On Polynomial System Solving and Multidimensional Realization Theory

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To the memory of Jan Willems.
A giant. A mentor. A friend.

Abstract

We describe how systems of multivariate polynomial equations can be solved by formulating the task as a problem in multidimensional realization theory. The Macaulay matrix formulation is used to represent the system of polynomial equations in a linear algebra setting. The null space of the Macaulay matrix, which is spanned by Vandermonde vectors constructed from the roots, is a multidimensional observability matrix. This allows for a natural representation of the problem in realization theory. The formulation leads to an eigenvalue-based root-finding algorithm. By considering the problem in the projective space, a description is obtained for both the affine solutions as well as the solutions at infinity. The latter lead to an interpretation as multidimensional descriptor system realizations.

Keywords: realization theory, polynomial system solving, multidimensional systems, eigenvalue problems,

1. Introduction

Univariate polynomials play an important role in the system theory of one-dimensional linear dynamical systems that are governed by difference or differential equations. The link between the defining difference or differential equations and polynomials is classically made by means of the Laplace transform or the Z-transform [18]. The roots of the corresponding polynomials have the system theoretical meaning of zeros or poles of the system, and are central in understanding the dynamics.

Multidimensional systems can be treated in a similar way, the analysis of which involves multivariate polynomials. The last decades have witnessed an increased research activity in multidimensional systems [1, 5, 6, 11, 12, 16], but a simple and accessible unified theory is lacking. Perhaps one of the causes is that the algebraic methods to deal with multivariate polynomials are more intricate and are largely based on symbolic computer algebra or differential algebra [7, 16].

The current article aims at exploring multidimensional systems from a linear algebra point-of-view. We will highlight natural and accessible links between multivariate polynomials and multidimensional realization theory that are, surprisingly, rather unknown. In particular, we will illustrate that finding the roots of a system of multivariate polynomial equations can be solved using tools of numerical linear algebra and realization theory. In the language of system theory, the root-finding approach boils down to constructing a state-space realization of a set of multidimensional difference equations.

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Earlier work by Hanzon and Hazewinkel [17] led to the observation that a system of multivariate polynomial equations and its corresponding Groebner basis have a straightforward interpretation as in terms of the state-space realization of the associated difference equations. The current article is in the same spirit, but we will use the link with realization theory to devise an eigenvalue-based root-finding method, avoiding the use of Groebner bases.

We have aimed to keep the exposition as simple and pure as possible, requiring only the most elementary notions of linear algebra and state-space system theory. Throughout the paper we will consider systems defined by sets of multidimensional difference equations. We assume that the corresponding system of polynomials describes a zero-dimensional solution set in the projective space [8], a fact that our approach will also reveal as the dimensions of certain null spaces will stabilize in the case the solution set is zero-dimensional.

2. Univariate polynomials and one-dimensional systems

In the current section we will review 'by example' a few well-known facts from linear algebra and system theory that tie together polynomial root-finding and realization theory. These examples serve to introduce the tools we will use in the remainder of the paper.

Example 1. *With a difference equation* $w[k+2] - 3w[k+1] + 2w[k] = 0$*, one may associate the polynomial equation* $p(x) = x^2 - 3x + 2 = 0$*. The roots of* $p(x)$ *are* $x^{(1)} = 1$ *and* $x^{(2)} = 2$ *and they can be computed by means of linear algebra. We write* $p(x)$ *as its coefficient matrix multiplied by a Vandermonde*

monomial vector $\boldsymbol{v}$ *as* $\boldsymbol{p}^\top \boldsymbol{v} = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & x^2 \end{bmatrix}^\top$. *The two solutions generate two vectors that span the right null space of* $\boldsymbol{p}^\top$, *which we call the Vandermonde basis of the null space* $\boldsymbol{V}$. *We have* $\boldsymbol{p}^\top \boldsymbol{V} = \boldsymbol{0}^\top$ *with*

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 \\ x^{(1)} & x^{(2)} \\ (x^{(1)})^2 & (x^{(2)})^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}. \tag{1}$$

The Vandermonde basis of the null space $\boldsymbol{V}$ *has a multiplicative shift structure, allowing us to write*

$$\underline{\boldsymbol{V}} \boldsymbol{D} = \overline{\boldsymbol{V}}, \tag{2}$$

where $\boldsymbol{D} = \operatorname{diag}(x^{(1)}, x^{(2)})$ *and* $\overline{\boldsymbol{V}}$ *and* $\underline{\boldsymbol{V}}$ *denotes* $\boldsymbol{V}$ *with its first and last row removed, respectively. This is a direct application of realization theory in the null space of* $\boldsymbol{p}^\top$.

It should be obvious that the shift structure is a property of the column space of $\boldsymbol{V}$, *and is independent of the choice of basis. Thus, the shift relation holds for any basis of the null space* $\boldsymbol{Z}$, *which is related to* $\boldsymbol{V}$ *by a nonsingular transformation* $\boldsymbol{T}$ *as* $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}$, *leading to the generalized eigenvalue equation*

$$\underline{\boldsymbol{Z}} \boldsymbol{T} \boldsymbol{D} = \overline{\boldsymbol{Z}} \boldsymbol{T}. \tag{3}$$

Another choice of basis is obtained by putting a numerical basis of the null space $\boldsymbol{Z}$ *in its column echelon form* $\boldsymbol{H}$ *(Appendix A). The column echelon form is related to* $\boldsymbol{V}$ *by the relation* $\boldsymbol{V} = \boldsymbol{H}\boldsymbol{U}$, *with*

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4}$$

Notice that the columns of $\boldsymbol{U}$ *have the form* $\begin{bmatrix} 1 & x \end{bmatrix}^\top$, *evaluated in the solutions* $x^{(1)} = 1$ *and* $x^{(2)} = 2$. *The shift relation is*

$$\boldsymbol{U}\boldsymbol{D} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \boldsymbol{U}. \tag{5}$$

Remark that, when one uses the column echelon basis of the null space $\boldsymbol{H}$, *the construction results in the well-known Frobenius companion matrix.*

The fact that for the column echelon basis of the null space $\boldsymbol{H}$ *the eigenvectors are monomial vectors allows for a direct interpretation as a state-space realization of the difference equation* $w[k+2] - 3w[k+1] + 2w[k] = 0$. *We have*

$$\begin{bmatrix} w[k+1] \\ w[k+2] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}. \tag{6}$$

Additionally, this prescribes how $w[k]$ *can be computed from initial conditions* $w[0]$ *and* $w[1]$.

The same procedure can be used to find a trajectory w that satisfies a set of difference equations. In terms of polynomial algebra this is equivalent to finding the greatest common divisor of a set of univariate polynomials. We will illustrate this in the following example.

Example 2. *Consider two difference equations*

$$\begin{aligned} w[k+3] + 2w[k+2] - 5w[k+1] - 6w[k] &= 0, \\ w[k+2] - w[k+1] - 2w[k] &= 0, \end{aligned} \tag{7}$$

that can be associated with the polynomial equations

$$\begin{aligned} p(x) &= x^3 + 2x^2 - 5x - 6 = 0, \\ q(x) &= x^2 - x - 2 = 0. \end{aligned} \tag{8}$$

The common roots of p *and* q *are* $x^{(1)} = -1$ *and* $x^{(2)} = 2$. *In order to describe trajectories* $w[k]$ *that satisfy both equations one needs to find the greatest common denominator (GCD) of* p *and* q. *This leads to the Sylvester matrix construction*

$$\begin{bmatrix} -6 & -5 & 2 & 1 & 0 \\ 0 & -6 & -5 & 2 & 1 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{9}$$

which is obtained by multiplying p *and* q *by powers of* x.

The common roots of p *and* q *give rise to Vandermonde-structured vectors in the null space. From the system theoretic point of view, the Sylvester matrix construction generates additional equations that impose constraints on* $w[k]$. *Remark that* any *vector in the null space of the Sylvester matrix defines a valid trajectory* $w[k]$.

It is important to realize that we have arrived again at the situation where the solution to the problem involves some basis of a null space that has a Vandermonde structure.

The column echelon basis of the null space $\boldsymbol{H}$ *is for this example*

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 2 & 3 \\ 6 & 5 \end{bmatrix}. \tag{10}$$

The shift relation leads to a rectangular generalized eigenvalue problem. We find $\underline{\boldsymbol{H}}\boldsymbol{U}\boldsymbol{D} = \overline{\boldsymbol{H}}\boldsymbol{U}$ *as*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}. \tag{11}$$

It suffices to reduce the above relation to the square eigenvalue problem from which $\boldsymbol{U}$ *and* $\boldsymbol{H}$ *can be obtained as well. A square generalized eigenvalue problem can be obtained by selecting the first linear independent rows of* $\boldsymbol{H}$ *as to obtain a square invertible matrix, leading to*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \tag{12}$$

where again a Frobenius companion matrix shows up: it is the companion matrix of the GCD of p *and* q.

3. Multivariate polynomials: affine case

3.1. Macaulay's construction

We will generalize the results of Section 2 to the multivariate/multidimensional case. A system of multivariate polynomials defines trajectories $w[k_1,\ldots,k_n] \in \mathbb{R}$, for $k_1,\ldots,k_n \in \mathbb{N}$, of a multidimensional system represented in the representation $\boldsymbol{r}(\boldsymbol{x})w = \boldsymbol{0}$ [22, 23, 24], where $\boldsymbol{r} \in \mathbb{R}^{n\times 1}$ and $\boldsymbol{x} = (x_1,\ldots,x_n)$ denotes the multidimensional shift operator

$$x_i : (x_i w)[k_1,\ldots,k_i,\ldots,k_n] = w[k_1,\ldots,k_i+1,\ldots,k_n]. \quad (13)$$

We denote the corresponding system of multivariate polynomial equations as

$$\begin{array}{rcl} f_1(x_1,\ldots,x_n) &=& 0, \\ &\vdots& \\ f_n(x_1,\ldots,x_n) &=& 0, \end{array} \quad (14)$$

having degrees $d_1,\ldots,d_n$. To start with, we assume that (14) has a zero-dimensional solution set. We also assume that the solutions are simple (i.e., no multiplicity) and there are no solutions at infinity. This is for the moment just for didactical purposes, and we will remove these restrictions in Section 3.5 and Section 4. Under these assumptions the number of solutions m is given by the Bezout number $m = \prod_i d_i$ [8].

It is necessary to carefully order monomials, for which we have chosen to use the degree negative lexicographic ordering, but the method can be generalized to any (graded) monomial ordering.

Definition 1 (Degree negative lexicographic order)**.** *Let $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^n$ be monomial exponent vectors. Then two monomials are ordered $\boldsymbol{x}^{\boldsymbol{\alpha}} < \boldsymbol{x}^{\boldsymbol{\beta}}$ by the degree negative lexicographic order if $|\boldsymbol{\alpha}| < |\boldsymbol{\beta}|$, or $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and in the vector difference $\boldsymbol{\beta} - \boldsymbol{\alpha} \in \mathbb{Z}^n$, the left-most non-zero entry is negative.*

Example 3. *The monomials of maximal degree three in two variables are ordered by the degree negative lexicographic order as*

$$1 < x_1 < x_2 < x_1^2 < x_1x_2 < x_2^2 < x_1^3 < x_1^2x_2 < x_1x_2^2 < x_2^3. \quad (15)$$

We will now introduce the Macaulay matrix $\boldsymbol{M}_d$ and the multivariate Vandermonde monomial vector $\boldsymbol{v}_d$ to represent a system of polynomial equations as a system of homogeneous linear equations.

Definition 2 (Vandermonde monomial vector)**.** *The multivariate Vandermonde monomial vector $\boldsymbol{v}_d$ is defined as*

$$\boldsymbol{v} := \left[\begin{array}{c|cccc|ccccc|ccc} 1 & x_1 & x_2 & \ldots & x_n & x_1^2 & x_1x_2 & x_1x_3 & \ldots & x_n^2 & \ldots & x_1^d & \ldots & x_n^d \end{array}\right]^\top. \quad (16)$$

The polynomial $f_i(x_1,\ldots,x_n)$ can in this way be represented as a row vector containing the coefficients multiplied by a Vandermonde vector of a suitable degree as $\boldsymbol{f}_i^\top \boldsymbol{v}$.

The Macaulay matrix contains as its rows such coefficient vectors that are obtained by multiplying the equations $f_i(x_1,\ldots,x_n)$ by monomials such that at most some predefined degree d is not exceeded.

Definition 3 (Macaulay matrix)**.** *The Macaulay matrix $\boldsymbol{M}_d$ contains as its rows the vector representations $\boldsymbol{x}^{\boldsymbol{\alpha}_i}\boldsymbol{f}_i^\top$ as follows*

$$\boldsymbol{M}_d := \left[\begin{array}{c} \{\boldsymbol{x}^{\boldsymbol{\alpha}_1}\}\,\boldsymbol{f}_1^\top \\ \vdots \\ \{\boldsymbol{x}^{\boldsymbol{\alpha}_n}\}\,\boldsymbol{f}_n^\top \end{array}\right]. \quad (17)$$

where each f_i, for $i = 1,\ldots,n$ is multiplied by all monomials $\boldsymbol{x}^{\boldsymbol{\alpha}_i}$ of degrees $\leq d - d_i$.

The rows of the Macaulay matrix for degree d represent polynomial consequences of the polynomials $f_1,\ldots,f_n$ that can be obtained by elementary row operations.[1]

For the case there are only affine roots, the Macaulay matrix is constructed for degree $d = \sum_i d_i - n + 1$ [15, 19]. Dependence of $\boldsymbol{M}_d$ and $\boldsymbol{v}_d$ on d is often left out for notational convenience, i.e., $\boldsymbol{M} = \boldsymbol{M}_d$ and $\boldsymbol{v} = \boldsymbol{v}_d$.

It is important to observe that every solution of (14), denoted $\left(x_1^{(k)},\ldots,x_n^{(k)}\right)$, for $k = 1,\ldots,m$, gives rise to a Vandermonde vector

$$\left[\begin{array}{c|ccc|ccc} 1 & x_1^{(k)} & \cdots & x_n^{(k)} & \cdots & (x_1^{(k)})^d & \cdots & (x_1^{(k)})^d \end{array}\right]^\top \quad (18)$$

in the null space of $\boldsymbol{M}$. The collection of all such vectors into matrix $\boldsymbol{V}$ is called the Vandermonde basis of the null space $\boldsymbol{V}$.

Example 4. *Consider the system of equations*

$$\begin{array}{rcl} f_1(x_1,x_2) &=& 4x_1^2 - 16x_1 + x_2^2 - 2x_2 + 13 \;=\; 0, \\ f_2(x_1,x_2) &=& 2x_1 + x_2 - 7 \;=\; 0, \end{array} \quad (19)$$

which describes the intersection of an ellipse and a line. It can be verified that the solutions are $\left(x_1^{(1)},x_2^{(1)}\right) = (3,1)$ and $\left(x_1^{(2)},x_2^{(2)}\right) = (2,3)$. The system is represented as

$$\left[\begin{array}{rr|rr|ccc} 13 & -16 & -2 & 1 & 4 & 0 & 1 \\ -7 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -7 & 0 & 0 & 2 & 1 \end{array}\right] \left[\begin{array}{c} 1 \\ \hline x_1 \\ x_2 \\ \hline x_1^2 \\ x_1x_2 \\ x_2^2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]. \quad (20)$$

The Macaulay matrix has dimensions 4×6, rank four and nullity two. The Vandermonde basis of the null space is

$$\boldsymbol{V} = \left[\begin{array}{cc} 1 & 1 \\ \hline x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \\ \hline x_1^{(1)}x_1^{(1)} & x_1^{(2)}x_1^{(2)} \\ x_1^{(1)}x_2^{(1)} & x_1^{(2)}x_2^{(2)} \\ x_2^{(1)}x_2^{(1)} & x_2^{(2)}x_2^{(2)} \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ \hline 2 & 3 \\ 3 & 1 \\ \hline 4 & 9 \\ 6 & 3 \\ 9 & 1 \end{array}\right], \quad (21)$$

where the columns are multivariate Vandermonde monomial vectors evaluated at the two solutions $(2,3)$ and $(3,1)$.

[1] Remark that the row span of the Macaulay matrix does *not* necessarily coincide with the elements of the ideal $I = \langle f_1,\ldots,f_n\rangle$ of degree d or less (denoted $I_{\leq d}$) [8]. It is possible that by reductions of degree $\Delta > d$ polynomials, degree $\delta \leq d$ equations are are obtained that cannot be reached by the row space of total degree $\delta \leq d$ shifts of the f_i.

3.2. System theoretic interpretation

The Macaulay construction has a natural interpretation in multidimensional system theory. Polynomials are associated with difference equations through the use of the (multidimensional) shift operator $\boldsymbol{x} = (x_1, \ldots, x_n)$ defined in (13). A system of n multivariate polynomial equations in n equations can thus be associated with a set of n difference equations

$$\boldsymbol{r}(\boldsymbol{x})w = \boldsymbol{0}, \tag{22}$$

where $\boldsymbol{r}(\boldsymbol{x}) \in \mathbb{R}^{n\times 1}$ is a vector with polynomial entries. The Macaulay matrix construction can be interpreted as a way to generate equations that $w[k_1, \ldots, k_n]$ has to satisfy: the rows of the Macaulay matrix are shifts of the difference equations (22). Components of a vector in the null space of the Macaulay matrix can be in this way be seen as shifted $w[k_1, \ldots, k_n]$.

Example 5. *The system in Example 4 can be associated with the difference equations*

$$\begin{aligned} 4w[k_1+2,k_2] - 16w[k_1+1,k_2] + w[k_1,k_2+2] \\ -2w[k_1,k_2+1] + 13w[k_1,k_2] &= 0, \\ 2w[k_1+1,k_2] + w[k_1,k_2+1] - 7w[k_1,k_2] &= 0. \end{aligned} \tag{23}$$

The above equations, shifted up to indices $k_i + 2$, are expressed by the Macaulay matrix $\boldsymbol{M}_2$. We have

$$\begin{bmatrix} 13 & -16 & -2 & 4 & 0 & 1 \\ -7 & 2 & 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 2 & 1 & 0 \\ 0 & 0 & -7 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} w[k_1,k_2] \\ w[k_1+1,k_2] \\ w[k_1,k_2+1] \\ w[k_1+2,k_2] \\ w[k_1+1,k_2+1] \\ w[k_1,k_2+2] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

Trajectories $w[k_1, k_2]$ that are compatible with (23) are linear combinations of the two basis vectors in (21).

3.3. From shift structure to eigendecompositions

Let us study the multiplicative shift structure of the null space. Multiplication by monomial x_i maps a degree δ monomial to a degree $\delta + 1$ monomial. In general this is expressed in the Vandermonde monomial vector $\boldsymbol{v}$ as $\boldsymbol{S}_0\boldsymbol{v}x_i = \boldsymbol{S}_i\boldsymbol{v}$, where $\boldsymbol{S}_0$ selects monomial rows of degrees 0 through $d-1$ and $\boldsymbol{S}_i$ selects the rows onto which they are mapped by multiplication with x_i. The shift relation for the entire Vandermonde basis of the null space $\boldsymbol{V}$ is

$$\boldsymbol{S}_0\boldsymbol{V}\boldsymbol{D}_i = \boldsymbol{S}_i\boldsymbol{V}, \tag{25}$$

where $\boldsymbol{D}_i = \operatorname{diag}\left(x_i^{(1)}, \ldots, x_i^{(m)}\right)$ contains on the diagonal the evaluation of the shift monomial x_i at the m roots.

In general, the Vandermonde basis of the null space $\boldsymbol{V}$ cannot be obtained directly, but instead a numerical basis $\boldsymbol{Z}$ can be computed, for instance with the singular value decomposition (see Appendix A). The shift relation (25) holds for any basis of the null space $\boldsymbol{Z}$, which leads to the affine root-finding procedure.

Theorem 1 (Affine root-finding)**.** *Let $\boldsymbol{Z}$ be a basis of the null space of $\boldsymbol{M}$, which is related to the Vandermonde basis by $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}$. The shift relation (25) reduces to the generalized eigenvalue problem*

$$\boldsymbol{S}_0\boldsymbol{Z}\boldsymbol{T}\boldsymbol{D}_i\boldsymbol{T}^{-1} = \boldsymbol{S}_i\boldsymbol{Z}, \tag{26}$$

where $\boldsymbol{S}_0$ selects the rows of $\boldsymbol{Z}$ that correspond to the monomials of degrees 0 through $d-1$, and $\boldsymbol{S}_i$ selects the rows onto which these monomials are mapped under multiplication by x_i. The eigenvalues (i.e., the diagonal elements of $\boldsymbol{D}_i$) correspond to the x_i components of the solutions of (14).

Remark that $\boldsymbol{S}_0\boldsymbol{Z}$ needs to have full column rank in order to ensure that the eigenvalue problem is not degenerate (i.e., it has infinite eigenvalues). In general $\boldsymbol{S}_0\boldsymbol{Z}$ is nonsquare (tall), which leads to a rectangular generalized eigenvalue problem. We can convert it to a square regular eigenvalue problem by means of the pseudoinverse as $(\boldsymbol{S}_0\boldsymbol{Z})^\dagger \boldsymbol{S}_i\boldsymbol{Z} = \boldsymbol{T}\boldsymbol{D}_i\boldsymbol{T}^{-1}$.

Corollary 1 (Reconstructing Vandermonde basis of the null space from $\boldsymbol{Z}$)**.** *The Vandermonde basis of the null space $\boldsymbol{V}$ can be recovered (up to column-wise scaling) from*

$$\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}, \tag{27}$$

in which all solutions $(x_1^{(k)}, \ldots, x_n^{(k)})$ can be read off.

Let us conclude by reviewing the key properties of the null space of the Macaulay matrix $\boldsymbol{M}$. First, for any choice of basis of the null space, we can use the shift structure to set up an eigenvalue problem that will reveal the common roots. Special choices are the Vandermonde basis $\boldsymbol{V}$, in which case the eigenvectors are the columns of the identity matrix, and the column echelon basis $\boldsymbol{H}$, in which case the shift matrix is in Frobenius companion form.

Another peculiar property is that the indices of the linearly independent rows, starting from the top, are the same for any choice of basis in the null space of the Macaulay matrix. Because of the second property, a generalized square eigenvalue problem can be obtained by letting $\boldsymbol{S}_0$ select the m first linearly independent rows of the null space.

Remark that the column echelon basis will not be used in practice; instead, one employs a numerical basis of the null space $\boldsymbol{Z}$ to compute the roots, which can reliably be computed using the singular value decomposition (Appendix A).

3.4. Multidimensional realization

Let us again consider the column echelon basis of the null space, which we will denote by $\boldsymbol{H}$. In $\boldsymbol{H}$ each of the m columns contains as the first nonzero element a "1" in the rows that correspond to linearly independent monomials. More specifically, they are the lowest-degree linearly independent monomials, ordered by the degree negative lexicographic order. It can be verified that the transformation $\boldsymbol{U}$ has a particular structure in this case.

Proposition 1. *Let $\boldsymbol{V} = \boldsymbol{H}\boldsymbol{U}$ express the relation between the column echelon basis of the null space $\boldsymbol{H}$ and the Vandermonde*

basis of the null space $\boldsymbol{V}$*. The k-th column of* $\boldsymbol{U}$ *is a monomial vector containing the linearly independent monomials, evaluated at the k-th solution* $\left(x_1^{(k)}, \ldots, x_n^{(k)}\right)$.

Proof. Let $\boldsymbol{x}^{\alpha_1}, \boldsymbol{x}^{\alpha_2}, \ldots, \boldsymbol{x}^{\alpha_m}$ denote the linearly independent monomials. Then for a single Vandermonde vector $\boldsymbol{v}$ we have $\boldsymbol{v} = \boldsymbol{H}\boldsymbol{u}$ such that

$$\begin{bmatrix} \boldsymbol{x}^{\alpha_1} \\ \vdots \\ \boldsymbol{x}^{\alpha_2} \\ \vdots \\ \boldsymbol{x}^{\alpha_m} \\ \vdots \\ \boldsymbol{x}^{\alpha_{m-1}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \textcircled{1} & 0 & \cdots & 0 & 0 \\ \times & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & \textcircled{1} & \ddots & \vdots & \vdots \\ \times & \times & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \textcircled{1} & 0 \\ \times & \cdots & \times & \times & \mathbf{0} \\ 0 & \cdots & 0 & 0 & \textcircled{1} \\ \times & \cdots & \times & \times & \times \end{bmatrix} \begin{bmatrix} \boldsymbol{x}^{\alpha_1} \\ \boldsymbol{x}^{\alpha_2} \\ \vdots \\ \boldsymbol{x}^{\alpha_{m-1}} \\ \boldsymbol{x}^{\alpha_m} \end{bmatrix}, \tag{28}$$

where the circled ones are in the linearly independent monomial rows. In the affine case we have $\boldsymbol{x}^{\alpha_1} = 1$. □

The column echelon basis of the null space allows for a natural interpretation in a multidimensional systems setting.

Theorem 2 (Canonical realization)**.** *The difference equations* $\boldsymbol{r}(z)w = \boldsymbol{0}$ *admit the state-space realization*

$$\begin{aligned} \boldsymbol{\xi}_{\boldsymbol{H}}[k_1+1, k_2, \ldots, k_{n-1}, k_n] &= \boldsymbol{A}_1 \boldsymbol{\xi}_{\boldsymbol{H}}[k_1, \ldots, k_n], \\ &\vdots \\ \boldsymbol{\xi}_{\boldsymbol{H}}[k_1, k_2, \ldots, k_{n-1}, k_n+1] &= \boldsymbol{A}_1 \boldsymbol{\xi}_{\boldsymbol{H}}[k_1, \ldots, k_n], \end{aligned} \tag{29}$$

where the $\boldsymbol{A}_i$*, for* $i = 1, \ldots, n$*, are defined as*

$$\boldsymbol{A}_i := (\boldsymbol{S}_0 \boldsymbol{H})^{\dagger} \boldsymbol{S}_i \boldsymbol{H}, \tag{30}$$

where $\boldsymbol{H}$ *denotes the column echelon basis of the null space, and* $\boldsymbol{\xi}_{\boldsymbol{H}}[k_1, \ldots, k_n]$ *contain the* $w[k_1, \ldots, k_n]$ *corresponding to the linearly independent monomials (Proposition 1). The initial conditions that are necessary to iterate the state-space realization (29) can be read off immediately from* $\boldsymbol{\xi}_{\boldsymbol{H}}[0, \ldots, 0]$.

Example 6. *We revisit Example 5. The column echelon basis of the null space* $\boldsymbol{H}$ *is computed as*

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 7 & -2 \\ -6 & 5 \\ 12 & -3 \\ 25 & -8 \end{bmatrix}, \tag{31}$$

in which the first two rows are corresponding to the linearly independent monomials 1 *and* x_1*. A state-space realization can be read off as*

$$\begin{bmatrix} w[k_1+1, k_2] \\ w[k_1+2, k_2] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} w[k_1, k_2] \\ w[k_1+1, k_2] \end{bmatrix}$$

$$\begin{bmatrix} w[k_1, k_2+1] \\ w[k_1+1, k_2+1] \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} w[k_1, k_2] \\ w[k_1+1, k_2] \end{bmatrix}. \tag{32}$$

The equivalence of polynomial system solving and eigenproblems is central in the Stetter approach [2, 21], where the matrices $\boldsymbol{A}_i := (\boldsymbol{S}_0 \boldsymbol{H})^{\dagger} \boldsymbol{S}_i \boldsymbol{H}$ correspond to the multiplication matrices that represent multiplication by x_i in the quotient space $\mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$. The linearly independent monomials $\boldsymbol{x}^{\alpha_i}$ correspond to the Groebner basis normal set elements, which form a basis of the quotient space, as discussed in [3, 10].

3.5. Multiple roots

Let us briefly discuss the case of multiple roots. For a system of multivariate polynomials, a μ-fold solution $\boldsymbol{x}^{\star}$ gives rise to a null space spanned by linear combinations of vectors of the form

$$\frac{1}{\alpha_1! \cdots \alpha_n!} \frac{\partial^{\alpha_1 + \cdots + \alpha_n} \boldsymbol{v}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}\bigg|_{\boldsymbol{x}^{\star}}, \tag{33}$$

where the factor $(\alpha_1! \cdots \alpha_n!)^{-1}$ serves as a normalization. For a thorough treatment of the so-called dual space of $\boldsymbol{M}$, we refer to [4, 9]. The shift relation in the Vandermonde basis involves in the case of multiple roots a Jordan-like normal form

$$\boldsymbol{S}_0 \boldsymbol{V} \boldsymbol{J}_i = \boldsymbol{S}_i \boldsymbol{V}, \tag{34}$$

where $\boldsymbol{J}_i$ is uppertriangular with x_i, evaluated at the m roots, on the diagonal. Some uppertriangular elements of $\boldsymbol{J}_i$ are nonzero, which can be analyzed by inspection of $\boldsymbol{V}$. In practice, the computation of the Jordan normal form is numerically ill-posed, but can be avoided by computing a Schur decomposition [4].

Let us illustrate the multiplicity structure of the null space for a system of polynomial equations having a four-fold root.

Example 7. *Consider the equations*

$$\begin{aligned} f_1(x_1, x_2) &= (x_2 - 2)^2 &&= 0, \\ f_2(x_1, x_2) &= (x_1 - x_2 + 1)^2 &&= 0, \end{aligned}$$

having a four-fold solution $(1, 2)$*. It can be verified that* $\boldsymbol{M}$ *has a four-dimensional null space that is spanned by the vectors* ∂_{00}*,* ∂_{10}*,* ∂_{01} *and* $2\partial_{20} + \partial_{11}$*, where we use a simplified notation* $\partial_{\alpha_1 \cdots \alpha_n}$ *for (33). We have*

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ x_1^2 & 2x_1 & 0 & 2 \\ x_1 x_2 & x_2 & x_1 & 1 \\ x_2^2 & 0 & 2x_2 & 0 \\ x_1^3 & 3x_1^2 & 0 & 6x_1 \\ x_1^2 x_2 & 2x_1 x_2 & x_1^2 & 2x_2 + 2x_1 \\ x_1 x_2^2 & x_2^2 & 2x_1 x_2 & 2x_2 \\ x_2^3 & 0 & 3x_2^2 & 0 \end{bmatrix},$$

with $x_1 = 1$ *and* $x_2 = 2$*.It can be verified that this leads to generalized shift relations* $\boldsymbol{S}_0 \boldsymbol{V} \boldsymbol{J}_i = \boldsymbol{S}_i \boldsymbol{V}$*, for* $i = 1, 2$*, where*

$$\boldsymbol{J}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textit{and} \quad \boldsymbol{J}_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{35}$$

4. Projective case and descriptor systems

In this section we will shift gears and take a closer look at, and refine where necessary, some of the results that we have obtained so far. We will study roots at infinity, which are 'special' solutions that are caused by algebraic relations among the coefficients. The system theoretic interpretation will lead to multidimensional descriptor systems.

4.1. Roots at infinity

Solutions at infinity are caused by algebraic relations among the coefficients (often due to the occurence of zero coefficients) in the equations (14). The Bezout number $m = \prod_i d_i$ counts both affine solutions and solutions at infinity, including multiplicities. Roots at infinity can be analyzed by embedding the system into the $(n+1)$-dimensional projective space. A homogenization variable x_0 is introduced to lift each of the terms of equations f_i to the same total degree d_i, denoted by f_i^h. The system of homogeneous equations $f_1^h(x_0, x_1, \ldots, x_n) = \cdots = f_n^h(x_0, x_1, \ldots, x_n) = 0$ describes a projective variety, where for affine roots $x_0 = 1$, while for roots at infinity $x_0 = 0$.

Example 8. *Consider the equations*

$$\begin{array}{rcl} f_1(x_1, x_2) & = & x_2 - x_1^2 \\ f_2(x_1, x_2) & = & x_1 - 3 \end{array} \begin{array}{c} = \\ = \end{array} \begin{array}{l} 0, \\ 0, \end{array} \tag{36}$$

which has a single affine solution $(3, 9)$*. Notice that the Bezout number is* $m = 2$*, which indicates that the system has two solutions in the projective case. Indeed, by homogenizing the system we have*

$$\begin{array}{rcl} f_1^h(x_0, x_1, x_2) & = & x_0 x_2 - x_1^2 \\ f_2^h(x_0, x_1, x_2) & = & x_1 - 3x_0 \end{array} \begin{array}{c} = \\ = \end{array} \begin{array}{l} 0, \\ 0, \end{array} \tag{37}$$

where the homogenization variable x_0 *is introduced. This system has two solutions: an affine solution* $(1, 3, 9)$ *and a solution at infinity* $(0, 0, 1)$*.*

The following proposition is essential to understand that roots at infinity are naturally showing up in the Macaulay construction.

Proposition 2. *The homogeneous Macaulay coefficient matrix, built from the homogeneous system* $f_1^h = \cdots = f_n^h = 0$ *is identical to the Macaulay matrix* $\boldsymbol{M}$ *in Definition 3.*

This fact can be understood by considering the monomials in $n = 2$ variables of degree $d = 3$ for both the nonhomogeneous and the homogeneous case. We have

$$\left\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\right\},$$

and

$$\left\{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_0x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\right\},$$

both of which are ordered by the degree negative lexicographic ordering. They are element-wise equal up to the substitution $x_0 = 1$.

Due to the property that $x_0 = 0$, roots at infinity give rise to linearly independent monomials of (affine) degree d. This can appreciated from the fact that a substitution $x_0 = 0$ in the homogenized system eliminates all but the terms of degree d.

Keeping in mind that the all variables $x_0, x_1, \ldots, x_n$ in the projective space are treated on equal footing, it is more suitable partition roots into regular and singular, denoted by subscripts R and S, respectively. For affine root finding, the regular roots are the affine roots, while the singular roots are the roots at infinity.

A simple mechanism to separate affine roots and roots at infinity is hence investigating the indices of the linearly independent monomials as d increases. The m_R affine linearly independent monomials are of degrees $\delta \leq d_R$ and stabilize as d increases. The m_S roots at infinity give rise to linearly independent monomials of degrees $\delta \geq d_S$ with $d_S > d_R$ that move along to higher degrees as d increases. This can be seen as a multidimensional variation of the Cayley-Hamilton theorem.

Proposition 3 (Mind-the-gap [10])**.** *Let the Bezout number* $m = m_R + m_S$*, where* m_R *denotes the number of affine roots, and* m_S *denotes the number of roots at infinity. Furthermore,* d_R *and* d_S *are the degrees defined as above. The degree at which* $d_S > d_R$*, i.e., an (affine) degree block is in between the linearly independent monomials corresponding to the affine roots and projective roots, is called the degree of regularity* $d^\star$*.*

Remark that linear dependence of monomials (rows of the basis of the null space) does not need to be checked row by row in order to determine $d^\star$. It suffices to monitor for a given degree d the increase in (numerical) rank of the (affine) degree δ blocks of the basis of the null space $\boldsymbol{Z}$, for $\delta = 0, \ldots, d$. As soon as the rank does not increase in consecutive degree blocks, we have $d^\star = d$.

4.2. Projective shift relation

Let us investigate how the shift relation can be generalized to the homogeneous case. The exposition is similar to that of [4], but we review here the main facts.

Proposition 4 (Homogeneous shift relation)**.** *Let* $\boldsymbol{V}$ *denote the homogeneous Vandermonde basis of the null space. We can write the shift relation as*

$$\boldsymbol{S}_{i/j}\boldsymbol{V}\boldsymbol{D}_i = \boldsymbol{S}_{j/i}\boldsymbol{V}\boldsymbol{D}_j, \tag{38}$$

with row selection matrices $\boldsymbol{S}_{i/j}$ *and* $\boldsymbol{S}_{j/i}$ *describing an up shift in* x_i *and a down shift in* x_j*, and* $\boldsymbol{D}_i$ *and* $\boldsymbol{D}_j$ *are diagonal matrices with the values of* x_i *and* x_j *on the diagonals.*

Figure 1 is a visual representation of the shift relation for the $n = 3$ and $d = 3$ case.

For a numerical basis of the null space $\boldsymbol{Z}$ with $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{T}$ this leads to the homogeneous eigenvalue problem

$$\boldsymbol{S}_{i/j}\boldsymbol{Z}\boldsymbol{T}\boldsymbol{D}_i = \boldsymbol{S}_{j/i}\boldsymbol{Z}\boldsymbol{T}\boldsymbol{D}_j. \tag{39}$$

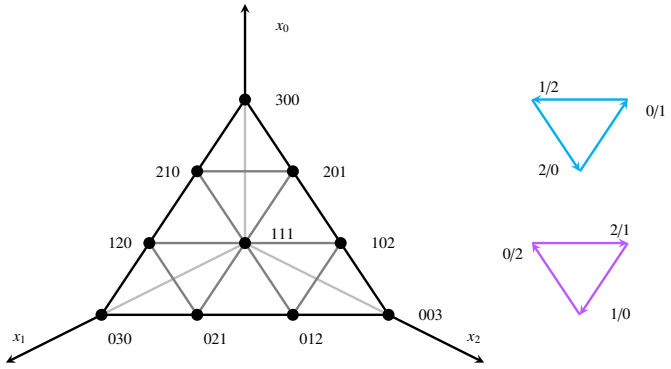

Figure 1: Shifts in the three-dimensional monomial grid. The black dots represent the homogeneous monomials of degree three. The label *abc* represents monomial $x_0^a x_1^b x_2^c$. The "i/j" shift relation in the homogeneous case amounts to increasing the exponent of x_i by one, and decreasing the exponent of x_j by one. In the monomial grid a shift is a move from a point to an adjacent point. The six possible shifts, i.e., 0/1, 0/2, 1/0, 1/2, 2/0 and 2/1, are denoted by the arrows. Notice that not all shifts are possible: monomials along the edges have exponents equal to zero that cannot be decreased.

Example 9. *The "1/2" shift relation of Proposition 4 for the case $n = 2$ and $d = 2$ is*

$$\begin{aligned}\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0^2x_1^0x_2^0 \\ x_0^1x_1^1x_2^0 \\ x_0^1x_1^0x_2^1 \\ x_0^0x_1^2x_2^0 \\ x_0^0x_1^1x_2^1 \\ x_0^0x_1^0x_2^2 \end{bmatrix} x_1 \\ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_0^2x_1^0x_2^0 \\ x_0^1x_1^1x_2^0 \\ x_0^1x_1^0x_2^1 \\ x_0^0x_1^2x_2^0 \\ x_0^0x_1^1x_2^1 \\ x_0^0x_1^0x_2^2 \end{bmatrix} x_2.\end{aligned} \tag{40}$$

As long as either $x_i \neq 0$ or $x_j \neq 0$ for all roots, the homogeneous eigenvalue problem (39) can always be reduced to an affine eigenvalue problem. If $x_j \neq 0$ we can write

$$\boldsymbol{S}_{i/j}\boldsymbol{Z}\boldsymbol{T}\boldsymbol{D}_{i/j}\boldsymbol{T}^{-1} = \boldsymbol{S}_{j/i}\boldsymbol{Z}, \tag{41}$$

with $\boldsymbol{D}_{i/j} = \boldsymbol{D}_i\boldsymbol{D}_j^{-1}$. However, if $x_j = 0$ then inverting $\boldsymbol{D}_j$ is not possible.

Remark 1. *In the case of affine root-finding (Section 3), we consider shifts up in the affine components $x_1, x_2, \ldots, x_n$, whereas roots at infinity are characterized by $x_0 = 0$, which is the degenerate component that is being shifted down.*

In agreement with the separation of the linearly independent monomials into regular and singular roots, the column echelon basis of the null space $\boldsymbol{H}$ can be partitioned accordingly. We have

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_R^{(1)} & \boldsymbol{0} \\ \boldsymbol{H}_R^{(2)} & \boldsymbol{0} \\ \times & \boldsymbol{H}_S \end{bmatrix}, \tag{42}$$

where $\boldsymbol{H}_R^{(1)}$ and $\boldsymbol{H}_S$ denote the (affine) degree blocks containing the regular and singular linearly independent monomials, respectively, and $\boldsymbol{H}_R^{(2)}$ denotes the (affine) degree block where there are no linearly independent monomials.

For a numerical basis of the null space $\boldsymbol{Z}$ the regular and singular columns are not separated. We perform a column compression to find $\boldsymbol{Z}_R$ having m_R columns and rank m_R on which the affine root-finding procedure can be applied.

Corollary 2 (Root-finding in the presence of singular roots)**.** *Consider a column compression (Appendix A) of the degree-blocks of $\boldsymbol{Z}$ corresponding to the $\boldsymbol{H}_R^{(1)}$ and $\boldsymbol{H}_R^{(2)}$, where*

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_R^{(1)} & \boldsymbol{0} \\ \boldsymbol{Z}_R^{(2)} & \boldsymbol{0} \\ \times & \times \end{bmatrix}, \tag{43}$$

where $\boldsymbol{Z}_R$ has m_R columns and is compatible with the shift relation involving the affine linearly independent monomial rows. Applying the affine root-finding procedure of Theorem 1 on $\boldsymbol{Z}_R$ returns the regular roots.

Remark that the column compression 'from the right' also requires a rank decision: This rank, as revealed by the column compression of the upper part of the null space, will reveal the number of affine roots.

4.3. Descriptor systems

Affine roots and roots at infinity have an elegant interpretation in terms of (multidimensional) systems. Let us first recall that a pencil $\lambda\boldsymbol{E} - \boldsymbol{A}$ can be realized as a descriptor system.

Lemma 1 (Weierstrass canonical form)**.** *A pencil $\lambda\boldsymbol{E} - \boldsymbol{A}$ with $\boldsymbol{A}, \boldsymbol{E} \in \mathbb{R}^{m\times m}$ can be decomposed into a regular and a singular part, denoted by $\boldsymbol{A}_R$ and $\boldsymbol{E}_S$, respectively, by the Weierstrass canonical form [13, 14, 18]*

$$\boldsymbol{P}(\lambda\boldsymbol{E} - \boldsymbol{A})\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{A}_R - \lambda\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \lambda\boldsymbol{E}_S - \boldsymbol{I} \end{bmatrix}, \tag{44}$$

for some $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{m\times m}$. The Weierstrass canonical form admits the state-space realization [20]

$$\begin{bmatrix} \boldsymbol{x}_R[k+1] \\ \boldsymbol{x}_S[k-1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_R & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}_S \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_R[k] \\ \boldsymbol{x}_S[k] \end{bmatrix}, \tag{45}$$

where $\boldsymbol{E}_S$ is nilpotent, and $\boldsymbol{x}_R$ and $\boldsymbol{x}_S$ denote the regular and singular parts of the state.

An iteration running forward-in-time is obtained for the regular part of the state, while an iteration running backward-in-time is obtained for the singular part. Notice that the forward and backward iterations are reminiscent of the up and down shifts that we encountered before. The separation of regular and singular parts can be interpreted in the polynomial system solving framework as the separation of the affine roots and roots at infinity.

Recall that the projective description allows for an equal treatment of all variables, and can also determine the roots at infinity. In order to include the roots at infinity, one chooses an up shift in x_0, implying that the one of the affine components $x_1, \ldots, x_n$ is shifted down.

Example 10. *Consider the system*

$$\begin{array}{rcl} f_1(x_1,x_2) &=& x_1^2 + x_1x_2 - 10x_0^2 = 0, \\ f_2(x_1,x_2) &=& x_2^2 + x_1x_2 - 15x_0^2 = 0, \end{array} \tag{46}$$

having two affine solutions $(1,2,3)$, $(1,-2,-3)$*, and a projective root* $(0,1,-1)$ *with multiplicity two. At degree* $d = 4$ *the linearly independent monomials are* 1, x_1, x_1^3 *and* x_1^4*, of which* 1 *and* x_1 *are associated with to the two affine roots and* x_1^3 *and* x_1^4 *with the double root at infinity. Notice that there are no linearly independent monomials of degree two, which means that there is a separation of one degree block between the two sets of linearly independent monomials. The column echelon basis of the null space is partitioned as in (42), and we find*

$$\left[\begin{array}{c} \boldsymbol{H}_R^{(1)} \\ \hline \boldsymbol{H}_R^{(2)} \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 3/2 \\ \hline 4 & 0 \\ 6 & 0 \\ 9 & 0 \end{array}\right], \text{ and } \left[\begin{array}{c} \boldsymbol{H}_S \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{array}\right]. \tag{47}$$

For the affine roots, we can immediately write the corresponding realization problems

$$\begin{bmatrix} w[k_1+1,k_2] \\ w[k_1+2,k_2] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} w[k_1,k_2] \\ w[k_1+1,k_2] \end{bmatrix}$$

$$\begin{bmatrix} w[k_1,k_2+1] \\ w[k_1+1,k_2+1] \end{bmatrix} = \begin{bmatrix} 0 & 3/2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} w[k_1,k_2] \\ w[k_1+1,k_2] \end{bmatrix} \tag{48}$$

where the action matrices $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *have the eigenvalue decompositions*

$$\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1},$$

$$\begin{bmatrix} 0 & 3/2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}^{-1}. \tag{49}$$

We recognize in (47) in the $\boldsymbol{H}_S$ *block the evaluation of* $(0,1,-1)$ *in the (homogeneous) Vandermonde vector* $\boldsymbol{v}$*, which has only nonzero elements in the highest (affine) degree block, and the evaluation of* $(0,1,-1)$ *in the differential* $\partial \boldsymbol{v}/\partial x_0$*, which is nonzero only in the* $d-1$ *degree block. The multiplicity structure of the root* $(0,1,-1)$ *can be read off immediately from the relations*

$$\begin{aligned} \boldsymbol{S}_{0/1} \begin{bmatrix} \boldsymbol{v} & \partial \boldsymbol{v}/\partial x_0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \boldsymbol{S}_{1/0} \begin{bmatrix} \boldsymbol{v} & \partial \boldsymbol{v}/\partial x_0 \end{bmatrix}, \\ \boldsymbol{S}_{0/2} \begin{bmatrix} \boldsymbol{v} & \partial \boldsymbol{v}/\partial x_0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} &= \boldsymbol{S}_{2/0} \begin{bmatrix} \boldsymbol{v} & \partial \boldsymbol{v}/\partial x_0 \end{bmatrix}, \end{aligned} \tag{50}$$

Let us finally remark that in the homogeneous setting it is possible to consider the "$0/i$" shift relation, where x_0 is shifted upwards and x_i is shifted downwards. Recall that x_0 is the homogenization variable: the eigenvalue decomposition $\boldsymbol{S}_{0/i}\boldsymbol{Z}\boldsymbol{D}_{0/i} = \boldsymbol{S}_{i/0}\boldsymbol{V}$ has as many nonzero eigenvalues as there are affine roots, and as many zero eigenvalues as there are roots at infinity. Even though we are shifting with the homogenization variable, it is in this case the component x_i that determines whether or not there are singular solutions: As long as $x_i \neq 0$ for all solutions, there are only regular solutions. In the case that x_i is zero for some of the solutions, there are both singular and regular roots. The regular solutions are the solutions for which x_i takes a nonzero value, and the singular solutions are the solutions for which x_i takes the value zero. Also in this case, the presented methods allow for computing the solutions.

5. Conclusions

The current article establishes a link between multivariate polynomial system solving and multidimensional realization theory. Strong similarities are manifest when the polynomial equations are viewed as a set of linear differential or difference equations. Simple (numerical) linear algebra tools allow for the constuction of an eigenvalue-based approach for solving systems of polynomial equations, without resorting to algebraic geometry concepts. A system theoretical interpretation leads to associating with the system of polynomial equations a multidimensional autonomous system realization, possibly in descriptor form.

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Appendix A. Algebra and numerical linear algebra tools

Let us first recall how a numerical basis of the null space of a matrix $\boldsymbol{M}$ is found using the singular value decomposition.

Lemma 2 (Numerical basis of the null space)**.** *A numerical basis of the null space $\boldsymbol{Z}$ can be obtained from the singular value decomposition as*

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma} & \\ & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{W}^\top \\ \boldsymbol{Z}^\top \end{bmatrix}.$$

The column echelon basis of the null space $\boldsymbol{H}$ can be constructed in a classical 'Gaussian elimination' fashion, or by means of numerical linear algebra as follows.

Lemma 3 (Column echelon form)**.** *Let $\boldsymbol{Z}$ be a numerical basis of the null space of $\boldsymbol{M}$, e.g., computed with the singular value decomposition. Let $\boldsymbol{Z}^\star$ be composed of the linearly independent rows of $\boldsymbol{Z}$, where linear independence is checked going from top to bottom rows, ordered by the degree negative lexicographic order. We the column reduced echelon form is given by $\boldsymbol{H} = \boldsymbol{Z}\,(\boldsymbol{Z}^\star)^\dagger$. Remark that computing $\boldsymbol{H}$ may be numerically ill-posed since it requires checking linear (in)dependence of single rows of $\boldsymbol{Z}$.*

The following tool is essential to separate the regular and singular roots.

Lemma 4 (Column compression)**.** *Let $\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_1^\top & \boldsymbol{W}_2^\top \end{bmatrix}^\top$ be a $q \times m$ matrix, which is partitioned into a $k \times m$ matrix $\boldsymbol{W}_1$ and an $q - k \times m$ matrix $\boldsymbol{W}_2$ with $\operatorname{rank}(\boldsymbol{Z}_1) = m_R < m$. Let the singular value decomposition of $\boldsymbol{W} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$. Then $\boldsymbol{Z} = \boldsymbol{W}\boldsymbol{Q}$ is called the column compression of $\boldsymbol{W}$ and can be partitioned as*

$$\boldsymbol{Z} = \begin{bmatrix} \boldsymbol{Z}_{11} & \boldsymbol{0} \\ \boldsymbol{Z}_{21} & \boldsymbol{Z}_{22} \end{bmatrix}, \tag{A.1}$$

where $\boldsymbol{Z}_{11}$ has size $k \times m_R$ (and the remaining blocks have compatible dimensions).