BACK TO THE ROOTS – POLYNOMIAL SYSTEM SOLVING USING LINEAR ALGEBRA AND SYSTEM THEORY*

PHILIPPE DREESEN[†‡], KIM BATSELIER[†‡], AND BART DE MOOR[†‡]

Abstract. We return to the algebraic roots of the problem of finding the solutions of a set of polynomial equations, and review this task from the linear algebra perspective. The system of polynomial equations is represented by a system of homogeneous linear equations by means of a structured Macaulay coefficient matrix multiplied by a vector containing monomials. Two properties are of key importance in the null spaces of Macaulay coefficient matrices, namely the correspondence between linear (in)dependent monomials in the polynomials and the linear (in)dependent rows, and secondly, the occurrence of a monomial multiplication shift structure. Both properties are invariant and occur regardless of the specific numerical basis of the null space of the Macaulay matrix. By exploiting the multiplication structure in the vector of monomials, a (generalized) eigenvalue problem is derived in terms of matrices built up from certain rows of a numerically computed basis for the null space of the Macaulay matrix. The main goal of the paper is to develop a simple solution approach, making the problem accessible to a wide audience of applied mathematicians and engineers.

Key words. polynomial system solving, polynomials, eigenvalue problems, algebraic geometry, realization theory, multidimensional system theory

AMS subject classifications. 13P05, 13P15, 15A03, 15A18, 15B05

1. Introduction.

1.1. Motivation. The problem considered in this paper is finding the roots of a system of multivariate polynomial equations

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) &= 0, \\ f_2(x_1, x_2, \ldots, x_n) &= 0, \\ & \vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0. \end{cases}$$

Polynomial system solving is typically studied in the field of algebraic geometry [10, 9], where it was the primary problem of attention until the end of the 19th century. Algebraic geometry took a turn to abstract algebra in the 20th century, and polynomial system solving came into focus again around the 1960s with the seminal work of B. Buchberger on Gröbner basis algorithms [6, 7], giving rise to the field of computer algebra [21]. The Gröbner basis solution approach is until today dominant in polynomial system solving, but suffers from poor numerical properties because it relies on symbolic operations and exact arithmetic.

Solving systems of polynomial equations is still a relevant task, showing up in a multitude of scientific and engineering applications [44]. In practical situations, floating-point arithmetic is desired and stable numerics are of paramount importance. For these reasons, Gröbner basis approaches are unsuitable. The current article addresses these issues by viewing polynomial system solving as a linear algebra task. Linear algebra turns out to be a natural setting for polynomial system solving: The *roots* of the problem have strong ties with linear algebra, *cf.,* Sylvester [47] and Macaulay

*Preliminary results of work presented in this article have previously appeared in [3, 12, 13].

[†]KU Leuven, Department of Electrical Engineering (ESAT), STADIUS Center for Dynamical Systems, Signal Processing and Data Analytics, Kasteelpark Arenberg 10, 3000 Leuven, Belgium, email: {philippe.dreesen,bart.demoor}@esat.kuleuven.be.

[‡]iMinds Medical IT Department, Kasteelpark Arenberg 10, 3000 Leuven, Belgium.

[35, 36]. We will develop a linear algebra based root-finding method starting from the work of these early 'linear' algebraists. The method is developed in the modern language of numerical linear algebra and dynamical systems theory. From numerical linear algebra [20] we borrow the notions of row/column spaces and null spaces, as well as the tools to solve homogeneous linear systems, eigenvalue problems and singular value decompositions. From dynamical systems theory we employ the tools of realization theory: we will study the null space of a Macaulay coefficient matrix, which exhibits an observability matrix structure. Applying realization theory will lead to the fact that the root-finding problem can be solved as an eigenvalue problem.

It should be mentioned that many of the presented elements are not new; similar ideas have been described in [2, 15, 16, 26, 32, 33, 37, 40, 45], among others. Yet, surprisingly, the linear algebra approach to polynomial system solving has remained largely unknown, perhaps due to the fact that the key results are scattered over the literature, and, to the best of the authors' knowledge, never have been collected into a conceptually simple framework.

1.2. Tools. We begin with a short overview of relevant concepts that we borrow from numerical linear algebra and dynamical system theory.

Numerical Linear Algebra. The second half of the 20th century has witnessed the maturation of numerical linear algebra which has turned into an established field of research. A multitude of reliable numerical linear algebra tools [20] is well understood and developed. We will outline an eigenvalue-based solution method, where linear algebra notions such as row and column space, null space, linear (in)dependence and matrix rank play an important role, as well as the essential numerical linear algebra *tools* such as the singular value decomposition and the eigenvalue decomposition.

The central object in the proposed method is the Macaulay matrix [35, 36], which is a Sylvester-like structured matrix built from the coefficients of the set of multivariate polynomials. The Macaulay matrix is obtained by considering multiplications (shifts) of the equations with monomials, such that the result has a certain maximal degree. It translates a system of polynomial equations into a system of homogeneous linear equations: the polynomials are represented as (rows of the) Macaulay matrix multiplied with a multivariate Vandermonde vector containing the monomials. The proposed method proceeds by iterating over increasing degrees for which the Macaulay matrix is built. We will study the dimensions, the rank and (co-)rank[1] of the Macaulay matrix for an increasing degree. The Macaulay matrix will become overdetermined from a certain degree on. Interpreting this as a homogeneous system allows us to divide the unknowns (monomials) into sets of linearly independent and linearly dependent unknowns. We will see that the corank of the Macaulay matrix corresponds to the number of linearly independent monomials, and moreover, is equal to the number of solutions.

System Theory. We will borrow elements from system theory [27] in general, and realization theory [25, 28, 49, 50, 51] in particular. Realization theory will enter the scene when we analyze the structure of the null space of the Macaulay matrix. The link between realization theory and polynomial systems should not come as a surprise. For one-dimensional LTI systems it is a well-known fact that the Sylvester matrix built from the denominator polynomial of a transfer function is the left annihalator of the observability matrix of the LTI system. Realization theory for multidimensional (nD)

[1]The corank of a matrix is defined as the dimension of its right null space, *i.e.,* the number of columns minus the rank of the matrix.

systems [18] has been linked to polynomial system solving in [22] using a Gröbner basis viewpoint. In multidimensional dynamical systems the dynamics depend not on a single independent variable (such as discrete time in linear time-invariant systems modelled by difference equations), but are dependent on several independent variables (such as space and time in PDEs). We will show that the null space of the Macaulay matrix can also be modeled as the output of an autonomous multidimensional (nD), possibly singular (*i.e.,* in descriptor form) system.

The null space of the Macaulay matrix has a multiplication structure that stems from the monomial structure of the unknowns. By combining the multiplication structure in the null space with the realization theory interpretation, we develop an algorithm to form a (generalized) eigenvalue problem that delivers all roots of the system.

1.3. Related work. Solving systems of polynomials has been central throughout the history of mathematics, and has been almost synonymous with algebra until the beginning of the 20th century. Sylvester [47] and Macaulay [35, 36] were the first to phrase root-finding into a (premature) linear algebra framework. This vantage point constitutes the heart of the proposed matrix approach. Unfortunately, these contributions were overshadowed by a shift to abstract algebra in the early 20th century, and were abandoned for nearly a century.

Around the 1960s, together with the advent of digital computers, computational algebra came into focus again with the development of Buchberger's Gröbner Basis algorithm [6, 7]. Although Gröbner bases have dominated computer algebra ever since, the inherent exact arithmetic (*i.e.,* symbolic calculations) of Buchberger's algorithm make its extension to floating point arithmetic cumbersome; a limited number of alternative approaches is available [17, 26, 45], not surprisingly, often involving (numerical) linear algebra.

The relevance of the work of Sylvester and Macaulay was only rediscovered during the 1980s independently by Lazard [32, 33] and Stetter [2, 45]. In the spirit of Buchberger's work on Gröbner bases [6, 7], Lazard [33] shows that a Gröbner basis can be computed by triangularization of a large Macaulay-like coefficient matrix. The seminal work by Stetter [2] and later [38, 45] elaborates on the root-finding problem and develops the natural link between the polynomial system solving problem and eigenvalue problems.

The work by Lazard [33] and Stetter can be seen as the re-birth of *numerical polynomial algebra*, as Stetter's eponymous monograph [45]. The approaches initiated by Lazard and Stetter were explored further in [16, 17, 37, 40], among others. However, in many of these approaches, the abstract notions of algebraic geometry and symbolically computed Gröbner bases persist.

The last decades have witnessed significant research interest in polynomial system solving [11, 46] and polynomial optimization methods [23, 29, 42]. Modern methods outperform symbolic methods, see *e.g.,* homotopy continuation methods [34, 48] and recent developments in real algebraic geometry and polynomial optimization [30, 31, 42]. Applications are found in applied mathematics, science and engineering, *e.g.,* systems and control [8], bioinformatics [15, 41], robotics [14], computer vision [43], and many more.

1.4. Our contribution. In this manuscript, we present a polynomial system solving method that we build up from elementary linear algebra notions such as rank, column and row spaces, null spaces, singular value decompositions and eigenvalue computations, without the need to resort to advanced algebraic geometry concepts.

We will outline a rudimentary eigenvalue-based algorithm, and provide a system theoretical interpretation using realization theory.

The proposed method will make use of two properties. Since the Macaulay matrix becomes overdetermined for a certain degree and is rank-deficient by construction, we can view some of the monomials as linearly independent and the others as linearly dependent. Together with the multiplicative structure of the set of monomials, the root-finding problem will be shown equivalent to an eigenvalue problem. Moreover, the null space of the Macaulay matrix can be modeled as the observability matrix of a multidimensional nD descriptor system, which provides a system theoretical interpretation that generalizes the link between polynomial equations and one-dimensional LTI systems realization theory to the multidimensional (nD) systems case.

Because we want to adhere to an informal, yet rigorous expository 'tutorial' style, we have chosen not to include formal proofs in this manuscript. Nevertheless, the geometrical, (numerical) linear algebraic and algorithmic point of view that we develop in this paper open a whole new avenue of research challenges that are not straightforward in the symbolical computer algebra framework.

1.5. Organization. The solution method will be built up by means of three didactical examples in Section 2. We then develop general theory dealing with the solution of systems of multivariate polynomial equations. In Section 3 we write the system of polynomial equations as a homogeneous linear system by constructing the Macaulay matrix. The null space of the Macaulay matrix and its properties are studied in Section 4. The roots will be obtained by applying realization theory in the null space of the Macaulay matrix, leading to the eigenvalue-based algorithm of Section 5. Section 6 is devoted to the conclusions and open questions. In Appendix A the linear algebraic tools and concepts relating to the solution of homogeneous linear equations are reviewed.

Throughout the remainder of this article, we will assume that the system of polynomial equations has as many equations as unknowns, and has a finite number of distinct solutions. We will assume that the coefficients of the polynomials are real, and as a consequence for every complex root, its complex conjugate is also a root. Generalizing the method beyond these assumptions is straightforward, but for the sake of simplicity this is reserved for future work.

2. Didactical examples. We will introduce a solution method by a few didactical examples that serve to illustrate the main ingredients of the proposed approach and reveal some of the linear algebra basics.

2.1. Example 1: Intersection of circle and ellipse. The intersection of a circle in the plane with center $(3, 3)$ and radius three and an ellipse in the plane with center $(4, 3)$ and half axes one and four is described by

$$\begin{array}{rcccccc} f_1(x,y) & = & (x-3)^2+(y-3)^2-9 & = & x^2+y^2-6x-6y+9 & = & 0, \\ f_2(x,y) & = & (x-4)^2+(y-3)^2/16-1 & = & 16x^2+y^2-128x-6y+249 & = & 0, \end{array}$$

which has four solutions $(3.3333, 0.0186)$, $(4.8000, 0.6000)$, $(3.3333, 5.9814)$ and $(4.8000, 5.4000)$.

Two-step Approach for Finding the Roots. We will develop a two-step approach to find its roots, *i.e.,* the pairs (x, y) that satisfy the equations $f_1 = 0$ and $f_2 = 0$.

In the first step, the system of equations is considered as a set of homogeneous linear equations in the 'unknown' monomials $1, x, y, x^2, xy, y^2, \ldots$ From this interpretation, we write dependent monomials as a linear combination of independent monomials. The first step will require to increase the maximal degree up to which we consider the equations/monomials that will constitute the Macaulay matrix. It will turn out that at a certain degree, all the information to compute the roots is contained in the Macaulay matrix (and its null space). The second step of the method employs the multiplicative structure of the monomials to derive an eigenvalue problem from which the roots can be calculated. Let us explain these steps in more details, using the example.

Degree iteration $d = 2$. Since the input equations are of degree two, we let the iteration count start at $d = 2$. We consider the two equations as a set of two homogeneous *linear* equations in the unknown monomials 1, x, y, x^2, xy, y as

$$\left(\begin{array}{c|cc|ccc} 9 & -6 & -6 & 1 & 0 & 1 \\ 249 & -128 & -6 & 16 & 0 & 1 \end{array}\right) \left(\begin{array}{c} 1 \\ \hline x \\ y \\ \hline x^2 \\ xy \\ y^2 \end{array}\right) = 0.$$

In doing so, we take the convention of ordering the monomials in the degree negative lexicographic ordering, which for two variables is given by

$$1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < xy^2 < y^3 < x^4 < \ldots$$

We constructed a coefficient matrix $M(2)$ ('M' from Macaulay, and '2' representing the maximal degree of the monomials taken into account) to rewrite the two equations in matrix-vector form. The rank of the coefficient matrix is two, hence its right corank (*i.e.,* the dimension of the null space) is four, and there are six unknowns. This means that we can take four unknowns as independent variables, and two as dependent. The idea is that we try to take as dependent variables, the monomials that are as high in the ranking as possible.

Let us now inspect the linearly dependence of columns of the coefficient matrix, starting from the right-most one. Clearly, as column five is a zero column, it is linearly dependent on column six. Column four is linearly independent of column six, so that the sub-matrix consisting of column four and six is of rank two, hence allowing us to write the two dependent variables uniquely as a linear function of the remaining four variables. As the sub-matrix that consists of column four and six of $M(2)$ (columns corresponding to the dependent variables x^2 and y^2) is of rank two, we find x^2 and y^2 from the relation

$$\begin{pmatrix} 1 & 1 \\ 16 & 1 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = - \begin{pmatrix} 9 & -6 & -6 & 0 \\ 249 & -128 & -6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ xy \end{pmatrix}.$$

We can now write the four solutions in the canonical null space matrix $V_{\mathrm{can}}(2)$, the columns of which form a basis for the null space of $M(2)$. The rows of $V_{\mathrm{can}}(2)$ corresponding to $1, x, y$ and xy form the identity matrix (bold-faced elements in the

matrix) as[2]

$$M(2)V_{\text{can}}(2) = \left(\begin{array}{c|cc|ccc} 9 & -6 & -6 & 1 & 0 & 1 \\ 249 & -128 & -6 & 16 & 0 & 1 \end{array}\right) \left(\begin{array}{cccc} \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \hline -16 & 8.1333 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 7 & -2.1333 & 6 & 0 \end{array}\right) = 0$$

Degree iteration $d = 3$**.** The next iteration starts by multiplying each of the original two equations with the two first order monomials x and y. This generates four more equations, each of degree three, with additional monomials x^3, x^2y, xy^2, y^3. Taking the two original equations together with these four new ones generates a set of six homogenous linear equations in the ten unknown monomials up to degree three, represented by

$$\left(\begin{array}{c|cc|ccc|cccc} 9 & -6 & -6 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 249 & -128 & -6 & 16 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 9 & 0 & -6 & -6 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 & -6 & -6 & 0 & 1 & 0 & 1 \\ 0 & 249 & 0 & -128 & -6 & 0 & 16 & 0 & 1 & 0 \\ 0 & 0 & 249 & 0 & -128 & -6 & 0 & 16 & 0 & 1 \end{array}\right) \left(\begin{array}{c} 1 \\ \hline x \\ y \\ \hline x^2 \\ xy \\ y^2 \\ \hline x^3 \\ x^2y \\ xy^2 \\ y^3 \end{array}\right) = 0\,.$$

The Macaulay coefficient matrix $M(3)$ is a 6×10 matrix, that contains as its rows the coefficients of the six equations $f_1 = 0, f_2 = 0, xf_1 = 0, yf_1 = 0, xf_2 = 0, yf_2 = 0$, and as its columns the coefficients of 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3 in these equations. It can be verified that its rank is six, hence its corank is four.

Checking linear independency of the columns of $M(3)$ starting from the right, we find that the columns ten, nine, eight, seven, six and four are linear independent. Recall that the reason why we check the linear dependence of the columns of $M(3)$ from right to left is because we are using the complementarity property (Appendix A): we wish to have in $V(3)$ the top-most rows as the linearly independent rows.

We find that the unknowns $1, x, y, xy$ are independent, and $x^2, y^2, x^3, x^2y, xy^2, y^3$ are dependent. This should come as no surprise, as in iteration $d = 2$ we already found that x^2 and y^2 are dependent, implying that also all monomials of higher degree that contain x^2 and/or y^2 as a factor, will be dependent! The canonical basis for the null space of $M(3)$, denoted by $V_{\text{can}}(3)$, is given by

$$V_{\text{can}}(3) = \left(\begin{array}{cccc} \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ \hline -16.0000 & 8.1333 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ 7.0000 & -2.1333 & 6.0000 & 0 \\ \hline -130.1333 & 50.1511 & 0 & 0 \\ 0 & 0 & -16.0000 & 8.1333 \\ 34.1333 & -10.3511 & 0 & 6.0000 \\ 42.0000 & -12.8000 & 43.0000 & -2.1333 \end{array}\right),$$

[2]For the time being we assume that the canonical null space is known; in Section 4.2.1 we show how it can be computed.

in which the identity matrix sits at the position of the independent variables 1, x, y, xy, as indicated by the bold-face numbers. Observe that the first six rows of $V_{\mathrm{can}}(3)$ are identical to those of $V_{\mathrm{can}}(3)$. Let us write out the results for one more iteration, which is degree iteration $d = 4$.

Degree iteration $d = 4$. We multiply the two original equations with the monomials x^2, xy, y^2, which generates another six equations, this time for fifteen unknown monomials $1, x, y, \ldots, x^4, \ldots, y^4$, with coefficient matrix $M(4)$ as

$$\left(\begin{array}{rrr|rrr|rrrr|rrrrr}
9 & -6 & -6 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
249 & -128 & -6 & 16 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 9 & 0 & -6 & -6 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 9 & 0 & -6 & -6 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 249 & 0 & -128 & -6 & 0 & 16 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 249 & 0 & -128 & -6 & 0 & 16 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 9 & 0 & 0 & -6 & -6 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 9 & 0 & 0 & -6 & -6 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & -6 & -6 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 249 & 0 & 0 & -128 & -6 & 0 & 0 & 16 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 249 & 0 & 0 & -128 & -6 & 0 & 0 & 16 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 249 & 0 & 0 & -128 & -6 & 0 & 0 & 16 & 0 & 1
\end{array}\right).$$

Notice that $M(2)$ and $M(3)$ are nested in the structure of $M(4)$. Hence, one can obtain the subsequents matrices recursively as the iteration number proceeds.

Matrix $M(4)$ is a 12×15 matrix, having rank eleven, so its right corank is four, as before. One can verify, by monitoring linear independency of the columns starting from the right, that the columns corresponding to the monomials x^2, y^2, x^3, x^2y, xy^2, y^3, x^4, x^3y, x^2y^2, xy^3, y^4 are linearly independent. This implies that the variables 1, x, y, xy are still the independent variables, the other ones being dependent, which can also be seen from the following canonical null space:

$$V_{\mathrm{can}}(4) = \left(\begin{array}{rrrr}
\mathbf{1} & 0 & 0 & 0 \\
\hline
0 & \mathbf{1} & 0 & 0 \\
0 & 0 & \mathbf{1} & 0 \\
\hline
-16.0000 & 8.1333 & 0 & 0 \\
0 & 0 & 0 & \mathbf{1} \\
7.0000 & -2.1333 & 6.0000 & 0 \\
\hline
-130.1333 & 50.1511 & 0 & 0 \\
0 & 0 & -16.0000 & 8.1333 \\
34.1333 & -10.3511 & 0 & 6.0000 \\
42.0000 & -12.8000 & 43.0000 & -2.1333 \\
\hline
-802.4178 & 277.7624 & 0 & 0 \\
0 & 0 & -130.1333 & 50.1511 \\
165.6178 & -50.0557 & -96.0000 & 48.8000 \\
204.8000 & -62.1067 & 34.1333 & 25.6489 \\
228.1822 & -69.6510 & 300.0000 & -25.6000
\end{array}\right). \tag{2.1}$$

For the subsequent iterations, where $d > 4$, the matrices $M(d)$ and $V_{\mathrm{can}}(d)$ are too large to print, so we summarize their properties in the stabilization diagram given in Table 2.1.

Notice that the number of rows of $M(d)$ grows faster than the number of columns. Indeed, for degree d we have that the number of rows $p(d)$ and the number of columns $q(d)$ of $M(d)$ is given by

$$\begin{aligned}
p(d) &= 2\binom{d}{d-2} = d^2 - d = \frac{d!}{(d-2)!}, \\
q(d) &= \binom{d+2}{d} = \frac{1}{2}d^2 + \frac{3}{2}d + 1 = \frac{(d+2)!}{2 \cdot d!},
\end{aligned}$$

TABLE 2.1

Stabilization diagram for Example 1, showing the dynamics of the properties of the Macaulay matrix $M(d)$ as a function of the maximal degree d of the monomials taken into account. It can be seen from the table that the matrix $M(d)$ becomes overdetermined at degree $d = 6$. The rank keeps increasing as d grows, however the corank stabilizes at four. Also the linear independent monomials stabilize, in this example, right away from $d = 2$ onwards.

d	size $M(d)$	rank $M(d)$	corank $M(d)$	lin. indep. mons
2	2×6	2	4	$1, x, y, xy$
3	6×10	6	4	$1, x, y, xy$
4	12×15	11	4	$1, x, y, xy$
5	20×21	17	4	$1, x, y, xy$
6	30×28	24	4	$1, x, y, xy$

which shows that from degree six on, the matrix $M(d)$ has more rows than columns. We also observe that the rank of $M(d)$ keeps increasing as d grows. However, the corank of $M(d)$ stabilizes at four, which is the number of solutions. The four linearly independent monomials stabilize as well, being 1, x, y, xy. The general expression for the rank of $M(d)$ in this example is given by $\text{rank}(M(d)) = \frac{1}{2}d^2 + \frac{3}{2}d - 3$.

Finding the roots. Let us now show how we can find the roots of the system of equations from the null space of the Macaulay matrix. Assume for the time being that we know the four true solutions $(3.3333, 0.0186)$, $(4.8000, 0.6000)$, $(3.3333, 5.9814)$ and $(4.8000, 5.4000)$, which we denote as $(x_1, y_1), \ldots, (x_4, y_4)$. Each of the solutions generates a vector in the basis of the null space of $M(d)$. Indeed, by evaluating the monomial basis vector $\left(\begin{array}{c|c|cc|c} 1 & x & y & x^2 & xy & y^2 & \ldots \end{array}\right)^T$ at each of the solutions $(x_1, y_1), \ldots, (x_4, y_4)$, we find four linearly independent vectors. By collecting these vectors in a matrix $V_{\text{Vdm}}(d)$ we find the generalized Vandermonde-structured basis of the null space as (shown here for $d = 4$):

$$\left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ \hline x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ \hline x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2y_1 & x_2^2y_2 & x_3^2y_3 & x_4^2y_4 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 \\ \hline x_1^4 & x_2^4 & x_3^4 & x_4^4 \\ x_1^3y_1 & x_2^3y_2 & x_3^3y_3 & x_4^3y_4 \\ x_1^2y_1^2 & x_2^2y_2^2 & x_3^2y_3^2 & x_4^2y_4^2 \\ x_1y_1^3 & x_2y_2^3 & x_3y_3^3 & x_4y_4^3 \\ y_1^4 & y_2^4 & y_3^4 & y_4^4 \end{array}\right) = \left(\begin{array}{cccc} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ \hline 3.3333 & 4.8000 & 3.3333 & 4.8000 \\ 0.0186 & 0.6000 & 5.9814 & 5.4000 \\ \hline 11.1111 & 23.0400 & 11.1111 & 23.0400 \\ 0.0619 & 2.8800 & 19.9380 & 25.9200 \\ 0.0003 & 0.3600 & 35.7774 & 29.1600 \\ \hline 37.0370 & 110.5920 & 37.0370 & 110.5920 \\ 0.2064 & 13.8240 & 66.4602 & 124.4160 \\ 0.0012 & 1.7280 & 119.2581 & 139.9680 \\ 0.0000 & 0.2160 & 213.9999 & 157.4640 \\ \hline 123.4568 & 530.8416 & 123.4567 & 530.8416 \\ 0.6880 & 66.3552 & 221.5342 & 597.1968 \\ 0.0038 & 8.2944 & 397.5270 & 671.8464 \\ 0.0000 & 1.0368 & 713.3333 & 755.8272 \\ 0.0000 & 0.1296 & 1280.0246 & 850.3056 \end{array}\right). \tag{2.2}$$

Let us now, starting from $V_{\text{can}}(4)$ and the fact that 1, x, y, and xy are the linear independent monomials of lowest degree, develop a method to find the roots (x_i, y_i), for $i = 1, \ldots, 4$. Rather than V_{Vdm}, we have the canonical basis for the null space V_{can} available. The canonical basis V_{can} for the null space of $M(4)$ is shown in (2.1). However, it is the Vandermonde structured basis for the null space (2.2) that reveals

the four roots and their mutual matching. We have

$$V_{\mathrm{Vdm}} = V_{\mathrm{can}} \cdot T_{\mathrm{Vdm}} = V_{\mathrm{can}} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix},$$

where T_{Vdm} is nonsingular.

So, how can we find from V_{can} the solutions (x_i, y_i)? We will make use of the multiplicative shift structure in the Vandermonde basis. Let us consider as a shift function the monomial x. We can write

$$S_1 \cdot V_{\mathrm{Vdm}} \cdot D_x = S_x \cdot V_{\mathrm{Vdm}},$$

where $D_x = \mathrm{diag}(x_1, x_2, x_3, x_4)$, S_1 selects the rows 1, x, y, xy from V_{Vdm} (and possibly others), and S_x selects the rows $1 \cdot x$, $x \cdot x$, $y \cdot x$, $xy \cdot x$ of V_{Vdm}. For the current example, we have

$$S_1 = \left(\begin{array}{c|cc|ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right), S_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

This trick leads to the formulation of a generalized eigenvalue problem

$$\begin{aligned} (S_1 \cdot V_{\mathrm{can}}) \cdot T_{\mathrm{Vdm}} \cdot D_x &= (S_x \cdot V_{\mathrm{can}}) \cdot T_{\mathrm{Vdm}}, \\ B \cdot T_{\mathrm{Vdm}} \cdot D_x &= A_x \cdot T_{\mathrm{Vdm}}. \end{aligned}$$

Let us make the following important observations.

1. The diagonal matrix D_x contains the eigenvalues, which correspond to the x-components of the solutions. The matrix T_{Vdm} contains the eigenvectors. When using V_{can} as a basis for the null space of M, the eigenvectors obey the Vandermonde structure. When using the Vandermonde basis for the null space V_{Vdm} to formulate the eigenvalue problem, the identity matrix contains the eigenvectors: $S_1 \cdot V_{Vdm} \cdot I \cdot D_x = S_x \cdot V_{\mathrm{Vdm}} \cdot I$.
2. The matrix S_1 may select more rows than only the linearly independent rows 1, x, y, xy; in which case a rectangular matrix pencil will be obtained, which is exactly solvable since the Vandermonde shift structure holds for all rows in V_{Vdm}. A square ordinary eigenvalue problem is found as

$$B^{\dagger} A_x = T \cdot D_x \cdot T^{-1},$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose pseudoinverse.

The generalized eigenvalue problem was derived for a shift with x. Now suppose that we are using y as a shift function. Again we let S_1 select the rows of V_{Vdm} corresponding to the rows 1, x, y, xy. Now $D_y = \mathrm{diag}(y_1, y_2, y_3, y_4)$ and S_y selects from V_{can} the rows corresponding to the multiplication of 1, x, y, xy with the shift function y. We now obtain

$$\begin{aligned} (S_1 \cdot V_{\mathrm{can}}) \cdot T_{\mathrm{Vdm}} \cdot D_y &= (S_y \cdot V_{\mathrm{can}}) \cdot T_{\mathrm{Vdm}}, \\ B \cdot T_{\mathrm{Vdm}} \cdot D_y &= A_y \cdot T_{\mathrm{Vdm}}, \end{aligned}$$

in which we observe that $B = S_1 \cdot V_{can}$ is the same as for the shift with x. Moreover, the eigenvectors are also the same as in the case of using the shift x. As a consequence, $B^{-1}A_x$ and $B^{-1}A_y$ commute. The commutation property allows for using

any polynomial function as a shift function instead of x or y. Consider for example the polynomial $g(x,y) = 3xy + 2y^2$. We now have

$$\begin{aligned} g(B^{-1}A_x, B^{-1}A_y) &= 3 \cdot T \cdot D_x \cdot T^{-1} \cdot T \cdot D_y \cdot T^{-1} + 2T \cdot D_y \cdot T^{-1} \cdot T \cdot D_y \cdot T^{-1} \\ &= T\left(3 \cdot D_x \cdot D_y + 2 \cdot D_y \cdot D_y\right) T^{-1} \\ &= T \cdot D_{g(x,y)} \cdot T^{-1} \end{aligned}$$

Now we will show that the derivation of the generalized eigenvalue problem holds for any arbitrary basis for the null space V. Let $V = V_{\text{can}}T^{-1}$ with T a nonsingular matrix denote a basis for the null space of M. We now have $V_{\text{can}} = VT$ and hence

$$\begin{aligned} S_1 \cdot V_{\text{can}} \cdot T_{\text{Vdm}} \cdot D_x &= S_x \cdot V_{\text{can}} \cdot T_{\text{Vdm}}, \\ S_1 \cdot V_{\text{can}} \cdot T_{\text{Vdm}} \cdot D_y &= S_y \cdot V_{\text{can}} \cdot T_{\text{Vdm}}, \end{aligned}$$

so we have

$$\begin{aligned} (S_1 \cdot V) \cdot (T \cdot T_{\text{Vdm}}) \cdot D_x &= (S_x \cdot V) \cdot (T \cdot T_{\text{Vdm}}), \\ (S_1 \cdot V) \cdot (T \cdot T_{\text{Vdm}}) \cdot D_y &= (S_y \cdot V) \cdot (T \cdot T_{\text{Vdm}}). \end{aligned}$$

This immediately leads to two important observations. Firstly, the eigenvalues D_x and D_y are not affected by the use of another basis for the null space. Secondly, only the eigenvectors change, and they become $T \cdot T_{\text{Vdm}}$.

Summary Example 1. We have developed the essentials of a root-finding method based on eigendecompositions. Summarizing, to find the roots we first compute a basis for the null space of a Macaulay matrix $M(d)$ as $V(d)$. The selection of rows of V corresponding to the linearly independent monomials results in the matrix $B = S_1V$. A shift function is chosen (*i.e.,* x or y, or any polynomial function $g(x,y)$), which defines the row-selection matrix S_g and $A = S_gV$. The generalized eigenvalue problem $B \cdot T \cdot D_g = A_g \cdot T$ returns as its eigenvalues the shift function g evaluated at the roots of the system.

Let us enumerate the following properties.

1. The row and column dimensions of the Macaulay coefficient matrix grow as a polynomial function as the iteration for creating additional equations proceeds. The rank of the Macaulay matrix increases with growing d, but the (right) corank stabilizes. In this example, the corank is four right away, but typically, the corank grows until it eventually stabilizes or keeps growing in a certain pattern.
2. The set of linear independent monomials stabilizes. There is a corresponding canonical basis for the null space, and we can also consider a Vandermonde structured basis or any basis for the null space of $M(d)$. In both bases, the row indices of the linear independent variables are the same. Applying the method on another basis for the null space results in different eigenvectors, but it does not alter the eigenvalues.
3. The matrices $B^{-1} \cdot A_x$ and $B^{-1} \cdot A_y$ commute and have common eigenspaces. As a consequence, any polynomial function of x and y can be used as a shift function.

2.2. Example 2: Internal shifting and iterating over the degree d. The current example serves to illustrate two new points. When not all equations are of the same degree, the initial Macaulay matrix should include also internal shifts of the equations of degrees lower than the maximal degree occuring in the system. Secondly

it is shown that that the corank sometimes stabilizes only after a few degree-iterations. Consider the equations

$$\begin{array}{rcccl} f_1(x,y,z) & = & x^2+5xy+4yz-10 & = & 0,\\ f_2(x,y,z) & = & y^3+3x^2y-12 & = & 0,\\ f_3(x,y,z) & = & z^3+4xyz-8 & = & 0, \end{array}$$

where $d_1 = 2$ and $d_2 = d_3 = 3$. We denote $d_\circ = \max(d_i) = 3$, and hence we start the Macaulay matrix construction at degree $d = 3$. As in Example 1 we consider the Macaulay matrix $M(3)$ with columns indexed by all monomials up to degree three. As equation f_1 is of degree two, we can also adjoin the shifted versions xf_1, yf_1 and zf_1 to the matrix $M(3)$ so that we generate a maximum number of polynomials of degree three. Including internal shifts we find $M(3)$ as

$$\left(\begin{array}{c|ccc|cccccc|cccccccccc} -10 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 4 & 0 & 0\\ 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 4 & 0\\ -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

of which the rows correspond to the equations f_1, xf_1, yf_1, zf_1, f_2 and f_3 and the columns to the monomials 1, x, y, z, x^2, xy, xz, y^2, yz, z^2, x^3, x^2y x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3, again ordered by the degree negative lexicographic order.

Matrix sizes, ranks, coranks and the indices of the linearly independent monomials of $M(d)$ for the consecutive degrees d are summarized in Table 2.2.

TABLE 2.2

Stabilization diagram for Example 2, showing the dynamics of the properties of the Macaulay matrix $M(d)$ as a function of degree d. The rank keeps increasing as d grows, however the corank stabilizes eighteen. Again, also the linear independent monomials stabilize.

d	size $M(d)$	rank $M(d)$	corank $M(d)$	lin. indep. mons (index)
3	6×20	6	14	$1,2,3,4,5,6,7,8,10,11,12,13,14,16$
4	18×35	18	17	$1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23$
5	40×56	38	18	$1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23,36$
6	75×84	66	18	$1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23,36$
7	126×120	102	18	$1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23,36$
8	196×165	147	18	$1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23,36$

For degree d we have that the number of rows $p(d)$ and the number of columns $q(d)$ of $M(d)$ is given by

$$\begin{array}{rcccccl} p(d) & = & \binom{d+1}{d-2} + 2\binom{d}{d-3} & = & \frac{(d+1)!}{2!\cdot(d-1)!} + 2\frac{d!}{3!\cdot(d-3)!} & = & \frac{1}{3}d^3 - d^2 + \frac{1}{2}d,\\ q(d) & = & \binom{d+3}{d} & = & \frac{(d+3)!}{3!\cdot d!} & = & \frac{1}{6}d^3 + d^2 + \frac{11}{6}d + 1. \end{array}$$

Note that there are in this example eighteen monomials that stabilize, which is also the product of the degrees of the input equations. This is indeed no coincidence, and it will turn out that the dimension of the null space of the Macaulay matrix corresponds to the Bézout number when the roots are isolated (*i.e.,* it describes a zero-dimensional variety [10]). For a system having n equations in n unknowns describing a zero-dimensional variety, the Bézout number is defined as the product of the degrees of the equations $f_i = 0$, for $i = 1, \ldots, n$ [10]. For a sufficiently large

degree d we have $\text{corank}(M(d)) = \prod_{i=1}^{n} d_i$, where d_i denotes the degree of polynomial f_i.

As in the previous example, we can now compute a basis for the null space of $M(6)$ as V. We set up the generalized eigenvalue problems using

$$\begin{array}{rcrcl} B & = & S_1 V & = & V([1,2,3,4,5,6,7,8,10,11,12,13,14,16,21,22,23,36],:), \\ A_x & = & S_x V & = & V([2,5,6,7,11,12,13,14,16,21,22,23,24,26,36,37,38,57],:), \\ A_y & = & S_y V & = & V([3,6,8,9,12,14,15,17,19,22,24,25,27,29,37,39,40,58],:), \\ A_z & = & S_z V & = & V([4,7,9,10,13,15,16,18,20,23,25,26,28,30,38,40,41,59],:), \end{array}$$

where MATLAB notation is used to index the selected rows of V. From the eigenvalues we correctly retrieve the eighteen solutions (x, y, z).

Summary Example 2. The current example illustrates the following two points.

1. When constructing the Macaulay matrix for the initial degree, it may be necessary to bring the initial equations to the same degree as the maximal degree occurring in the original equations. This is done by multiplying the equations of lower degree with monomials up to $d_\circ = \max(d_i)$.
2. The corank stabilizes only after a few iterations, together with all independent variables.

2.3. Example 3: Roots at infinity – Mind the gap. Let us now look at an example where the corank stabilizes, but only some of the indices of the independent variables stabilize, and others do not. It turns out that the indices that do not stabilize are caused by *roots at infinity*.

Consider the system of two equations

$$\begin{array}{rcrcl} f_1(x,y) & = & x^2 + xy - 2 & = & 0, \\ f_2(x,y) & = & y^2 + xy - 2 & = & 0. \end{array}$$

We construct for several iterations the Macaulay matrix and monitor its rank, corank and the indices of the linear independent monomials. The results are summarized in Table 2.3. We observe that there are four linear independent monomials in all

TABLE 2.3

Stabilization diagram for Example 3, showing the dynamics of the properties of the Macaulay matrix $M(d)$ as a function of the degree d. The rank keeps increasing as d grows, however the corank stabilizes at four. Observe that only two of the linear independent monomials stabilize, namely 1 and x, whereas the remaining two shift towards higher degrees as the overall degree of the Macaulay matrix increases.

d	size $M(d)$	rank $M(d)$	corank $M(d)$	lin. indep. mons
2	2×6	2	4	$1, x, y, x^2$
3	6×10	6	4	$1, x, x^2, x^3$
4	12×15	11	4	$1, x, x^3, x^4$
5	20×21	17	4	$1, x, x^4, x^5$
6	30×28	24	4	$1, x, x^5, x^6$

iterations, but only 1 and x stabilize, while the other two monomials are replaced by higher degree monomials as d increases.

Obviously there is a pattern in the two remaining monomials: they are always x^d and x^{d-1}. It turns out that the system has two affine roots, corresponding to the monomials 1 and x, and two roots at infinity, corresponding to the monomials at higher degrees. The roots at infinity can be analyzed by homogenizing the two

equations as

$$\begin{aligned} f_1^h(t,x,y) &= x^2+xy-2t^2 &= 0,\\ f_2^h(t,x,y) &= y^2+xy-2t^2 &= 0. \end{aligned}$$

Setting $t=0$ reveals the roots at infinity. We identify $x+y$ as a common factor, which confirms that there exists a root at infinity $(x,y,t)=(\alpha,-\alpha,0)$. The fact that the roots at infinity can be scaled with some $\alpha\neq 0$ readily follows from the fact that they are defined by a system of homogeneous equations.

One can observe that the existence of roots at infinity is also expressed in the Macaulay matrix: If there can be found linear independent monomials of degree d in $M(d)$, for any sufficiently large degree d, then there are roots at infinity. Indeed, setting the homogenization variable t to zero in the homogenized system is equivalent to retaining only the highest degree columns of the Macaulay matrix. If there is linear dependence among these columns, there are roots at infinity.

The dynamical behaviour of the structure of the null space as a function of d, when there are roots at infinity, can easily be understood by inspecting the canonical basis for the null space. At degree $d=4$ we see separation appearing between the affine roots and the roots at infinity. At degree $d=5$ the separation between the affine roots and the roots at infinity is increased by one degree-block, as shown in

$V_{\text{can}}(4) =$

affine		infinity	
1	0	0	0
0	**1**	0	0
0	1	0	0
1	0	**0**	**0**
1	0	**0**	**0**
1	0	**0**	**0**
0	0	**1**	0
0	2	−1	0
0	0	1	0
0	2	−1	0
0	0	0	**1**
2	0	0	−1
0	0	0	1
2	0	0	−1
0	0	0	1

↑ gap ↓

and

affine		infinity	
1	0	0	0
0	**1**	0	0
0	1	0	0
1	0	**0**	**0**
1	0	**0**	**0**
1	0	**0**	**0**
0	1	**0**	**0**
0	1	**0**	**0**
0	1	**0**	**0**
0	1	**0**	**0**
0	0	**1**	0
2	0	−1	0
0	0	1	0
2	0	−1	0
0	0	1	0
0	0	0	**1**
0	2	0	−1
0	0	0	1
0	2	0	−1
0	0	0	1
0	2	0	−1

↑ gap ↓

$= V_{\text{can}}(5).$

We term this effect the *mind-the-gap phenomenon*: In the canonical basis, as a function of the degree d, we see the appearance of zeros in the top part corresponding to the degrees 0, 1, 2, *etc.* of the columns 3 and 4. As d increases, a gap between the linear independent monomials corresponding to the affine roots and the linear independent monomials corresponding to the roots at infinity emerges. The *mind-the-gap phenomenon* will be used to separate the affine roots and the roots at infinity. In Figure 2.1 we visualize this observation. Although we illustrate it here for the canonical basis of the null space only,it is important to recall that the same rank properties hold for other bases of the null space of $M(d)$.

Realization theory. In the following paragraphs we will illustrate that the null space of the Macaulay matrix is an observability-like matrix in the dynamical systems sense. For a 1D LTI system described by a rational function $G(q)=b(q)/a(q)$ is known that the null space of the Sylvester matrix of $a(q)$ is an observability-like matrix. In

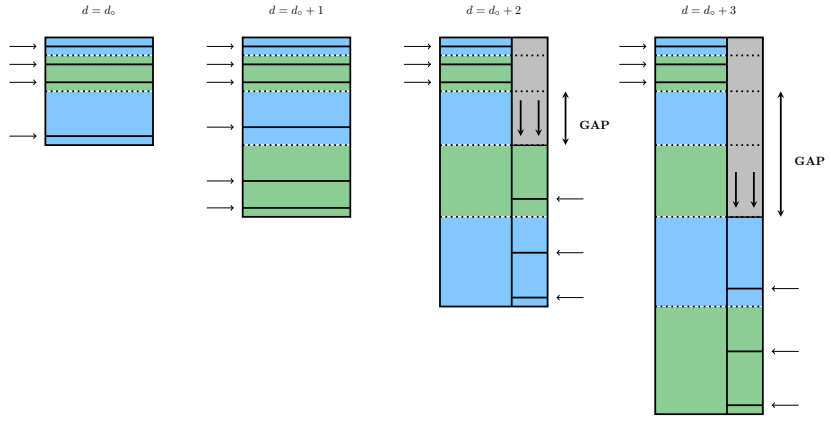

FIG. 2.1. *Visual representation of the mind-the-gap phenomenon as observed in the canonical basis for the null space* $V_{\text{can}}(d)$. *As the degree* d *increases, the linear independent monomials corresponding to the affine roots stabilize (indicated by the horizontal lines and the arrows on the left-hand-side of the matrix), whereas the linear independent monomials which are caused by the roots at infinity move along to high degrees (indicated by the horizontal lines in the matrix and the arrows on the right-hand-side of the matrix). Since we are considering the canonical basis for the null space, in the right-most columns the entries on the top are all zero (indicated by the grey block which grows in vertical dimension as* d *increases), and only entries in the bottom blocks are nonzero. Hence, at a certain degree* d *a separation emerges which allows us to separate the affine roots and the roots at infinity.*

the multivariate case this viewpoint is not entirely novel, and was suggested in [22] where a link between polynomial systems and nD Attasi models [1] was established using Gröbner bases.

In the following paragraphs, we will generalize the result of [22] to the case where there are roots at infinity. The result follows naturally when an nD descriptor system is employed. An additional merit of this viewpoint is that we will not need the Gröbner basis setting that is the starting point for [22].

We will begin with introducing the concept of a descriptor system in the familiar one-dimensional case. Later on it will be illustrated how the nD case corresponds to the roots at infinity case. In [5] a condition was derived under which the descriptor systems interpretation is possible.

Consider the state update equation of a one-dimensional autonomous system as

$$Ex(k+1) = Ax(k),$$

where E and A are square system matrices and $x(k)$ denotes the dimensional state vector at time k. Such a system is known as a descriptor system when the matrix E is rank-deficient and can always be written in the Kronecker canonical form [19, 39], where A and E occur as block-diagonal matrices

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & \overline{E}_S \end{array}\right)\left(\begin{array}{c} x_R(k+1) \\ \hline \overline{x}_S(k+1) \end{array}\right) = \left(\begin{array}{c|c} A_R & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c} x_R(k) \\ \hline \overline{x}_S(k) \end{array}\right),$$

where the I denote identity matrices and $\overline{E}_S$ is nilpotent. The indices R and S refer to the *regular* and *singular* parts. By introducing the substitution $x_S(k+1) = \overline{x}_S(k)$ the state update equations are found as

$$\begin{aligned} x_R(k+1) &= A_R x_R(k), \\ x_S(k-1) &= E_S x_S(k), \end{aligned}$$

where the regular part runs forward in time, while the singular part runs backward in time [39]. Generalizing the above to the two-dimensional case gives

$$\begin{array}{rclcrcl} u(k+1,\ell) & = & H_x u(k,\ell), & & v(k-1,\ell) & = & E_x v(k,\ell), \\ u(k,\ell+1) & = & H_y u(k,\ell), & & v(k,\ell-1) & = & E_y v(k,\ell), \end{array}$$

in which $u(k,\ell)$ denotes the regular part of the state vector and $v(k,\ell)$ is the singular part of the state vector. Inspection of the canonical null space easily leads to the following results:

$$\begin{array}{rclcrcl} A_x & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & & A_y & = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ E_x & = & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & & E_y & = & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{array}$$

Notice that E_x and E_y are nilpotent, which causes the block of zeros in the first rows-blocks of the canonical null space. We construct the eigenvalue problem as in the previous examples and correctly retrieve the roots as $(1,1)$ and $(-1,-1)$.

Summary Example 3. The current example provides us with the final ingredients to understand and solve the root-finding problem.

1. Algebraic relations between the coefficients cause roots at infinity. Variables that are independent may become dependent; variables that are dependent, stay dependent.
2. The nilpotency of the system matrices corresponding to the singular part confines the effect of the roots at infinity to the highest-degree blocks of the null space. The resulting mind-the-gap phenomenon allows us to separate affine roots and roots at infinity: linear independent monomials corresponding to roots at infinity shift towards higher degrees as the overal degree of the Macaulay matrix increases.
3. The shift structure property is the signature of realization theory, where it is used to compute the action matrix of a state space system.

3. The Macaulay matrix: properties and dynamics. Let us now formalize what we have learned from the examples and discuss in general terms the dynamics of the properties of the Macaulay matrix and derive the root-finding algorithm.

3.1. Prerequisites. It will often be convenient to order monomials, for which we have chosen to use the degree negative lexicographic ordering. This choice is to a certain extent arbitrary, as most of the results that will be developed can be easily generalized to any graded ordering.

Definition 3.1 (Degree Negative Lexicographic Order). *Let $\alpha, \beta \in \mathbb{N}^n$ be monomial exponent vectors. Then $\alpha < \beta$ by the degree negative lexicographic order, denoted $\alpha < \beta$), if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and in the vector difference $\beta - \alpha \in \mathbb{Z}^n$, the left-most non-zero entry is negative.* The monomials of maximal degree three in two variables are ordered by the degree negative lexicographic order as

$$1 < x_1 < x_2 < x_1^2 < x_1x_2 < x_2^2 < x_1^3 < x_1^2x_2 < x_1x_2^2 < x_2^3.$$

3.2. Construction, dimensions and matrix structure. The system of polynomial equations is represented as a Macaulay matrix multiplied with a multivariate Vandermonde monomial vector. In this way, the problem of finding the solutions of

a system of polynomial equations is translated into linear algebra: solving a system of homogeneous linear equations.

We consider the case of n polynomial equations $f_i = 0$ in n unknowns, having degrees $\deg(f_i) =: d_i$, for $i = 1, \ldots, n$. The maximal degree occurring in the equations is denoted $d_\circ := \max(d_i)$.

By multiplying all the polynomials f_i by monomials, such that the maximum degree of the polynomials multiplied by monomials is less or equal to d, we obtain rows of the Macaulay matrix, giving rise to a matrix equation of the form $M(d)\, v(d) = 0$.

DEFINITION 3.2 (Macaulay matrix and monomial vector). *The Macaulay matrix $M(d)$, defined for $d \geq d_\circ$, contains as its rows representations of the polynomials f_i, for $i = 1, \ldots, n$, and shifts of the polynomials represented as $x^{\sigma_i} f_i$, where x^{σ_i} is a monomial such that the total degree of $x^{\sigma_i} f_i$ is at most d. The Macaulay matrix is constructed by considering increasing degrees $d \geq d_\circ$ where in each iteration rows are added, resulting in a nested matrix. The multivariate Vandermonde monomial vector $v(d)$ is composed accordingly,* i.e.,

$$v(d) := \begin{pmatrix} 1 & x_1 & \ldots & x_n & x_1^2 & \ldots & x_n^2 & \ldots & x_1^d & \ldots & x_n^d \end{pmatrix}^T.$$

The properties of the Macaulay matrix exhibit an interesting behavior as the degree increases. Let us start off by investigating the dimensions and structure of $M(d)$ as d increases. The subsequent iterations of the Macaulay matrix are found by multiplying the equations $f_i = 0$ with monomials and including them as new rows. At the same time the number of columns increases, as monomials of a higher degree are taken into account. The following formulas express the number of monomials (either of total degree d or of total degree $\leq d$) by binomial coefficient expressions.

LEMMA 3.3 (Number of monomials). *The number of monomials of total degree d in n variables $x_1, \ldots, x_n$ is given as $\binom{n+d-1}{d} = \frac{(n+d-1)!}{(n-1)!\cdot d!}$. The number of monomials of total degree $\leq d$ in n variables $x_1, \ldots, x_n$ is given as $\binom{n+d}{d} = \frac{(n+d)!}{n!\cdot d!}$.*

LEMMA 3.4 (Dimensions Macaulay matrix). *Let $p(d)$ and $q(d)$ denote the number of rows and columns of $M(d)$, respectively. We have*

$$p(d) = \sum_{i=1}^{n} \binom{n+d-d_i}{d-d_i} = \sum_{i=1}^{n} \frac{(n+d-d_i)!}{n!\cdot(d-d_i)!} \quad \text{and} \quad q(d) = \binom{n+d}{d} = \frac{(n+d)!}{n!\cdot d!}.$$

The Macaulay matrix exhibits an interesting sparse matrix structure. In $M(d)$ the matrix $M(d-1)$ occurs as a submatrix in the top-left part; in $M(d-1)$ the matrix $M(d-2)$ occurs as a submatrix, *etc.* Due to this structure and its construction, one can identify for every degree nonzero blocks in which the coefficients of the polynomials f_i occur. These blocks are repeated in a quasi-Toeplitz structure: the blocks are repeated along the diagonals of $M(d)$. We call this a quasi-Toeplitz structure because the elements in the repeated blocks do not satisfy a strict Toeplitz structure, because for increasing degrees the blocks are growing in column dimensions.

3.2.1. Linear dependence, corank and number of solutions. It is instrumental for the root-finding method to distinguish between linearly independent and linearly dependent monomials. Linearly independent monomials correspond to linearly dependent columns of $M(d)$ when scanning the columns from right to left. The reason that the rank increases of the columns of M are checked from the right to the left is due to the complementarity between the column indices of M and the row indices of the null space V (see Appendix A).

As we monitor the corank as the degree d which the Macaulay matrix $M(d)$ is constructed, the corank may stabilize or not. Provided that we are dealing with a system of n equations in n unknowns, the number of solutions counting with multiplicity and including roots at infinity is in the zero-dimensional case given by the Bézout number $m_B = \prod_{i=1}^{n} d_i$ [9]. At a sufficiently large degree d, the nullity of the Macaulay matrix is equal to the Bézout number [4], *i.e.,*

$$\operatorname{corank}\left(M(d)\right) = \prod_{i=1}^{n} d_i.$$

Moreover, it turns out that the number of linearly independent monomials, *i.e.,* the (right) corank, corresponds to the number of solutions. Although the study of positive-dimensional systems is beyond the scope of this article, it is interesting to mention that, when the corank keeps increasing, it can be described as a polynomial function of d for a sufficiently large d. The degree of this polynomial corrresponds to the dimensionality of the solution space [10, 24].

3.2.2. Roots at infinity. We have encountered roots at infinity in Example 3, where we saw that the roots at infinity cause the the linearly independent monomials to shift towards higher degrees as the overall degree of the Macaulay matrix grows. Roots at infinity are caused by algebraic relations between the coefficients or the occurrence of zero coefficients in the system of polynomials. The roots at infinity can be analyzed using homogenization and projective space.

Definition 3.5 (Homogenization and dehomogenization). *The homogenization of an equation f, denoted f^h, is computed using*

$$f^h(x_0, x_1, \ldots, x_n) := x_0^d \cdot f\left(x_1/x_0, \ldots, x_n/x_0\right), \quad \textit{with} \quad d := \deg(f).$$

Dehomogenizing f^h yields f, or formally $f^h(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. A homogenized system of equations describes solutions in the $n+1$-dimensional projective space, in which $x_0, \ldots, x_n$ are called the homogeneous coordinates. Observe that in the projective space the roots at infinity are incorporated as regular points for which $x_0 = 0$.

The Macaulay matrix for the homogenized system $f_i^h(x_0, x_1, \ldots, x_n) = 0$, for $i = 1, \ldots, n$ is denoted by $M^h(d)$ and is defined such that the columns of $M^h(d)$ correspond to monomials of degree *equal to* d.

$$M^h(d) \equiv M(d).$$

Indeed, the difference between $M(d)$ and $M^h(d)$ lies in a mere relabeling of rows and columns of $M(d)$. The ordering of the monomials in this relabeling is consistent with the degree negative lexicographic monomial ordering.

The existence of roots at infinity is expressed by the existence of linearly independent monomials that do not stabilize as the degree iteration increases. This fact can be easily understood from the following proposition.

Proposition 3.6. *Consider the partitioning of $M(d)$ for a sufficiently large degree d as*

$$M(d) = \begin{pmatrix} M_0 & M_1 & M_2 & \ldots & M_d \end{pmatrix},$$

where the block M_i contains the columns of $M(d)$ indexed by the monomials of degree i, for $i = 0, \ldots, d$. The existence of roots at infinity is revealed by the column rank

deficiency of the block M_d The proof of this proposition relies on the fact that the column rank deficiency of M_d implies there exist solutions for which the homogenization variable x_0 is zero.

4. Modelling the null space. The null space of the Macaulay matrix plays an important role in the root-finding procedure. We will now describe the shift property that immediately leads to the formulation of the eigenvalue problem root-finding method. Next we will discuss three different bases for the null space. Finally, the link between polynomial system solving and nD realization theory will be established.

4.1. Shift structure in multivariate Vandermonde vectors. Let us for a moment assume that we would know the (affine) roots $(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)})$, for all $i = 1, \ldots, m_a$ of the system. For each affine solution we can now construct a vector which lies in the null space of $M(d)$ by evaluating the root at the monomials in the multivariate Vandermonde vector $v(d)$. Multiplication of a such a multivariate Vandermonde monomial vector $v(d)$ by a monomial or a polynomial exhibits a multiplicative shift property.

PROPOSITION 4.1 (Multiplication property in monomial vectors). *Multiplication of entries in a multivariate Vandermonde monomial vector $v := v(d)$ with a monomial x^γ maps the entries of v of degrees 0 up to $d-|\gamma|$ to entries in v of degrees $|\gamma|$ up to d. This is expressed by means of row selection matrices operating on v as $S_1 v x^\gamma = S_\gamma v$, where S_1 selects all monomials in v of degrees 0 up to $d-|\gamma|$ and S_γ selects the rows of v onto which the monomials $S_1 v$ are mapped by multiplication by x^γ.*

This property can be generalized directly to an arbitrary polynomial shift function $g(x)$ with $\deg(g) \leq d$.

PROPOSITION 4.2. *Multiplication of a multivariate Vandermonde monomial vector $v(d)$ with a polynomial $g(x) := \sum_\gamma c_\gamma x^\gamma$ gives $S_1 v g(x) = S_g v$, where $S_g := \sum_\gamma c_\gamma S_\gamma$. In this case, the row combination matrix S_g takes linear combinations of rows of v.*

The shift property in the multivariate Vandermonde vectors can be applied to the generalized Vandermonde structured null space V_{Vdm} which is composed of the Vandermonde monomial vectors evaluated at all affine roots $x^{(i)}$, for all $i = 1, \ldots, m_a$. This is done by means of the introduction of $D_g := \mathrm{diag}\left(g(x^{(1)}), g(x^{(2)}), \ldots g(x^{(m_a)})\right)$, leading to the important relation

$$S_1 V_{\mathrm{Vdm}} D_g = S_g V_{\mathrm{Vdm}}, \tag{4.1}$$

from which the link to eigendecompositions is clearly starting to take shape.[3]

4.2. Three bases for the null space. In practice, the Vandermonde basis V_{Vdm} is not known, however, a basis for the null space of the Macaulay matrix can be computed, *e.g.,* using SVD. For didactical purposes we consider different bases for the null space of the Macaulay matrix, each of which have their merit either in understanding the dynamics of the properties or for finding the solutions themselves.

4.2.1. Canonical basis and linearly independent monomials. In the case there are roots at infinity, it is useful to consider a canonical basis for the null space of M. The canonical basis for the null space of the Macaulay matrix contains a '1' in each linearly independent row, such that an identity matrix sits in the linearly independent rows.

[3]Multiple roots are beyond the scope of this article, but will result in the Jordan canonical form.

For the roots at infinity, some of the linearly independent rows of V_{can} will be located in the highest-degree blocks as predicted by Proposition 3.6. The linearly independent rows of V_{can} corresponding to the affine roots stabilize at low degrees. The monomials corresponding to the linearly independent rows of V_{can} can be interpreted as the linearly independent monomials during an elimination procedure as in Example 1. This allows one to separate the affine roots from the roots at infinity: a 'gap' between the two emerges at a sufficiently high degree as in Figure 2.1.

4.2.2. Numerically computed bases. Although the properties of the Vandermonde structured basis V_{Vdm} and the canonical null space V_{can} are instrumental for understanding the root-finding algorithm, they cannot be computed directly from $M(d)$. However, the properties are not dependent on the specific basis, but they are properties of the null space itself. In practical situations, one can only find a numerical basis for the null space of M, *e.g.,* by an SVD, denoted V_{SVD}. From V_{SVD} the canonical null space V_{can} can be computed as

$$V_{\text{can}} = V_{\text{SVD}} \left(V^{\star}_{\text{SVD}}\right)^{-1},$$

where $V^{\star}_{\text{SVD}}$ is composed of the linearly independent rows of V_{SVD}.

5. Finding the roots.

5.1. From shift structure to eigenvalue problems. In order to compute the roots, we need a basis for the null space of $M(d)$. A natural choice is V_{can} since it reveals the separation between the affine roots and the roots at infinity, however, its computation may be ill-posed. Therefore, one performs a column compression on V_{SVD} in order to obtain the block of zero elements in the top-right corner, after which the separation between the affine roots and the roots at infinity can be obtained.

PROPOSITION 5.1 (Column compression). *Let r_∞ denote the row number of V_{SVD} of the first index of the roots at infinity and let m_a denote the number of affine roots. Let $Z^\star$ denote the matrix composed by the first $r_\infty - 1$ rows of V_{SVD}. Compute the SVD of $Z^\star$ as $Z^\star = U \cdot \Sigma \cdot V^T$. The column compressed basis for the null space, denoted V_{CC}, is defined as the matrix composed of the first m_a columns of $Z \cdot V$, and has dimensions $q \times m_a$.*

Using the column compression of Proposition 5.1 we now have $V_{\text{CC}} \cdot T = V_{\text{Vdm}}$, with T a square invertible matrix of size $m_a \times m_a$. As a consequence, we have that $S_1 \cdot V_{\text{CC}} \cdot T \cdot D_g \cdot T^{-1} = S_g \cdot V_{\text{CC}} \cdot T \cdot T^{-1}$, or $T \cdot D_g \cdot T^{-1} = (S_1 \cdot V_{\text{CC}})^{\dagger} S_g \cdot V_{\text{CC}}$.

There are essentially two ways to phrase the eigenvalue problem. The first way is to obtain the regular square eigenvalue problem by letting S_1 select the first m_a linearly independent rows of V_{CC}. Secondly, it is possible to let S_1 select all degree-blocks of V_{CC} which have degrees 0 up to $d_\infty - \deg(g)$, where d_∞ denotes the degree at which the first row-index of a monomial corresponding to the roots at infinity occurs.

In both cases the generalized eigenvalue problem

$$(S_1 \cdot V_{\text{CC}}) \cdot \left(T \cdot D_g \cdot T^{-1}\right) = S_g \cdot V_{\text{CC}},$$

is found, which is either square or rectangular. The individual components x_i can be reconstructed from (a column-wise rescaled) $V_{\text{Vdm}} = V_{\text{CC}} \cdot T$. To ensure that the resulting eigenvalue problem has no multiple eigenvalues, the shift polynomial $g(x)$ should be defined in such a way that the evaluation of $g(x)$ is different for each of the m_a roots.

5.2. Eliminating roots at infinity. When a system has roots at infinity, this can be detected by means of Proposition 3.6. However, as explained in the previous paragraphs, it would generally not suffice to simply dismiss the linearly independent monomials of degree d alone.

Assume that the Macaulay matrix $M := M(d)$ is constructed for a sufficiently large degree d, *i.e.,* for which a gap is detected between the affine roots and the roots at infinity, and that a numerically computed basis for the null space of $M(d)$ is available as $V_{\text{SVD}} := V_{\text{SVD}}(d)$. We now have two methods for discarding the solutions at infinity using the 'mind-the-gap' property.

1. By using the column compressed version of V_{SVD}, the null space vectors corresponding to the roots at infinity are discarded – the affine roots are preserved.
2. Alternatively one can remove the roots at infinity by removing from M the columns corresponding to the monomials associated to the roots at infinity, leading to a reduced Macaulay matrix $M^\star(d)$. The root-finding algorithm can then be applied to the matrix $M^\star$, which has only null space vectors corresponding to affine roots.

6. Conclusions and open questions. This article presents a conceptually simple and accessible framework to use linear algebra methods to tackle problems in polynomial algebra. The problem of solving a system of polynomial equations is translated into a system of homogeneous linear equations by using the Macaulay matrix. The multivariate Vandermonde structure in the monomials leads to a multiplication invariance property in the null space of the Macaulay matrix, which leads to the formulation of an eigenvalue problem that returns the roots. The method involves only basic notions of (numerical) linear algebra and relates the problem to multidimensional nD system theory.

A dual version of the presented algorithm has been derived in [12] that operates on the column space of the Macaulay matrix, rather than its null space and will be elaborated upon in a follow-up paper. Furthermore a detailed discussion of the link between system solving and multidimensional realization theory is being prepared. Apart from the nD systems theory interpretation, the mind-the-gap phenomenon suggests a multidimensional system theoretic interpretation of the Cayley-Hamilton theorem: the null space of the Macaulay matrix will extend in a predictable/deterministic way as its degree grows. Finally, the link with multidimensional systems raises the question how an nD subspace identification algorithm would operate. The interpretation of the multidimensional observability matrix would seem to play an important algorithmic role.

Other important challenges for future work include the development of tailored algorithms that exploit the sparsity and matrix structure, for instance by iteratively computing a basis for the null space of the Macaulay matrix as the degree grows, in much the same way as the rank of an extended controllability/observability matrix converges to the system order for an LTI system.

Moreover, very often one is only interested in one single solution. For instance when solving polynomial optimization problems one is interested only in the solution of the KKT equations that returns the global optimizer. This fits into the framework by taking as the shift function the objective function itself: the task reduces to solving an eigenvalue problem of which we are only interested in the minimal (real) eigenvalue. Our ultimate desire is thus to develop a method that is inspired on power iterations to solve the eigenvalue problem. Ideally such an algorithm would operate by only using

the coefficients of the polynomials, where the step of explicitly building the Macaulay matrix and computing its null space are avoided; this would likely involve FFT-like computations.

Obtaining quantitative insights into conditioning and sensitivity of root-finding is another research direction that is natural to pursue in the numerical linear algebra framework. Certain problems that are beyond the grasp of classical symbolic methods are feasible, perhaps even straightforward in the current framework: For instance, approximate solutions of overdetermined systems can be computed directly with the proposed method, which is cumbersome using symbolic methods.

Acknowledgments. Research supported by Research Council KUL: GOA/10/09 MaNet, CoE PFV/10/002 (OPTEC); PhD/Postdoc grants; Flemish Government: IOF: IOF/KP/SCORES4CHEM; iMinds Medical Information Technologies SBO 2014; Belgian Federal Science Policy Office: IUAP P7/19 (DYSCO, Dynamical systems, control and optimization, 2012-2017). The scientific responsibility is assumed by its authors.

REFERENCES

[1] S. Attasi, *Modelling and recursive estimation for double indexed sequences*, System Identification: Advances and Case Studies, Academic Press, New York, 1976, pp. 289–348.
[2] W. Auzinger and H. J. Stetter, *An elimination algorithm for the computation of all zeros of a system of multivariate polynomial equations*, in Proc. Int. Conf. Num. Math., Birkhäuser, 1988, pp. 11–30.
[3] K. Batselier, *A Numerical Linear Algebra Framework for Solving Problems with Multivariate Polynomials*, PhD thesis, Faculty of Engineering Science, KU Leuven, September 2013.
[4] K. Batselier, P. Dreesen, and B. De Moor, *The canonical decomposition of C_d^n and numerical Gröbner border bases*, SIAM J. Mat. Anal. Appl., 35 (2014), pp. 1242–1264.
[5] K. Batselier and N. Wong, *Computing the state recursion polynomials for discrete linear m-d systems*, tech. report, The University of Hong Kong, 2014.
[6] T. Becker and V. Weispfenning, *Gröbner Bases: A Computational Approach to Commutative Algebra*, Springer Verlag, New York, 1993.
[7] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, PhD thesis, University of Innsbruck, 1965.
[8] ———, *Gröbner bases and systems theory*, Multidimens. Syst. Signal Process., 12 (2001), pp. 223–251.
[9] D. A. Cox, J. B. Little, and D. O'Shea, *Using Algebraic Geometry*, Springer-Verlag, New York, second ed., 2005.
[10] ———, *Ideals, Varieties and Algorithms*, Springer-Verlag, third ed., 2007.
[11] A. Dickenstein and I. Z. Emiris, eds., *Solving Polynomial Equations*, vol. 14 of Algorithms and Computation in Mathematics, Springer, 2005.
[12] P. Dreesen, *Back to the Roots – Polynomial System Solving Using Linear Algebra*, PhD thesis, Faculty of Engineering Science, KU Leuven, September 2013.
[13] P. Dreesen, K. Batselier, and B. De Moor, *Back to the roots: polynomial system solving, linear algebra, systems theory*, in Proc. 16th IFAC Symp. Syst. Ident. (SYSID 2012), 2012, pp. 1203–1208.
[14] I. Z. Emiris, *Sparse Elimination and Applications in Kinematics*, PhD thesis, UC Berkeley, Dec 1994.
[15] I. Z. Emiris and B. Mourrain, *Computer algebra methods for studying and computing molecular conformations*, Algorithm., 25 (1999), pp. 372–402.
[16] ———, *Matrices in elimination theory*, J. Symb. Comput., 28 (1999), pp. 3–44.
[17] J. C. Faugère, *A new efficient algorithm for computing Gröbner bases (F4)*, J. Pure Appl. Algebra, 139 (1999), pp. 61–88.
[18] K. Gałkowski, *State-space Realizations of Linear 2-D Systems with Extensions to the General nD ($n > 2$) case*, Lecture Notes in Control and Information Sciences, Springer, 2001.
[19] F. Gantmacher, *The Theory of Matrices, volume 2*, Chelsea Publishing Company, New York, 1960.

[20] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, USA, third ed., 1996.
[21] G.-M. Greuel, *Computer algebra and algebraic geometry – achievements and perspectives*, J. Symb. Comput., 30 (2000), pp. 253–289.
[22] B. Hanzon and M. Hazewinkel, *An introduction to constructive algebra and systems theory*, in Constructive Algebra and Systems Theory, B. Hanzon and M. Hazewinkel, eds., Royal Netherlands Academy of Arts and Sciences, 2006, pp. 2–7.
[23] B. Hanzon and D. Jibetean, *Global minimization of a multivariate polynomial using matrix methods*, J. Glob. Optim., 27 (2003), pp. 1–23.
[24] D. Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann., 36 (1890), pp. 473–534.
[25] B. L. Ho and R. E. Kalman, *Effective construction of linear state-variable models from input/output functions*, Regelungstechnik, 14 (1966), pp. 545–548.
[26] G. F. Jónsson and S. A. Vavasis, *Accurate solution of polynomial equations using Macaulay resultant matrices*, Math. Comput., 74 (2004), pp. 221–262.
[27] T. Kailath, *Linear Systems*, Prentice-Hall information and system sciences series, Prentice-Hall International, 1998.
[28] S. Y. Kung, *A new identification and model reduction algorithm via Singular Value Decomposition*, Proc. 12th Asilomar Conf. Circuits, Syst. Comput., Pacific Grove, CA, 1978, pp. 705–714.
[29] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.
[30] J. B. Lasserre, M. Laurent, B. Mourrain, P. Rostalski, and P. Trébuchet, *Moment matrices, border basis and real radical computation*, J. Symb. Comput., (2012).
[31] M. Laurent and P. Rostalski, *The approach of moments for polynomial equations*, in Handbook on Semidefinite, Conic and Polynomial Optimization, vol. 166 of International Series in Operations Research & Management Science, Springer-Verlag, 2012.
[32] D. Lazard, *Résolution des systèmes d'équations algébriques*, Theor. Comput. Sci., 15 (1981), pp. 77–110.
[33] ———, *Groebner bases, Gaussian elimination and resolution of systems of algebraic equations*, in Computer Algebra, J. van Hulzen, ed., vol. 162 of Lecture Notes in Computer Science, Springer Berlin / Heidelberg, 1983, pp. 146–156.
[34] T. Y. Li, *Numerical solution of multivariate polynomial systems by homotopy continuation methods*, Acta Numer., 6 (1997), pp. 399–436.
[35] F. S. Macaulay, *On some formulae in elimination*, Proc. London Math. Soc., 35 (1902), pp. 3–27.
[36] ———, *The algebraic theory of modular systems*, Cambridge University Press, 1916.
[37] D. Manocha, *Solving systems of polynomial equations*, IEEE Comput. Graph. Appl., 14 (1994), pp. 46–55.
[38] H. M. Möller and H. J. Stetter, *Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems*, Numer. Math., 70 (1995), pp. 311–329.
[39] M. Moonen, B. De Moor, J. Ramos, and S. Tan, *A subspace identification algorithm for descriptor systems*, Syst. Control Lett., 19 (1992), pp. 47–52.
[40] B. Mourrain and V. Y. Pan, *Multivariate polynomials, duality, and structured matrices*, J. Complex., 16 (2000), pp. 110–180.
[41] L. Pachter and B. Sturmfels, *Algebraic Statistics for Computational Biology*, Cambridge, 2005.
[42] P. A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, PhD thesis, California Institute of Technology, May 2000.
[43] S. Petitjean, *Algebraic geometry and computer vision: Polynomial systems, real and complex roots*, J. Math. Imaging Vis., 10 (1999), pp. 191–220.
[44] A. J. Sommese and C. W. Wampler, *The Numerical solution of systems of polynomials arising in engineering and science*, vol. 99, World Scientific Singapore, 2005.
[45] H. J. Stetter, *Numerical Polynomial Algebra*, SIAM, 2004.
[46] B. Sturmfels, *Solving systems of polynomial equations*, no. 97 in CBMS Regional Conference Series in Mathematics, Providence, 2002, American Mathematical Society.
[47] J. J. Sylvester, *On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's function and that of the greatest algebraical common measure*, Trans. Roy. Soc. Lond., (1853).
[48] J. Verschelde, *Algorithm 795: PHCpack: a general-purpose solver for polynomial systems by homotopy continuation*, ACM Trans. Math. Softw., 25 (1999), pp. 251–276.
[49] J. C. Willems, *From time series to linear system – Part I. Finite dimensional linear time invariant systems*, Automatica, 22 (1986), pp. 561–580.

[50] ———, *From time series to linear system – Part II. Exact modeling*, Automatica, 22 (1986), pp. 675–694.
[51] ———, *From time series to linear system – Part III. Approximate modeling*, Automatica, 23 (1987), pp. 87–115.

Appendix A. Homogeneous linear equations.

Let A be a $p \times q$ matrix for which we consider the problem of finding all vectors x of length q that satisfy $Ax = 0$. Let the columns of A be denoted by $a_i, i = 1, \ldots, q$ and the components of x be denoted by ξ_i. We can thus write $sum_{i=1}^{q} a_i \xi_i = 0$. Provided that x is not identically zero, this means that some columns of A can be written as a linear combination of other columns: there is a linear dependency between the column vectors of A. This can obviously only be the case if A is not of full column rank. Denote the rows of A as $b_j^T, j = 1, \ldots, p$. Then we can write $b_j^T x = 0$, $\forall j = 1, \ldots, p$. The solution vectors x are therefore orthogonal to all rows of the matrix A, and obviously orthogonal to the row space of A. As the dimension of the row space of A is its rank $r_A = \text{rank}(A)$, the orthogonal complement of the row space is a $(q - r_A)$-dimensional vector space. Hence, the set of solutions to $AX = 0$ is a subspace of dimension $q - r_A$.

In the particular case that $r_A = q$ (*i.e.,* A is of full column rank), the only solution to $AX = 0$, is the trivial solution $X = 0$. Equivalently, $AX = 0$ can only have non-trivial solutions, provided that $r_A < q$ (and this statement is independent from the number of equations p. In particular, it does not matter whether $p < q$ (underdetermined) or $p > q$ (overdetermined) or $p = q$: only the row rank of A matters. The number $q - r_A$ is called the *right corank* of the matrix A, as it is the dimension of the right null space of A. We have therefore shown that $Ax = 0$ has exactly $q - r_A$ linear independent solutions. These solutions generate a vector space of dimension $q - r_A$. The SVD of A is numerically the best way to determine the null space: there is no better way to estimate the (numerical) rank of a matrix than by counting the nonzero singular values, and the corresponding basis for the null space will be an orthonormal matrix. In general, the SVD of A will look like

$$A = U\Sigma V^T = \begin{matrix} \\ p \end{matrix} \begin{matrix} r_A & p - r_A \\ \end{matrix} \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{matrix} r_A & q - r_A \\ \end{matrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} q \\ \end{matrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}.$$

The matrices U and V are orthonormal, $U^T U = I_p = UU^T$ and $V^T V = I_q = VV^T$, and their sub-matrices satisfy $U_1^T U_1 = I_{r_A} = V_1^T V_1$, $U_2^T U_2 = I_{p-r_A}$, $V_2^T V_2 = I_{q-r_A}$, $U_2^T U_1 = 0$, $V_2^T V_1 = 0$. The $r_A \times r_A$ matrix Σ_1 contains the nonzero singular values, and obviously V_2 is an orthonormal basis of the null space of A as $AV_2 = 0$. Such an orthonormal basis is certainly not unique, as it can be post-multiplied by any orthonormal $(q - r_A) \times (q - r_A)$ matrix.

Let now X be a $q \times (q - r_A)$ matrix such that

$$AX = 0 \text{ with } \text{rank}(X) = q - r_A,$$

i.e., the columns of X form a basis for the right null space of A.

One can always reorder the columns of A and partition them as in $A = (A_1 \;\; A_2)$, where A_1 is $p \times (q - r_A)$ and A_2 is $p \times r_A$, in which A_2 contains r_A linear independent columns. This reordering and partitioning is generally not unique, but can always be done. Reorder and partition the rows of X accordingly so that

$$AX = (A_1 \;\; A_2) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A_1 X_1 + A_2 X_2 = 0.$$

Then we have the following property.

LEMMA A.1. *Complementarity of rank in column space and null space:*

$$\text{rank}(X_1) = q - r_A \Leftrightarrow \text{rank}(A_2) = r_A.$$

This can be proved as follows.

$\Leftarrow$ If $\text{rank}(A_2) = r_A$, then $A_2^T A_2$ is invertible, so that $X_2 = -(A_2^T A_2)^{-1} A_2^T A_1 X_1$, which shows that $\text{rank}(X) = q - r_A = \text{rank} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} I_{q-r_A} \\ -(A_2^T A_2)^{-1} A_2^T A_1 \end{pmatrix} X_1$.

$\Rightarrow$ Since X_1 is square and invertible, we have $A_1 = -A_2 X_2 X_1^{-1}$, so that $A = (A_1 \;\; A_2) = A_2 \, (-X_2 X_1^{-1} \;\; I_{r_A})$, which shows that $\text{rank}(A) = \text{rank}(A_2) = r_A$.

This result shows that the set of unknowns in homogeneous linear equations can always be partitioned in dependent variables (X_2) and independent variables (X_1), meaning that X_2 can be written as a linear combination of X_1. Alternatively, we can partition the matrix A in columns that are linear independent (the matrix A_2) and columns that can be written as linear combinations of the independent ones (the columns of the matrix A_1). The partitioning of A into A_1 and A_2, where $\text{rank}(A_2) = r_A$, is certainly not unique, but these results hold for any partitioning. Consequently we have the following fact concerning the complementarity of indices.

PROPERTY A.2. *The sets of indices of linear independent columns in A and linear independent rows of X, are complementary.*

In this paper, we typically use an ordering on the variables, *i.e.,* the unknowns x in the problem, in which ξ_1 preceeds ξ_2, ξ_3 *etc.* Therefore, we would like to have the set of linear independent variables to have indices that are as small as possible in the particular ordering we are using. This implies that we are interested in finding the first $q - r_A$ rows of X that are linear independent. The indices of the columns in A that are linearly independent follow from the complement of indices: They correspond to the r_A linear independent columns of A that one can find, when starting from the right to the left of A.

The result also means that, for a selection of r_A columns of A, in a submatrix A_2 that is *not* of full column rank, the corresponding sub-matrix of X formed from the complementary selection of rows, will *not* be of full rank.