NUMERICAL ALGEBRAIC GEOMETRY II: BACK TO THE ROOTS – POLYNOMIAL SYSTEM SOLVING USING LINEAR ALGEBRA*

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Abstract. Polynomial system solving is an old mathematical problem pervading science and engineering. We return to the very roots of the problem, and view the task of solving a system of polynomial equations from the linear algebra perspective. The system of polynomial equations is represented by a structured Macaulay matrix containing the coefficients of the polynomials multiplied with a vector containing monomials. We develop two algorithms which phrase the polynomial root-finding problem as eigenvalue problems. The first algorithm employs the multiplication structure in the vector of monomials to phrase an eigenvalue problem in terms of matrices built up from certain rows of a numerically computed basis for the null space of the Macaulay matrix. The second algorithm works in the column space of the Macaulay matrix, and phrases an eigenvalue problem via a QR decomposition on certain blocks of a column reordered Macaulay matrix.

Key words. polynomial system solving, polynomials, eigenvalue problems, algebraic geometry

AMS subject classifications. 13P05, 13P15, 15A03, 15A18, 15B05

1. Introduction. In this article, we consider zero-dimensional systems of n multivariate polynomial equations in n unknowns $f_i(x_1, \ldots, x_n) = 0$, for $i = 1, \ldots, n$, describing simple solutions (also called roots). Polynomial system solving is an old but central problem in algebraic geometry underlying many applications in applied mathematics, science and engineering [6, 7, 8, 10, 32, 33].

A well-known fact from linear algebra is that the roots of an univariate polynomial can be computed as the eigenvalues of its Frobenius matrix. Indeed, consider an

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univariate polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and the matrix equation

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-2} \\ x^{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{n-2} \\ x^{n-1} \end{pmatrix} x, \quad (1.1)$$

from which it is clear that all roots of $f(x)$ are eigenvalues of the Frobenius matrix. In this paper we will study the generalization of this notion to the multivariate case. Since the publication of the groundbreaking work [40], the Möller-Stetter eigenvalue approach [31] to multivariate polynomial system solving has become prevalent and is currently considered to be the most reliable way to approach the problem. In this approach first a Gröbner basis [4, 5] of the input system is computed, which is often a time-consuming and unwanted step.

There is a large body of literature in (computational) algebraic geometry, but it is only accessible after intensive study of the subject and often beyond the grasp of applied mathematicians and engineers. The authors strongly believe that bridging the gaps between applied mathematics and algebraic geometry is of paramount importance. This article collects important results which are scattered across a large body of literature and aims at presenting them in a didactical and accessible framework.

In the current article the problem at hand is approached from the linear algebra point-of-view. We present two algorithms for finding the solutions of a given zero-dimensional system, without requiring Gröbner basis computations, solely by making use of straightforward numerical linear algebra techniques, such as eigenvalue computations, singular value decompositions and QR decompositions. Apart from avoiding the often unwanted step of computing a Gröbner basis, there are several advantages of phrasing the problem as a linear algebra question. Numbers are in computers represented and manipulated in finite precision, which demand a careful consideration of the numerical aspects involved. On the other hand, an equation may be obtained from a noisy experimental setting, which requires taking into account the limited accuracy. The numerical linear algebra framework has reached a mature state and provides us with well-established numerical tools for handling with such issues. Moreover, the framework naturally allows for certain related generalizations which would become rather cumbersome in the symbolic case. An instance is the case of (noisy) overdetermined systems of multivariate polynomials, which would likely do not have any exact solutions, but might have approximate solutions.

We are aware of the fact that the methods presented here do not necessarily outperform existing methods in computational algebraic geometry, and that similar approaches have been described earlier, see Section 2. However, it is to the authors' knowledge the first time that the presented results have been written down in this simple form, with the intention of remaining as close as possible to the familiar language and (numerical) implementations of linear algebra [18]. As such, the current article would be the ideal starting point to get introduced to more technical computational algebraic geometry literature.

The remainder of this paper is organized as follows. In Section 2 the relation of the presented approach to the existing methods is presented. Section 3 illustrates how the common roots of two univariate polynomials can be obtained using Sylvester's matrix. This provides the main ingredients to tackle the multivariate case. Section 4 discusses

how systems of polynomial equations can be represented as a matrix-vector product by the use of the Macaulay matrix. In Section 5 the null space of the Macaulay matrix is discussed, including the description of affine roots and roots at infinity. Two flavors of root-finding algorithms are developed in Section 6, both of which translate the root-finding problem into eigenvalue problems. Finally, Section 7 summarizes the results of this article and points out some open questions for future work.

2. Historical Notes and Related Work. In this manuscript, we will study techniques which can be seen as a mix of resultant theory [17] and numerical linear algebra [18]. The notion of resultants can be traced back to important results due to Sylvester [42] who showed that checking whether two univariate polynomials have common roots is equivalent to checking whether a certain matrix built from the coefficients is singular. Macaulay [27, 28] generalized this result to the case of multivariate homogeneous polynomials. **Sylvester and Macaulay** discovered compelling results in polynomial algebra which are still of interest today because of their close link to linear algebra interpretations. However, due to their inherent limiting computational complexity, many of their insights have been neglected during most of the 20th century, as at the end of the 19th century the focus of algebraic geometry shifted away from polynomial system solving to abstract algebra.

Only in the 1960's the computational aspects of algebraic geometry came into focus again with the development of **Buchberger's algorithm**, which computes a so-called **Gröbner basis** of a system of polynomial equations [4, 5]. Since then, the Gröbner basis approach has dominated computer algebra. A major disadvantage of Buchberger's algorithm is that it is described using exact (*i.e.,* symbolic) manipulations on the input equations and its extension to floating point arithmetic is known to be cumbersome [37, 38] with limited alternatives available [15, 16, 20, 40, 39]. Among these approaches, Faugère's methods employ linear algebra manipulations on large sparse matrices and are currently the most reliable and efficient. Surprisingly, in most of the Gröbner basis literature, the specific problem of solving a system of multivariate polynomials is not of particular interest.

In the 1980's, the relevance of the work of Sylvester and Macaulay was rediscovered by **Lazard and Stetter** who employ Macaulay-like matrices for solving polynomial systems [1, 2, 24, 25]. In [25], it is shown that a Gröbner basis can be computed by triangularization of a large Macaulay-like coefficient matrix. The seminal papers [1, 2] and later [31] return to the root-finding problem rather than computing a Gröbner basis and develop the **natural link between polynomial system solving problem and eigenvalue problems**. In this approach the root-finding problem breaks down to finding a basis for the polynomial ring modulo the ideal generated by the input equations, and then expressing multiplication in this space by means of multiplication matrices. The eigenvalue decomposition of such multiplication matrices provides the (affine) solutions of the system. The research efforts initiated by Lazard and Stetter were further explored by Emiris [12], Faugère [15, 16], Manocha [29], Mourrain [34], among others.

The recent years have witnessed an **increased research interest in polynomial system solving and optimization** [10, 41], with a myriad of applications in applied mathematics, science and engineering, such as systems and control [6], bioinformatics [14, 35], robotics [13], *etc.* Homotopy continuation methods [26, 43] and recent developments in **real algebraic geometry** [22, 23, 36] and **polynomial optimization** [19, 21, 36] outperform the classic methods and the increasing computer power makes it possible to tackle new classes of problems, bearing the evidence that

the old problem of solving polynomial equations is still very relevant today.

3. Univariate Root-finding using the Sylvester Matrix. In Section 1 we have reviewed the fact that the eigenvalues of the Frobenius matrix correspond to the roots of a univariate polynomial. We will now discuss a related concept which employs linear algebra to finding the common roots of univariate polynomials.

The well-known construction due to Sylvester [42] tests whether two polynomials $f_1(x)$ and $f_2(x)$ have common roots by investigating the determinant of a structured square matrix built from the coefficients of the equations. Consider the system

$$\left\{ \begin{array}{rcl} f_1(x) & = & a_r x^r + a_{r-1}x^{r-1} + \ldots + a_0, \\ f_2(x) & = & b_s x^s + b_{s-1}x^{s-1} + \ldots + b_0, \end{array} \right. \tag{3.1}$$

having m single common roots (*i.e.,* no common roots with multiplicity).

DEFINITION 3.1 (Sylvester matrix). *Multiplying $f_1(x)$ and $f_2(x)$ with powers of x gives rise to a square system of linear equations $M\,k = 0$, or*

$$\begin{array}{l} s \text{ rows} \left\{ \right. \\ \\ \\ \\ r \text{ rows} \left\{ \right. \\ \\ \\ \\ \end{array} \left(\begin{array}{ccccccc} a_0 & a_1 & \ldots & a_r & & & \\ & a_0 & a_1 & \ldots & a_r & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \ldots & a_r \\ \hline b_0 & b_1 & \ldots & b_s & & & \\ & b_0 & b_1 & \ldots & b_s & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \ldots & b_s \end{array} \right) \left(\begin{array}{c} x^0 \\ \\ x^1 \\ \\ \\ \vdots \\ \\ x^{r+s-1} \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array} \right). \tag{3.2}$$

The coefficient matrix M is called the Sylvester matrix, and has size $(r+s)\times(r+s)$. The f_1-block of the Sylvester matrix has s rows, and the f_2-block has r rows. The monomial basis vector k contains as its elements the monomials x^d for $d = 0, \ldots, r+s-1$.

PROPOSITION 3.2. *The Sylvester matrix M has a zero determinant if the polynomials $f_1(x)$ and $f_2(x)$ have a common root.*

Indeed, evaluating the monomial basis vector $k = \begin{pmatrix} 1 & x & x^2 & \ldots & x^{r+s-1} \end{pmatrix}^T$ at a common root $x^{(i)}$ of $f_1(x)$ and $f_2(x)$ gives rise to a non-zero vector in the null space of the Sylvester matrix M.

Interestingly, Sylvester's construction can be employed to determine the common roots. An important tool in understanding how this can be achieved is the canonical null space of the Sylvester matrix composed of the (for the moment unknown) evaluations of the common roots in the monomial basis vector k.

DEFINITION 3.3 (Univariate Canonical Null Space). *Assume that $f_1(x)$ and $f_2(x)$ have m single common roots $x^{(i)}$, for $i = 1, \ldots, m$, with $x^{(i)} \neq x^{(j)}$, for $j \neq i$. Let K be the $r+s \times m$ matrix which contains as its columns the vectors k evaluated at the m common roots $x^{(i)}$,*

$$K := \begin{pmatrix} | & & | \\ \cdots & k|_{x^{(i)}} & \cdots \\ | & & | \end{pmatrix},$$

which is called the canonical null space of M. (Note that the canonical null space of M is a Vandermonde structured matrix).

The next ingredient is the observation that a multiplication property holds in the monomial vectors k, *e.g.,* one has

$$\begin{pmatrix} 1 & x & x^2 & \dots & x^{r+s-2} \end{pmatrix}^T x = \begin{pmatrix} x & x^2 & x^3 & \dots & x^{r+s-1} \end{pmatrix}^T \tag{3.3}$$

which can be represented as $S_1 k\, x = S_x k$, where S_1 is a row selection matrix selecting the rows 1 up to $r+s-1$, and S_x is a row selection matrix selecting the rows 2 up to $r+s$. This relation also holds for the monomial basis vectors evaluated at each of the m roots, when multiplying with the i-th root $x^{(i)}$, for $i = 1, \dots, m$, *i.e.,* $S_1\ k|_{x^{(i)}}\, x^{(i)} = S_1\ k|_{x^{(i)}}$. Applying the multiplication property to all m roots gives

$$S_1 K D_x = S_x K, \tag{3.4}$$

where K represents the canonical null space of M, and D_x is a diagonal matrix containing the m roots, *i.e.,* $D_x = \operatorname{diag}\left(x^{(1)}, x^{(2)}, \dots, x^{(m)}\right)$. Unfortunately the null space of M is not directly available with the canonical structure K. Instead a basis for the null space of M can be computed as Z, which is related to the canonical null space by $K = ZV$, with V non-singular. Combining the previous observations easily leads to an eigenvalue problem from which the common roots can be obtained.

THEOREM 3.4 (Sylvester matrix univariate root-finding). *Together with (3.4) the problem of finding the common roots of f_1 and f_2 is reduced to the eigendecomposition $VD_xV^{-1} = (S_1Z)^\dagger S_xZ$, with $\cdot^\dagger$ denoting the Moore-Penrose pseudoinverse. The eigenvalues of $(S_1Z)^\dagger S_xZ$ are the m common roots of $f_1(x)$ and $f_2(x)$.*

From the computation of ZV and consequently normalizing the result column-wise such that the first row consists of ones, a reconstruction of the canonical kernel exhibiting the Vandermonde structure is obtained. All solutions (and their powers) can now be read off directly.

EXAMPLE 3.5. *Consider the polynomials*

$$\begin{cases} f_1(x) &=& (x-1)(x-2) &=& x^2 - 3x + 2 \\ f_2(x) &=& (x-1)(x-2)(x+1) &=& x^3 - 2x^2 - x + 2 \end{cases} \tag{3.5}$$

having two common roots $x^{(1)} = 1$ and $x^{(2)} = 2$. The Sylvester matrix M is constructed as

$$M \quad = \quad \begin{array}{c} \\ f_1 \\ xf_1 \\ x^2f_1 \\ f_2 \\ xf_2 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 2 & -3 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & -3 & 1 \\ 2 & -1 & -2 & 1 & 0 \\ 0 & 2 & -1 & -2 & 1 \end{pmatrix} \end{array}. \tag{3.6}$$

The null space of M has a dimension of two (the singular values are 5.4434, 4.8990, 2.8931, 0.0000 *and* 0.0000*), and a numerical basis for the null space is computed using a singular value decomposition. The eigenvalue decomposition of $(S_1Z)^\dagger S_xZ$ reveals the common roots of $f_1(x)$ and $f_2(x)$ as $x^{(1)} = 1$ and $x^{(2)} = 2$ (exact up to machine precision).*

4. Matrix-vector representation of multivariate polynomial systems.

4.1. Preliminaries. Throughout the remainder of this paper, a monomial is represented as $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and is uniquely determined by its exponent vector $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} := \{0, 1, \dots\}$. A polynomial is defined as a linear combination of monomials. It will often be necessary to carefully order monomials, for which we have chosen to use the degree negative lexicographic (dnlex) ordering. [1]

DEFINITION 4.1 (Degree Negative Lexicographic Order). *Let $\alpha, \beta \in \mathbb{N}^n$ be monomial exponent vectors. Then $\alpha < \beta$ by the degree negative lexicographic order, denoted $\alpha <_{dnlex} \beta$ (simplified as $\alpha < \beta$), if $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and in the vector difference $\beta - \alpha \in \mathbb{Z}^n$, the left-most non-zero entry is negative.*

EXAMPLE 4.2. *The monomials of maximal degree three in two variables are ordered by the degree negative lexicographic order as*

$$1 < x_1 < x_2 < x_1^2 < x_1x_2 < x_2^2 < x_1^3 < x_1^2x_2 < x_1x_2^2 < x_2^3.$$

4.2. Problem Formulation. We consider the problem of finding the solutions of a system of n multivariate polynomial polynomials f_i where $i = 1, \dots, n$ in n unknowns $x_1, \dots, x_n$, having total degrees $d_1, \dots, d_n$. The system of equations is represented formally as

$$\begin{cases} f_1(x_1, \dots, x_n) &= 0, \\ f_2(x_1, \dots, x_n) &= 0, \\ & \vdots \\ f_n(x_1, \dots, x_n) &= 0. \end{cases} \tag{4.1}$$

The maximal total degree is denoted as $d_\circ = \max(d_1, \dots, d_n)$. It is assumed that (4.1) describes a zero-dimensional solution set in the projective space with simple affine solutions.

4.3. The Macaulay matrix. It is well-known that the space of multivariate polynomials with coefficients in the field $\mathbb{C}$ in n variables has the structure of a vector space if one considers its elements up to a given degree d. From the linear algebra perspective it is indeed natural to think of an equation f_i as a vector containing its coefficients multiplied with a vector $k(d_\circ)$ containing all possible monomials with total degrees from 0 up to $d_\circ$. The elements of $k(d_\circ)$ are ordered by the degree negative lexicographic monomial ordering as in Definition 4.1.

EXAMPLE 4.3. *The equation $x_1^2 - 3x_2^2 + 2x_1 - 5 = 0$ can be written in this way as*

$$\begin{pmatrix} -5 & 2 & 0 & 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}^T = 0.$$

By multiplying all the equations f_i in (4.1) by monomials, polynomial equations are found which compose the rows of a so-called Macaulay matrix. This gives rise to a matrix equation of the form

$$M(d)\, k(d) = 0.$$

DEFINITION 4.4 (Macaulay matrix and monomial vector). *The Macaulay matrix $M(d)$ contains as its rows the vector representations of $x^{\sigma_i} \cdot f_i$, for all i, where x^{σ_i}*

[1] Many results presented here can easily be generalized to other (admissible) monomial orders.

represents a monomials having $\deg(x^{\sigma_i}) \leq d$*, and the columns of* $M(d)$ *are indexed by all monomials of degree at most* d*, represented as follows,*

$$M(d) := \begin{pmatrix} \{x^{\sigma_1}\} \cdot f_1 \\ \hline \{x^{\sigma_2}\} \cdot f_2 \\ \hline \vdots \\ \hline \{x^{\sigma_n}\} \cdot f_n \end{pmatrix}, \tag{4.2}$$

where each equation f_i*, for* $i = 1, \ldots, n$ *is multiplied by all monomials of degrees* $\leq d - d_i$*, denoted by* $\{x^{\sigma_i}\}$*. The rows of* M *are ordered by considering the monomials shifting* f_i *by the degree negative lexicographic order.*

The monomial vector $k(d)$ *is composed accordingly,* i.e.,

$$k(d) := \begin{pmatrix} 1 & x_1 & \ldots & x_n & x_1^2 & \ldots & x_n^2 & \ldots & x_1^d & \ldots & x_n^d \end{pmatrix}^T.$$

The Macaulay matrix construction leads to a very sparse quasi-Toeplitz structured matrix where each row in the f_i-block contains only as many non-zero elements as there are non-zero coefficients in f_i.

EXAMPLE 4.5. *Consider the simple system*

$$\begin{cases} f_1(x_1, x_2) &= x_1^2 + x_1 x_2 + 4 \\ f_2(x_1, x_2) &= 2x_1^3 + 2x_1 x_2^2 + 8. \end{cases}$$

The Macaulay matrix $M(4)$ *is*

$$\begin{array}{c} \\ f_1 \\ x_1 f_1 \\ x_2 f_1 \\ x_1^2 f_1 \\ x_1 x_2 f_1 \\ x_2^2 f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{array} \\ \begin{pmatrix} 4 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \end{pmatrix} \end{array}.$$

The following formulas express the number of monomials (either of total degree d or of total degree $\leq d$) by binomial coefficient expressions.

LEMMA 4.6 (Number of monomials). *The number of monomials of total degree* d *in* n *variables* $x_1, \ldots, x_n$ *is given as* $\binom{n+d-1}{d}$*. The number of monomials of total degree* $\leq d$ *in* n *variables* $x_1, \ldots, x_n$ *is given as* $\binom{n+d}{d}$*.*

The size of the Macaulay matrix can now be easily calculated as follows.

LEMMA 4.7 (Dimensions Macaulay matrix). *Let* $p(d)$ *and* $q(d)$ *denote the number of rows and columns of* $M(d)$*, respectively. From Lemma 4.6 we find* $p(d) = \sum_{i=1}^{n} \binom{n+d-d_i}{d-d_i}$ *and* $q(d) = \binom{n+d}{d}$*.*

5. Null space of the Macaulay matrix.

5.1. Generic case: simple affine roots. We start with describing the case in which the input system (4.1) has only simple affine roots, which we call the generic case. This situation occurs *e.g.,* if all the possible coefficients in the system (4.1) are random numbers. Although the genericity assumption often does not always hold in practice, this case will be instrumental as a baseline setting for introducing the main ideas of this article.

5.1.1. Set of standard monomials. A central object in our exposition is the set of standard monomials. In the classical literature, the set of standard monomials is defined as a basis of polynomial ring modulo the ideal generated by the system of polynomial equations, and is usually determined by means of Gröbner basis computations [7, 40].

We define the standard monomials by means of numerical rank properties of the Macaulay matrix, as adopted from [3].

DEFINITION 5.1 (Set of standard monomials and its complement). *The set of standard monomials $B(d)$ for degree d of the system (4.1) is defined as the complement of the monomials of degrees $\delta = 0, \ldots, d$ indexing the right-most linear independent columns of $M(d)$. The set $\overline{B(d)}$ is defined as the set of all monomials indexing the columns of $M(d)$ which are not standard monomials.*

A rudimentary numerical procedure for numerically determining the standard monomials is to iterate over the columns of M and monitor increases in the numerical rank as more columns are added, starting from the right-most column. The columns which leave the rank unchanged are in the set of standard monomials B, whereas columns which increase the rank are in the complement of the set of standard monomials $\overline{B}$. Numerical rank tests can be implemented in a numerically reliable way by the singular value decomposition [18] and a suitable threshold value for deciding whether a singular value is small enough to be considered as zero, or when a sufficiently large decay in consecutive singular values is observed. The notion of standard monomials is illustrated using an example.

EXAMPLE 5.2. *We continue with the equations from Example 4.5. Let the standard monomials $B(d)$ be monomials indexing the right-most linear dependent columns as in the definition. One can easily check that $B(4) = \{1, x_1, x_2, x_1^2, x_2^3, x_2^4\}$.*

5.1.2. Canonical null space of the Macaulay matrix. From the construction of the Macaulay matrix it immediately follows that each solution of (4.1), denoted as $x^{(i)} := \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\right)$, for $i = 1, \ldots, m_B$, composes a vector in the null space. The canonical null space is defined as the collection of all such vectors.

DEFINITION 5.3 (Canonical null space). *Evaluating k at the m_B solutions gives rise to m_B independent vectors in the null space of M. We denote by $k|_{x^{(i)}}$, for $i = 1, \ldots, m_B$ the evaluation of k at the m_B solutions. The canonical null space of M is defined as the collection of the vectors $k|_{x^{(i)}}$ into the generalized Vandermonde structured matrix K of size $q \times m_B$,* i.e.,

$$K := \begin{pmatrix} | & | & | \\ \cdots & k|_{x^{(i)}} & \cdots \\ | & | & | \end{pmatrix}.$$

The canonical null space K will only be used implicitly in the derivations and will never be constructed explicitly. Therefore, the specific order in which the vectors k are placed in K is not of any relevance for the remainder of our exposition.

The following lemmas describe well-known facts regarding the number of standard monomials, the number of solutions of the system (4.1) and the nullity of the Macaulay matrix.

LEMMA 5.4 (Bézout number m_B [7]). *The number of solutions of (4.1) in projective space (counting multiplicities) is given by the Bézout number*

$$m_B = \prod_{i=1}^{n} d_i.$$

LEMMA 5.5 (Nullity of Macaulay matrix equals Bézout number [3]). *At a sufficiently large degree d_c, the nullity of the Macaulay matrix is equal to the Bézout number,* i.e.,

$$\text{nullity}\left(M(d)\right) = \prod_{i=1}^{n} d_i, \quad d \geq d_c. \tag{5.1}$$

Proof. The cardinality of the set of standard monomials $B(d_c)$ as defined in Definition 5.1 is equal to the nullity of M, which immediately follows from the definition. From [7] it follows that this number corresponds to the dimension of the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$ in the generic case. An immediate consequence is that this holds for the projective case. □

5.2. Roots at infinity. In most cases, the genericity assumption of Section 5.1 that the solution set contains only affine roots does not hold. In addition to affine solutions, so-called roots at infinity may exist, which are caused by algebraic relations between the coefficients or the occurrence of zero coefficients in (4.1).

5.2.1. The homogeneous Macaulay matrix. In order to describe the roots at infinity, we use the well-known concepts of homogenization and projective space. The construction of the Macaulay matrix of the system (4.1) as developed in Section 4 can alternatively be interpreted as the Macaulay matrix of the homogenized system.

DEFINITION 5.6 (Homogenization and dehomogenization). *The homogenization of an equation f, denoted f^h, is computed using the formula*

$$f^h = x_0^d \cdot f\left(x_1/x_0, \ldots, x_n/x_0\right).$$

Dehomogenizing f^h yields f, or formally $f^h(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$.

A homogenized system of equations describes solutions in the $n+1$-dimensional projective space, and $x_0, \ldots, x_n$ are called the homogeneous coordinates. In the projective space the roots at infinity are incorporated as regular points for which $x_0 = 0$.

DEFINITION 5.7 (Homogeneous Macaulay matrix and monomial vector). *The Macaulay matrix for the homogenized system $f_i^h(x_0, x_1, \ldots, x_n) = 0$, for $i = 1, \ldots, n$ is denoted by $M^h(d)$ and is defined as*

$$M^h(d) := \left(\begin{array}{c} \{x^{\sigma_1}\} \cdot f_1^h \\ \hline \{x^{\sigma_2}\} \cdot f_2^h \\ \hline \vdots \\ \hline \{x^{\sigma_n}\} \cdot f_n^h \end{array} \right),$$

where each equation f_i^h, for $i = 1, \ldots, n$ is multiplied by all monomials in the unknowns $x_0, \ldots, x_n$ of degree $d - d_i$, denoted by $\{x^{\sigma_i}\}$. The columns of $M^h(d)$ are indexed by all monomials in the unknowns $x_0, \ldots, x_n$ of degree d, which are placed in the monomial basis vector $k^h(d)$ to obtain the equation

$$M^h(d)\, k^h(d) = 0. \tag{5.2}$$

The monomial vector $k^h(d)$ consists of all monomials in $n+1$ unknowns of degree d,

$$k^h(d) := \begin{pmatrix} x_0^d & x_0^{d-1}x_1 & \ldots & x_0^{d-1}x_n & \ldots & x_0x_1^{d-1} & \ldots & x_0x_n^{d-1} & x_1^d & \ldots & x_n^d \end{pmatrix}^T.$$

PROPOSITION 5.8. *The Macaulay matrix for the system of homogeneous equations $f_i^h(x_0, x_1, \ldots, x_n) = 0$, for $i = 1, \ldots, n$, is identical to the Macaulay matrix for the system (4.1),* i.e.,

$$M^h(d) \equiv M(d).$$

It is easily seen that the difference between $M(d)$ and $M^h(d)$ lies in a mere relabeling of rows and columns of $M(d)$. The ordering of the monomials in this relabeling is consistent with the degree negative lexicographic monomial ordering.

5.2.2. The set of affine standard monomials. From the previous paragraphs we learn that solutions at infinity are defined as non-zero solutions for which the homogenization variable $x_0 = 0$. Lemma 5.5 and Proposition 5.8 imply that the roots at infinity compose vectors in the null space of $M(d_c)$.

PROPOSITION 5.9. *Consider the partitioning of $M(d_c)$ as*

$$M(d_c) = \begin{pmatrix} M_0 & M_1 & M_2 & \ldots & M_{d_c} \end{pmatrix}, \tag{5.3}$$

where the block M_i contains the columns of $M(d_c)$ indexed by the monomials of degree i, for $i = 0, \ldots, d_c$. The existence of roots at infinity is revealed by the column rank deficiency of the block M_{d_c}

Indeed, column rank deficiency of M_{d_c} implies there exist solutions for which the homogenization variable x_0 is zero. Also observe that evaluating $x_0 = 0$ would reduce all blocks M_d for $d < d_c$ to zero.

An immediate consequence is that when determining the standard monomials $B(d)$ for a sufficiently large degree d, we will find some monomials which are caused by the non-zero solutions with $x_0 = 0$. However, it would generally not suffice to simply dismiss the standard monomials of degree d alone (and, *e.g.,* remove the corresponding columns from $M(d)$) to resolve the roots at infinity, since roots at infinity may have — and often *do have* — a complicated multiplicity structure which protrudes into degrees smaller than d. Based on the insights obtained in [3] we adopt the following definition of the set of affine standard monomials.

DEFINITION 5.10 (Set of affine standard monomials). *Let d_G be the smallest degree for which the following properties hold:*

1. *$B^\star(d_G) \subseteq B(d_G)$,*
2. *if $x^\gamma \in B^\star(d_G)$, then $x^\gamma \in B^\star(d_G + 1)$,*
3. *for each $i = 1, \ldots, n$, there exists a monomial $x_i^{\delta_i} \in \overline{B(d_G)}$,*
4. *for each $x^\gamma \in B^\star(d_G)$, one has that $\gamma_i < \delta_i$, for $i = 1, \ldots, n$.*

We call the monomials in $B^\star(d_G)$ the affine standard monomials for degree d_G.

From the definition we propose the following procedure for determining the affine standard monomials. One inspects the elements in $B(d)$ as the degree d increases. The standard monomials corresponding to the affine roots will 'stabilize', whereas the ones corresponding to the roots at infinity will always appear at high degrees (and move along as d increases). From a sufficiently large degree d_G on, the set of affine standard monomials will be bordered by monomials of the form $x_i^{\delta_i}$, for each $i = 1, \ldots, n$.

An immediate implication is that the number of affine roots of (4.1) equals the number of affine standard monomials.

LEMMA 5.11 (Number of affine roots [3]). *The number of affine standard monomials corresponds to the number of affine roots, or*

$$m_a = \#B^\star(d_G),$$

where $\#(\cdot)$ denotes the cardinality of the set.

5.3. Removing the roots at infinity. Assume that the Macaulay matrix for degree d_G is constructed and the set of affine standard monomials $B^\star(d_G)$ is determined. The definition of the affine standard monomials leads to two methods for discarding the solutions at infinity.

1. By removing from $M(d_G)$ the standard monomial columns of the highest degrees, *i.e.,* the monomials $B(d_G) \setminus B^\star(d_G)$, the roots at infinity are annihalated. This leads to the reduced Macaulay matrix $M^\star(d_G)$.
2. One can also reason that the construction of the Macaulay matrix $M(d)$ for increasing degrees d 'separates' the affine roots and the roots at infinity as suggested in Definition 5.10. This observation can be employed to search for the affine roots only when devising a root-finding algorithm (see the column compression alternative in Section 6.2.2).

6. Finding the roots. We now have all the necessary tools to develop two algorithms which translate the root-finding task into eigenvalue problems. The first algorithm employs a numerically computed basis for the null space of M and makes use of the structure in the monomial vector k. The second algorithm directly operates on the columns of the Macaulay matrix and employs a Q-less QR decomposition of certain re-partitioned blocks of M.

6.1. From multiplication structure to eigenvalue problems. In a similar fashion as in the univariate case described in Section 3, multiplication of a monomial basis vector k by a monomial or a polynomial obeys a shift property which is an essential ingredient for phrasing the root-finding problem as an eigenvalue problem.

PROPOSITION 6.1 (Monomial Shift in Monomial Vectors k). *Multiplication of entries in a monomial basis vector $k := k(d)$ with the monomial x^γ maps the entries of k of degrees 0 up to $d - |\gamma|$ to entries in k of degrees $|\gamma|$ up to d. This is expressed by means of row selection matrices operating on k as*

$$S_1 k x^\gamma = S_\gamma k,$$

where S_1 selects all monomials in k of degrees 0 up to $d - |\gamma|$ and S_γ selects the rows of k onto which the monomials $S_1 k$ are mapped by multiplication by x^γ.

This property can be generalized directly to an arbitrary polynomial shift function $g(x)$ with $\deg(g) \leq d$.

PROPOSITION 6.2 (Polynomial Shift in Monomial Vectors k). *Multiplication of a monomial basis vector $k(d)$ with a polynomial $g(x) := \sum_\gamma c_\gamma x^\gamma$ gives $S_1 kg(x) = S_g k$, where $S_g := \sum_\gamma c_\gamma S_\gamma$, in accordance with Proposition 6.1. The selection matrix S_g takes in this case linear combinations of rows of k.*

EXAMPLE 6.3. *Consider a monomial vector k of degree three in two variables x_1 and x_2, given by*

$$k = \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{pmatrix}^T, \tag{6.1}$$

and a shift function $g(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2$. We can write

$$\begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}^T \left(3x_1^2 + 2x_2^2\right) = \tag{6.2}$$

$$\begin{pmatrix} 3x_1^2 + 2x_2^2 & 3x_1^3 + 2x_1x_2^2 & 3x_1^2x_2 + 2x_2^3 \end{pmatrix}^T, \tag{6.3}$$

which can alternatively be expressed as $S_1 kg(x) = S_g k$, with

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \textit{and} \tag{6.4}$$

$$S_g = \begin{pmatrix} 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 \end{pmatrix}. \tag{6.5}$$

The above shift property in the monomial basis vectors can be applied to the canonical null space K at once by means of the introduction of $D_g := \operatorname{diag}\left(g(x^{(1)}), g(x^{(2)}), \ldots g(x^{(m_B)})\right)$. This leads to

$$S_1 K D_g = S_g K, \tag{6.6}$$

from which the link to eigendecompositions is starting to take shape. In the following paragraphs, we will establish how the shift property can be used to phrase the root-finding problem as an eigenvalue problem.

6.2. Null space based root-finding.

6.2.1. Generic case. After the construction of the suitably sized Macaulay matrix $M := M(d_G)$ having $\text{nullity}(M) = m_B$, the canonical null space K is not available directly, but instead a basis for the null space of M can be computed as Z. A numerically reliable way to obtain Z is by means of a singular value decomposition [18]. Observe now that K can be written as $K = ZV$ where V is non-singular. In combination with the previous results this brings us to the first main theorem which phrases the root-finding problem as an eigenvalue problem.

THEOREM 6.4 (Null space based root-finding). *Assume that the system (4.1) has only affine roots and consider a shift polynomial $g(x) \neq 1$. Let S_1 select the rows from Z corresponding to the m_B standard monomials, and let S_g select their corresponding mappings after multiplication with the shift function $g(x)$. The eigenvalues of the generalized eigenvalue problem*

$$S_1 Z \left(V D_g V^{-1}\right) = S_g Z, \tag{6.7}$$

are the evaluations of the roots of (4.1) at the shift function $g(x)$. The matching between the individual components x_i can be obtained by computing $K = ZV$ and then rescaling the columns of K such that the first row entries equal one.

Note that the shift polynomial $g(x)$ needs to be defined in such a way that the evaluation of $g(x)$ is different for each of the m_B roots to ensure that (6.7) has no multiple eigenvalues. If (6.7) has multiple eigenvalues, the evaluation of $g(x)$ at the roots correctly corresponds to the eigenvalues, but the individual components x_i cannot be reconstructed from $K = ZV$. Mutual matching of the solution components can in such cases only be achieved by solving the eigenvalue problem for consecutive shift functions $g(x) = x_i$, for $i = 1, \ldots, n$, and exhaustively combining the results by evaluating them in the system (4.1).

6.2.2. Roots at infinity. If there are roots at infinity, the set of affine standard monomials $B^\star(d_G)$ as established in Definition 5.10 must be considered. After dismissing the columns of $M(d^G)$ corresponding to the standard monomials of the roots at infinity, one obtains $M^\star(d^G)$, which can be used in Theorem 6.4 to find the affine solutions.

Since Z by definition contains both null space vectors generated by affine solutions and null space vectors generated by roots at infinity, Z needs to be compressed to obtain W having m_a columns corresponding to the roots at infinity only. This is achieved by computing the so-called column compression of Z.

DEFINITION 6.5 (Column compression). *Let r_∞ denote the row number of the first element of $B(d_G) \setminus B^\star(d_G)$ in $k(d)$, where $\cdot \setminus \cdot$ denotes the set difference operator. Let $Z^\star$ denote the matrix composed by the first $r_\infty - 1$ rows of Z. Compute the singular value decomposition of $Z^\star$ as $Z^\star = U\Sigma V^T$. Let $W := ZV$ and let $W^\star$ denote the matrix composed of the first m_a columns of W, having size $q \times m_a$.*

The root-finding method described in Section 6.2.1 can now be used on $W^\star$ to find the affine roots of (4.1).

The complete null space based root-finding procedure is summarized in Algorithm 1.

ALGORITHM 1. (Affine null space based root-finding)

input: *system of n equations $f_i = 0$ having total degrees d_i in n unknowns x_i*
output: *m_a affine evaluations of the solutions at at the function $g(x)$*

1. *Determine $B^\star(d_G)$, and let $m_a = \#B^\star(d_G)$*
2. *Let $M := M(d_G)$ and let $Z := Z(d_G)$*
3. *Column compression of Z yields $W^\star$ (size $q(d_G) \times m_a$)*
   (Note that if $m_a = m_B$, $W^\star := Z$)
4. *The m_a affine roots evaluated at the shift function $g(x)$ are the eigenvalues of*

$$S_1 W^\star \left(V D_g V^{-1} \right) = S_g W^\star$$

   where S_1 selects the m_a affine standard monomials rows of $W^\star$ and S_g the mapping by $g(x)$
5. *Reconstruct the mutual matching between the solution components x_i from*

$$K = W^\star V,$$

   and consecutive column rescaling

EXAMPLE 6.6. *Consider the polynomial system*

$$\left\{ \begin{array}{rcl} f_1(x_1, x_2, x_3) & = & x_1 x_2 - 3 = 0 \\ f_2(x_1, x_2, x_3) & = & x_1^2 - x_3^2 + x_1 x_3 - 5 = 0 \\ f_3(x_1, x_2, x_3) & = & x_3^3 - 2 x_1 x_2 + 7 = 0, \end{array} \right. \tag{6.8}$$

with the roots $(1.857 \mp 0.176i, 1.600 \pm 0.151i, 0.500 \pm 0.866i)$, $(-2.000, -1.500, -1.000)$, $(-2.357 \pm 0.689i, -1.172 \mp 0.343i, 0.500 \mp 0.866i)$, $(3.000, 1.000, -1.000)$. *The set of affine standard monomials is found at degree* $d_G = 6$ *as* $B^\star(6) = \{1, x_1, x_2, x_3, x_1^2, x_1x_3\}$. *We construct the Macaulay matrix* $M(6)$ *having dimension* 90×84 *and find a basis for the null space as* Z *having dimension* 84×12. *The remaining standard monomials are* $B(d_G) \setminus B^\star(d_G) = \{x_2^3, x_2^4, x_2^5, x_2^6, x_2^4x_3, x_2^5x_3\}$, *with the row of* x_2^3 *in* $k(d_G)$ *being* $r_\infty = 17$. *Using* $m_a = 6$ *and* $r_\infty = 17$ *we perform a column compression on* Z *to find* $W^\star$ *with size* 84×6. *We choose as a shift function* $g(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3$ *and determine the selection matrices* S_1 *and* S_2 *according to Proposition 6.2. The eigenvalues of* $S_1W^\star \left(VD_gV^{-1}\right) = S_2W^\star$ *are the affine solutions evaluated at the function* $g(x)$. *From* $K = W^\star V$ *we correctly reconstruct the six affine solutions and their mutual matching (with absolute errors of the order* 10^{-13}*).*

6.3. Column space based Root-finding.

6.3.1. Generic case. The method starts from a Macaulay matrix $M := M(d_G)$ with size $p \times q$. We will consider as the null space of M the canonical null space K of size $q \times m_B$ as defined earlier, where column i of K contains the canonical monomial basis vector k evaluated at the i-th root, for all $i = 1, \ldots, m_B$. As it will turn out, the computation of a numerical basis for the null space of M will not be required.

Consider the column reordering of M as $\begin{pmatrix} M_1 & M_2 \end{pmatrix}$, such that

$$\begin{pmatrix} M_1 & M_2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0, \tag{6.9}$$

where K_1 is of size $m_B \times m_B$ containing the standard monomials rows of K, $\operatorname{rank}(M_2) = \operatorname{rank}(M) = r$ (full column rank), such that M_1 is of size $p \times q - r = p \times m_B$ (note that $q - r = m_B$ is the number of solutions), M_2 is of size $p \times r$, and K_2 is of size $r \times m_B$. Since M_2 is full column rank we have $M_1K_1 + M_2K_2 = 0$, and

$$K_2 = -M_2^\dagger M_1 K_1.$$

Now, Proposition 6.2 can be rephrased accordingly with the reordering (6.9) as

$$S_1KD_g = K_1D_g = \begin{pmatrix} \Sigma_1 & \Sigma_2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \tag{6.10}$$

$$= \begin{pmatrix} \Sigma_1 & \Sigma_2 \end{pmatrix} \begin{pmatrix} I_{m_B} \\ -M_2^\dagger M_1 \end{pmatrix} K_1. \tag{6.11}$$

Two kinds of multiplications ('shifts') can be distinguished in this expression.

1. 'Trivial shifts' are represented by Σ_1 and cause monomials in the block K_1 to be mapped onto (higher degree) monomials in K_1.
2. 'Non-trivial shifts' are represented by Σ_2 and map monomials from K_1 to monomials of K_2.

The selection matrices Σ_1 (size $m_B \times m_B$) and Σ_2 (size $m_B \times q - m_B$) are obtained as in Proposition 6.2, while respecting the reordering of monomials of M of (6.9).

Observe that zero columns in Σ_2 represent the rows of K_2 (and hence $M_2^\dagger M_1$) which are not reached by multiplying the standard monomials with $g(x)$. Denote by z the number of zero columns in Σ_2 and let Σ_2' denote the matrix Σ_2 with the zero columns removed, having size $m_B \times q - m_B - z$.

By means of a QR decomposition we can determine the rows of interest $\Sigma_2' M_2^\dagger M_1$ as follows

$$\begin{array}{l} {\scriptstyle p}\begin{pmatrix} {\scriptstyle z} & {\scriptstyle q-m_B-z} & {\scriptstyle m_B} \\ M_{22} & M_{21} & M_1 \end{pmatrix} = \\ \\ {\scriptstyle p}\begin{pmatrix} {\scriptstyle z} & {\scriptstyle q-m_B-z} & {\scriptstyle m_B} \\ Q_1 & Q_2 & Q_3 \end{pmatrix} \quad \begin{matrix} {\scriptstyle z} \\ {\scriptstyle q-m_B-z} \\ {\scriptstyle p-q+m_B} \end{matrix} \begin{pmatrix} {\scriptstyle z} & {\scriptstyle q-m_B-z} & {\scriptstyle m_B} \\ R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & & R_{33} \end{pmatrix} \end{array}, \tag{6.12}$$

where the columns of M_{22} correspond to the monomials of interest selected by Σ_2', the columns of M_1 correspond to the standard monomials, and, the columns of M_{22} represent the remaining monomials (*i.e.,* the zero columns of Σ_2). The sizes of the matrix blocks are indicated for the sake of clarity. Observe that due to the rank deficiency of M, R_{33} is either (numerically) zero, or has zero rows if $\operatorname{rank}(M) = p$. Moreover, it will turn out that Q will not be required to determine $\Sigma_2' M_2^\dagger M_1$.

THEOREM 6.7 (Column space based root-finding). *Assume that the system (4.1) has only affine roots and consider a shift polynomial $g(x) \neq 1$. The root-finding problem is phrased as the eigendecomposition*

$$K_1 D_g = \left(\Sigma_1 - \Sigma_2' R_{22}^{-1} R_{23}\right) K_1 \tag{6.13}$$

where Σ_2' is composed of the non-zero columns of Σ_2. The matching between the components x_i can be reconstructed if the components x_i are standard monomials. This is achieved by rescaling the eigenvectors such that the entries corresponding to the monomials 1 are scaled to 1.

6.3.2. Roots at infinity. If there are roots at infinity, the set of affine standard monomials $B^\star(d_G)$ as established in Definition 5.10 must be considered. After dismissing the columns of $M(d^G)$ corresponding to the standard monomials of the roots at infinity, one obtains $M^\star(d^G)$ and correspondingly $K^\star(d_G)$, which can be used in Theorem 6.7 to find the affine solutions.

The complete column space based algorithm for finding the affine roots is summarized in Algorithm 2.

ALGORITHM 2. (Affine column space based root-finding)

input: *a system of n equations f_i having total degrees d_i in n unknowns x_i*
output: *the m_a affine solutions for a user-chosen shift function $g(x)$*

1. *Determine $B^\star(d_G)$ and set $m_a = \#B^\star(d_G)$*
2. *Remove from $M := M(d_G)$ the columns corresponding to the roots at infinity, leading to $\overline{M}\ \overline{K} = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = 0$ where K_1 represents the m_a affine standard monomials*
3. *The trivial and non-trivial shifts are determined for the given shift function $g(x)$*
4. *Partition $\overline{M}$ into $(M_{22}\ M_{21}\ M_1)$, where M_1 contains the linear dependent columns, M_{21} contains the non-trivial shift monomials, and M_{22} contains the remaining columns.*
5. *Compute the QR decomposition of $(M_{22}\ M_{21}\ M_1)$*

$$\begin{pmatrix} M_{22} & M_{21} & M_1 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ & R_{22} & R_{23} \\ & & R_{33} \end{pmatrix} \tag{6.14}$$

6. *The m_a roots evaluated at the shift function $g(x)$ are found from*

$$K_1 D_g = \left(\Sigma_1 - \Sigma_2' R_{22}^{-1} R_{23}\right) K_1 \tag{6.15}$$

7. *The mutual matching between the solution components x_i is reconstructed by computing the eigenvectors of $\Sigma_1 - \Sigma_2' R_{22}^{-1} R_{23}$ and rescaling them such that the entries $\prod_{i=1}^n x_i^0 = 1$ are scaled to 1*

EXAMPLE 6.8. *We revisit Example 6.6 and consider the column reordering (6.9). Again we choose the polynomial $g(x_1, x_2, x_3) := x_1 + 2x_2 + 3x_3$. The monomials reached by multiplying the standard monomials $B^\star$ with $g(x)$ are*

$$\{x_1x_2, x_2^2, x_2x_3, x_3^2, x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_3^2\},$$

which index the columns of M_{22} in the column reordering

$$\begin{pmatrix} M_{22} & M_{21} & M_1 \end{pmatrix}.$$

The eigenvalue problem $K_1 D_g = \left(\Sigma_1 - \Sigma_2' R_{22}^{-1} R_{23}\right) K_1$ is solved and the rescaling of the eigenvector correctly reveals the six affine solutions (with absolute errors of the order 10^{-13}).

7. Conclusions and Open Problems. We have presented a framework to tackle polynomial system solving from the linear algebra point-of-view. Two root-finding algorithms have been developed which rephrase the task at hand as eigendecompositions, from which all solutions can be obtained.

In future work, the relation of the multiplicity of a solution to the multiple eigenvalue problem as in [9, 30, 31] will be worked out in the presented framework. Another open research problem is the analysis and solution of overdetermined polynomial systems, for which the presented framework seems a natural starting point. Moreover, the case of real root-finding and directly solving for the minimizing root of a certain objective is of particular interest in polynomial optimization; the work by [19, 22, 23, 36] is likely to be relevant in this context.

Finally it must be emphasized that nearly all methods in polynomial system solving and polynomial optimization suffer from the well-known dramatic increases of

number of monomials as more variables or higher degrees are considered. Although the monomial expansion is an inherent property of the task, it is of great interest to alleviate these effects by considering the relation to sparse resultants [12], and devising ways to exploit the quasi-Toeplitz structure and sparsity in the matrix computations.

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