mathcal{R}_\Delta(\Delta P) &= \mathcal{R}(\hat{P},\Sigma_0) - \mathcal{R}(\hat{P},\Sigma_0+\Delta\Sigma_0) \\ &= \mathbf{A}^\top(\hat{P}_\otimes+\Delta\hat{P}_\otimes)(\hat{Y}+\Delta\hat{Y})^{-1}\mathbf{A} - \mathbf{A}^\top\hat{P}_\otimes\hat{Y}^{-1}\mathbf{A} \\ &= \mathbf{A}^\top[\Delta\hat{P}_\otimes(\hat{Y}+\Delta\hat{Y})^{-1} \\ &\quad + \hat{P}_\otimes(\hat{Y}+\Delta\hat{Y})^{-1} - \hat{P}_\otimes\hat{Y}^{-1}]\mathbf{A} \\ &\overset{Eq.\ (29)}{=} \mathbf{A}^\top[\Delta\hat{P}_\otimes(\hat{Y}+\Delta\hat{Y})^{-1} \\ &\quad + \hat{P}_\otimes\overbrace{\hat{Y}^{-1}(I-\Delta\hat{Y}(\hat{Y}+\Delta\hat{Y})^{-1})} - \hat{P}_\otimes\hat{Y}^{-1}]\mathbf{A} \\ &= \mathbf{A}^\top[\Delta\hat{P}_\otimes - \hat{P}_\otimes\hat{Y}^{-1}\Delta\hat{Y}](\hat{Y}+\Delta\hat{Y})^{-1}\mathbf{A} \\ &= \mathbf{A}^\top[I-\hat{P}_\otimes\hat{Y}^{-1}\mathbf{S}]\Delta\hat{P}_\otimes(\hat{Y}+\Delta\hat{Y})^{-1}\mathbf{A},\end{aligned}$$

where we used the definition of $\Delta\hat{Y}$ for the final equality. We can further simplify the quantity between square brackets by applying the matrix inversion lemma once more:

$$I - \hat{P}_\otimes\hat{Y}^{-1}\mathbf{S} = I - \hat{P}_\otimes(I+\mathbf{S}\hat{P}_\otimes)^{-1}\mathbf{S} = \hat{Y}^{-\top}.$$

Therefore we have proven (28). Using the simplification of $\mathcal{R}_0$ and $\mathcal{R}_\Delta$ in (26) and (28) respectively we can move on to proving the bound in (14).

b) Bound on $\Phi(\Delta P)$: By the definition of Φ and by the definition of the operator norm of a linear matrix function we can write:

$$\|\Phi(\Delta P)\| \le \|\mathcal{L}^{-1}\|(\|\mathcal{R}_0(\Delta P)\| + \|\mathcal{R}_\Delta(\Delta P)\|).$$

[2]Consider the matrices X, Y, U, V of appropriate dimensions, with X and Y invertible. The matrix inversion lemma states: If $(Y^{-1}+VX^{-1}U)$ is invertible then the inverse of $(X+UYV)$ exists and is given by $(X+UYV)^{-1} = X^{-1} - X^{-1}U(Y^{-1}+VX^{-1}U)^{-1}VX^{-1}$ [21].

Therefore we will, in order, focus on bounding $\|\mathcal{R}_0(\Delta P)\|$, $\|\mathcal{R}_\Delta(\Delta P)\|$ and $\|\mathcal{L}^{-1}\|$. To do so we will need the following Lemma, which is a generalisation of [4, Lemma 7]:

Lemma A.1. *Let $M, N \in \mathbb{S}^n$ and $(I + MN)$ invertible. Then*

(i) $(I + MN)^{-1}M = M(I + NM)^{-1}$,
(ii) $MN(N^\dagger + M)NM \succeq 0 \Leftrightarrow (I + MN)^{-1}M \preceq M$,
(iii) $MN(2N^\dagger + M)NM \succeq 0$
$\Leftrightarrow (I + MN)^{-1}M(I + NM)^{-1} \preceq M$.

Proof. We first prove ***i)***. To do so we apply the matrix inversion lemma:

$$\begin{aligned}(I + MIN)^{-1}M &= (I - M(I + NM)^{-1}N)M \\ &= M(I + NM)^{-1}(I + NM) - M(I + NM)^{-1}NM \\ &= M(I + NM)^{-1} = ((I + MN)^{-1}M)^\top.\end{aligned}$$

Therefore ***i)*** is verified. To prove ***ii)*** we write the matrix inequality as:

$$x^\top Mx \geq x^\top (I + MN)^{-1}Mx, \quad \forall x. \tag{30}$$

Let $y = (I + NM)^{-1}x$, then we can substitute it into (30) to get an equivalent condition:

$$\begin{aligned}&y^\top (I + MN)M(I + NM)y \\ &\quad \geq y^\top M(I + NM)y, \forall y \\ \Leftrightarrow &y^\top (MNM + MNMNM)y \geq 0, \quad \forall y.\end{aligned} \tag{31}$$

Using the fact that a pseudoinverse is a weak inverse (*i.e.,* $NN^\dagger N = N$), we can then show that (31) is equivalent to

$$y^\top MN(N^\dagger + M)NMy \geq 0, \quad \forall y,$$

showing the required result in ***i)***. The proof for ***iii)*** follows a similar procedure. □

To bound $\|\mathcal{R}_0(\Delta P)\|$ we first employ Lemma A.1*(ii)* and the fact that $\mathbf{S} \succeq 0$ and $P^\star_\otimes + \Delta P_\otimes \succeq 0$ to state:

$$\Delta P_\otimes (I + \mathbf{S}P^\star_\otimes + \mathbf{S}\Delta P_\otimes)^{-1}\mathbf{S}\Delta P_\otimes \preceq \Delta P_\otimes \mathbf{S}\Delta P_\otimes.$$

Therefore we can write:

$$\|\mathcal{R}_0(\Delta P)\| \leq \|\mathbf{A}_\star^\top \Delta P_\otimes \mathbf{S}\Delta P_\otimes \mathbf{A}_\star\| \leq \kappa_{\mathcal{R}_0}\|\Delta P\|^2, \tag{32}$$

where the final inequality follows from the definition of $\kappa_{\mathcal{R}_0} := \|\mathbf{A}_\star\|^2\|\mathbf{S}\|\|\Sigma_0\|^2$.

We can apply a similar procedure to find a bound for $\|\mathcal{R}_\Delta(\Delta P)\|$, the simplification however is slightly more involved. First consider the following:

$$\begin{aligned}\hat{Y}^{-\top}\Delta\hat{P}_\otimes(\hat{Y} + \Delta\hat{Y})^{-1} &= \hat{Y}^{-\top}\Delta\hat{P}_\otimes(I + \hat{Y}^{-1}\Delta\hat{Y})^{-1}\hat{Y}^{-1} \\ &= \hat{Y}^{-\top}\Delta\hat{P}_\otimes(I + (I + \mathbf{S}\hat{P}_\otimes)^{-1}\mathbf{S}\Delta\hat{P}_\otimes)^{-1}\hat{Y}^{-1} \\ &\preceq \hat{Y}^{-\top}\Delta\hat{P}_\otimes\hat{Y}^{-1},\end{aligned}$$

where the inequality follows from Lemma A.1*(i)* followed by Lemma A.1*(ii)*. To prove that this is allowed, note that $N = (I + \mathbf{S}\hat{P}_\otimes)^{-1}\mathbf{S}$ and $M = \Delta\hat{P}_\otimes$. We need to prove that $N \in \mathbb{S}^{n_x n_w}$ and that $MN(N^\dagger + M)NM \succeq 0$. Symmetry follows from Lemma A.1*(i)*. The matrix inequality is then verified by noting that $\mathbf{S}(N^\dagger + M)\mathbf{S} = \mathbf{S}\mathbf{S}^\dagger\mathbf{S} + \mathbf{S}\mathbf{S}^\dagger\mathbf{S}\Delta\hat{P}_\otimes\mathbf{S} + \mathbf{S}\Delta\hat{P}_\otimes\mathbf{S} = \mathbf{S}(\mathbf{S}^\dagger + \hat{P}_\otimes + \Delta\hat{P}_\otimes)\mathbf{S} \succeq 0$, where we used $\mathbf{S}\mathbf{S}^\dagger\mathbf{S} = \mathbf{S}$. The next step involves getting rid of $\Delta\Sigma_0$, for which we will invoke Lemma IV.3:

$$\begin{aligned}\mathcal{R}_\Delta(\Delta P) &\preceq \mathbf{A}^\top\hat{Y}^{-\top}(\Delta\Sigma_0 \otimes \hat{P})\hat{Y}^{-1}\mathbf{A} \\ &\preceq \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}\mathbf{A}^\top\hat{Y}^{-\top}(\Sigma_0 \otimes \hat{P})\hat{Y}^{-1}\mathbf{A},\end{aligned}$$

To produce the final bound we still need to get rid of the inverses of $\hat{Y}$. More specifically we can prove that

$$\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}\mathbf{A}^\top\hat{Y}^{-\top}(\Sigma_0 \otimes \hat{P})\hat{Y}^{-1}\mathbf{A} \preceq \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}\mathbf{A}^\top(\Sigma_0 \otimes \hat{P})\mathbf{A} \tag{33}$$

by using Lemma A.1*(iii)* and $\mathbf{S} \succeq 0$ and $\hat{P}_\otimes \succeq 0$. Remember how $\mathcal{F}(P) = \mathbf{A}^\top(\Sigma_0 \otimes P)\mathbf{A}$, hence we can write:

$$\begin{aligned}\|\mathcal{R}_\Delta(\Delta P)\| &\leq \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}\|\mathcal{F}(P^\star + \Delta P)\| \\ &\leq \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}(\|\mathcal{F}(P^\star)\| + \|\mathcal{F}(\Delta P)\|) \\ &= \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}(\eta_\mathcal{F} + \kappa_\mathcal{F}\|\Delta P\|),\end{aligned} \tag{34}$$

where we used the definitions of $\eta_\mathcal{F}$ and $\kappa_\mathcal{F}$ for the final equality. Therefore, by combining (32), (34) and the definitions of $\eta^\star_\mathcal{F}$ and $\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}$, we have proven that, $\forall \Delta P \in \mathcal{D}_P$:

$$\begin{aligned}\|\Phi(\Delta P)\| &\leq h(\|\Delta P\|, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}) \\ &= \kappa^\star_\mathcal{L}{}^{-1}(\boldsymbol{\alpha}_{\boldsymbol{\Sigma}}(\eta_\mathcal{F} + \kappa_\mathcal{F}\|\Delta P\|) + \kappa_{\mathcal{R}_0}\|\Delta P\|^2).\end{aligned}$$

This is easily extended to (14) by noting that $h(\|\Delta P\|, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}})$ increases monotonically for increasing values of $\|\Delta P\|$. Therefore for all $\|\Delta P\| \leq \boldsymbol{\epsilon}_{\boldsymbol{P}}$ we have that $h(\|\Delta P\|, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}}) \leq h(\boldsymbol{\epsilon}_{\boldsymbol{P}}, \boldsymbol{\alpha}_{\boldsymbol{\Sigma}})$, which implies (14).

c) Symmetry of $\Phi(\Delta P)$: The final step is to show that $\Phi(\Delta P)$ is symmetric whenever ΔP is symmetric. To see this first note that $\mathcal{R}_0(\Delta P)$ and $\mathcal{R}_\Delta(\Delta P)$ are symmetric, since it follows from inspection that $\mathcal{L}_\star$ and $\mathcal{R}$ preserve the symmetry of the input. Therefore if we prove that $\mathcal{L}_\star^{-1}$ preserves symmetry then Φ does so as well. To do so note that $\mathcal{L}_\star(P) = P - \mathcal{F}(P)$. It is then easy to verify that $P = \sum_{k=0}^\infty \mathcal{F}^k(Q)$, where $\mathcal{F}^k(Q)$ implies that $\mathcal{F}$ is applied k times, solves $\mathcal{L}_\star(P) = Q$. After all $P - \mathcal{F}(P) = \sum_{k=0}^\infty \mathcal{F}^k(Q) - \sum_{k=1}^\infty \mathcal{F}^k(Q) = Q$. Since $\mathcal{F}$ clearly preserves symmetry, $\mathcal{L}_\star^{-1}$ does so as well. □