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LOSSLESS AND DISSIPATIVE DISTRIBUTED SYSTEMS*

HARISH K. PILLAI† AND JAN C. WILLEMS‡

Abstract. This paper deals with linear shift-invariant distributed systems. By this we mean systems described by constant coefficient linear partial differential equations. We define dissipativity with respect to a quadratic differential form, i.e., a quadratic functional in the system variables and their partial derivatives. The main result states the equivalence of dissipativity and the existence of a storage function or a dissipation rate. The proof of this result involves the construction of the dissipation rate. We show that this problem can be reduced to Hilbert's 17th problem on the representation of a nonnegative rational function as a sum of squares of rational functions.

Key words. quadratic differential forms, linear multidimensional systems, behavioral theory, polynomial matrices, lossless systems, positivity, dissipativeness, storage functions

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1. Introduction.

One of the very useful concepts in systems theory is the notion of a dissipative system. It lies at the root of most of the stability results and on the synthesis of robust controllers. The theory of dissipative systems has been developed until now as a system theoretic concept for dynamical systems, i.e., for systems in which the independent variable is time. However, many if not most models of physical systems are distributed, involving both time and space variables. The purpose of this paper is to develop the theory of dissipative systems for systems described by partial differential equations.

The central problem in the theory of dissipative systems is the construction of an internal function called the storage function. Instances of functions that play the role of storage functions are Lyapunov functions in stability analysis, the internal energy, and entropy in thermodynamics, etc. The construction of storage functions for dynamical systems is reasonably well understood [23, Part 1] for general nonlinear systems and in much detail for linear systems with quadratic supply rates [23, Part 2] [25]. As we shall see, analogous results may be obtained, as far as existence is concerned, for distributed systems described by linear constant coefficient partial differential equations and with quadratic differential forms (QDFs) as supply rates. However, there are important differences in the resulting theory, the most important one being the fact that for distributed systems the storage functions need to be (in general) a function of unobservable ("hidden") latent variables.

Several recent papers [2, 12, 13] dealing with conservative and dissipative systems have been brought to our notice. In these papers, the authors consider an input/state/output framework for the multidimensional systems involved. The results in these papers are clearly related to the results presented in this paper. While the

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†ISIS Research Group, Department of Electronics and Computing Science, University of Southampton, SO17 1QP, Southampton, UK. Current address: Department of Electrical Engineering, Indian Institute of Technology, Bombay, Powai, Mumbai 400076, India (hp@ee.iitb.ac.in).

‡Institute for Mathematics and Computing Science, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. Current address: Department of Electrical Engineering, ESAT/SISTA, Univeristy of Leuven, B-3001 Leuven-Haverlee, Belgium (Jan.Willems@esat.kuleuven.ac.be).

results in [2, 12, 13] are more general (in the sense that they consider more general signal spaces—Hilbert spaces), they are far less structured (in the sense that they tackle only problems that admit a type of state formulation—the Roesser model). On the other hand, the results in this paper are more structured in the sense that it deals with systems that arise as solutions of constant coefficient partial differential equations (without assuming "states," etc.), though the signal spaces used are not as general. The mathematics involved in the two approaches are also substantially different.

An interesting feature of the results presented in this paper is the mathematics that underlies the construction of the storage function (for linear systems with quadratic supply rates). In the context of lumped dynamical systems the construction of a storage function involves, as we shall see, the factorization of a real polynomial matrix Φ in one indeterminate into the product $\Phi(\xi) = F^T(-\xi)F(\xi)$ with F also a real polynomial matrix. This factorization is readily seen to be possible if and only if $\Phi(\xi) = \Phi^T(-\xi)$ and $\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. However, in the case of distributed systems, Φ is a polynomial matrix in n indeterminates. In this case, the factorization $\Phi(\xi) = F^T(-\xi)F(\xi)$ is not always possible with F as a real polynomial matrix but it is possible with F as a matrix of rational functions. This factorization, it turns out, is known as *Hilbert's* 17*th problem*, and it is most stimulating indeed to see this problem emerge in a basic system theoretic question!

First, a few words about notation. We use the standard notation $\mathbb{R}^n$, $\mathbb{R}^{n_1 \times n_2}$, etc., for finite-dimensional vectors and matrices. When the dimension is not specified (but, of course, finite), we write $\mathbb{R}^\bullet$, $\mathbb{R}^{n \times \bullet}$, $\mathbb{R}^{\bullet \times \bullet}$, etc. In order to enhance readability, we typically use the notation $\mathbb{R}^{\mathbb{w}}$ when functions taking their values in that vector space are denoted by w. Real polynomials in the indeterminates $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ are denoted by $\mathbb{R}[\xi]$ and real rational functions by $\mathbb{R}(\xi)$, with obvious modifications for the matrix case. The space of infinitely differentiable functions with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^{\mathbb{w}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$ and its subspace containing elements with compact support by $\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$.

The proofs of the results are collected in the appendix.

2. Multidimensional systems. We view a system as a family of trajectories mapping a set of "independent" variables into a set of "dependent" variables. See [20] for an elaboration of this with examples. Thus a *system* Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the indexing set, the set of independent variables, $\mathbb{W}$ is the signal space, the set of dependent variables, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ is the behavior. In the present paper we consider systems with $\mathbb{T} = \mathbb{R}$ (we call these *lumped dynamical* systems or one-dimensional (1D) systems) and systems with $\mathbb{T} = \mathbb{R}^n$ (we call these *distributed* systems—they are commonly called nD systems). Also, we assume throughout that $\mathbb{W}$ is a finite-dimensional real vector space, $\mathbb{W} = \mathbb{R}^{\mathbb{w}}$.

A system $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}, \mathfrak{B})$ is said to be *linear* if $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathbb{w}})^{\mathbb{R}^n}$ and *shift-invariant* if $\mathfrak{B} = \sigma^{x'}\mathfrak{B}$ for all $x' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n$, where $\sigma^{x'} : (\mathbb{R}^{\mathbb{w}})^{\mathbb{R}^n} \to (\mathbb{R}^{\mathbb{w}})^{\mathbb{R}^n}$ denotes the x'-shift defined for $x' = (x'_1, \ldots, x'_n)$ by $\sigma^{x'} f(x_1, \ldots, x_n) = f(x_1 + x'_1, \ldots, x_n + x'_n)$. We call Σ a linear shift-invariant differential system if $\mathfrak{B}$ is the solution set of a system of linear constant coefficient partial differential equations. More precisely, if there exists a real polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathbb{w}}[\xi]$ in n indeterminates, $\xi = (\xi_1, \ldots, \xi_n)$, such that $\mathfrak{B}$ consists of the $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$-solutions of

$$R\left(\frac{d}{d\mathbf{x}}\right) w = 0, \tag{1}$$

where $\frac{d}{d\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. The assumption that we consider only $\mathfrak{C}^\infty$-solutions is made for the ease of exposition, and the results remain valid for other solution concepts—for example, for distributions. We denote the family of linear shift-invariant differential systems $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ as $\mathfrak{L}_n^{\mathtt{w}}$. We also denote $(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}_n^{\mathtt{w}}$ as $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ since the indexing set and the signal space are then obvious from the context.

A system $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ is uniquely specified by its annihilators, defined by

$$\mathfrak{N}_{\mathfrak{B}} = \left\{ p \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid p\left(\frac{d}{d\mathbf{x}}\right)\mathfrak{B} = 0 \right\}.$$

It is easy to see that $\mathfrak{N}_{\mathfrak{B}}$ is a submodule of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$ viewed as a module over $\mathbb{R}[\xi]$. In fact, there is a one-to-one relation between $\mathfrak{L}_n^{\mathtt{w}}$ and the submodules of $\mathbb{R}^{1\times \mathtt{w}}[\xi]$. Thus, whereas $R \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ uniquely specifies a behavior $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ through (1) with $\mathfrak{N}_{\mathfrak{B}}$ the module generated by the rows of R, any other polynomial matrix whose rows generate the same submodule define the same behavior.

The family of systems $\mathfrak{L}_n^{\mathtt{w}}$ enjoys many convenient properties, and this has been studied in detail in [19]. An important feature is the elimination theorem, which is the consequence of the following. Let $F \in \mathbb{R}^{\mathtt{w}_1\times \mathtt{w}_2}[\xi]$. Then $\mathfrak{B}_2 \in \mathfrak{L}_n^{\mathtt{w}_2}$ implies $F(\frac{d}{d\mathbf{x}})\mathfrak{B}_2 \in \mathfrak{L}_n^{\mathtt{w}_1}$ and $\mathfrak{B}_1 \in \mathfrak{L}_n^{\mathtt{w}_1}$ implies $(F(\frac{d}{d\mathbf{x}}))^{-1}\mathfrak{B}_1 \in \mathfrak{L}_n^{\mathtt{w}_2}$. This, in particular, implies that if $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}_n^{\mathtt{w}}$, then $\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}_n^{\mathtt{w}}$ and $\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}_n^{\mathtt{w}}$. It also implies the elimination theorem that states that, for any $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}_1+\mathtt{w}_2}$, the set

$$\{w_1 \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1}) \mid \exists w_2 \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2}) : (w_1, w_2) \in \mathfrak{B}\}$$

is itself an element of $\mathfrak{L}_n^{\mathtt{w}_1}$. The elimination theorem and its variations follow from the important *fundamental principle* that states that the system of partial differential equations

$$A\left(\frac{d}{d\mathbf{x}}\right) f = g,$$

with $A \in \mathbb{R}^{\mathtt{w}_1\times \mathtt{w}_2}[\xi]$ and $g \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1})$ given, is solvable for $f \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2})$ if and only if whenever $p \in \mathbb{R}^{1\times \mathtt{w}_1}[\xi]$ satisfies $pA = 0$, then there must hold that $p(\frac{d}{d\mathbf{x}})g = 0$.

Whereas we have *defined* the behavior of a system in $\mathfrak{L}_n^{\mathtt{w}}$ as the set of solutions of a system of partial differential equations in the system variables, often, in practical applications, the specification of the behavior involves other, auxiliary variables, which we call *latent variables*. Specifically, consider the system of partial differential equations

$$R\left(\frac{d}{dx}\right) w = M\left(\frac{d}{dx}\right) \ell \tag{2}$$

with $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell})$ and with $R \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\bullet\times \ell}[\xi]$ polynomial matrices with the same number of rows. The set

$$\mathfrak{B}_f = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\ell}) \mid (2) \text{ holds}\} \tag{3}$$

obviously belongs to $\mathfrak{L}_n^{\mathtt{w}+\ell}$. It immediately follows from the elimination theorem that the set

$$\{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell}) : (w, \ell) \in \mathfrak{B}_f\} \tag{4}$$

belongs to $\mathfrak{L}_n^{\mathbb{w}}$. We call (2) a latent variable representation, with manifest variables w and latent variables ℓ, of the system with full behavior (3) and manifest behavior (4). Correspondingly, we call (1) a kernel representation of the system with the behavior $\ker(R(\frac{d}{d\mathbf{x}}))$. We shall soon meet another sort of representation, the image representations, in the context of controllability.

3. Controllability and observability. Two very influential classical properties of dynamical systems are those of controllability and observability. In [24] these properties have been lifted to lumped dynamical systems in a behavioral setting, while in [19] generalizations to distributed systems have been introduced. We discuss these concepts here exclusively in the context of systems described by linear constant coefficient partial differential equations.

DEFINITION 1. *A system $\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$ is said to be* controllable *if for all $w_1, w_2 \in \mathfrak{B}$ and for all sets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closure, there exists a $w \in \mathfrak{B}$ such that $w\,|_{U_1} = w_1\,|_{U_1}$ and $w\,|_{U_2} = w_2\,|_{U_2}$.*

Thus controllable partial differential equations are those in which the solutions can be "patched up" from solutions on subsets: in a sense there is no "action of a distance." There are a number of characterizations of controllability. In terms of its submodule of annihilators, $\mathfrak{N}_{\mathfrak{B}}$, $\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$, is controllable if and only if the module $\mathbb{R}^{1\times \mathbb{w}}[\xi]/\mathfrak{N}_{\mathfrak{B}}$ is torsion-free [19].

More useful for our purposes is the equivalence of controllability with the existence of an image representation. Consider the following special latent variable representation:

$$w = M\left(\frac{d}{d\mathbf{x}}\right)\ell \tag{5}$$

with $M \in \mathbb{R}^{\mathbb{w}\times \ell}[\xi]$. Obviously, by the elimination theorem, its manifest behavior $\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$. Such special latent variable representations often appear in physics, where the latent variables involved in such a representation are called *potentials*. Obviously, $\mathfrak{B} = \operatorname{im}(M(\frac{d}{d\mathbf{x}}))$ with $M(\frac{d}{d\mathbf{x}})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell)$ to $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$. For this reason, we call (5) an *image representation* of its manifest behavior. Whereas every $\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$ allows (by definition) a kernel representation and hence trivially a latent variable representation, not every $\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$ allows an image representation. In fact, see the following theorem.

THEOREM 2. *$\mathfrak{B} \in \mathfrak{L}_n^{\mathbb{w}}$ admits an image representation if and only if it is controllable.*

We denote the set of controllable systems in $\mathfrak{L}_n^{\mathbb{w}}$ by $\mathfrak{L}_{n,\mathrm{cont}}^{\mathbb{w}}$.

Observability is the property of systems that have two kinds of variables; the first set of variables are the "observed" set of variables, and the second set of variables are the ones that are "to-be-deduced" from the observed variables. Every variable that can be deduced uniquely from the manifest variables of a given behavior will be called an *observable*. So observability is not an intrinsic property of a given behavior. One has to be given a partition of the variables in the behavior into two classes before one can say whether one class of variables in the behavior can actually be deduced from the other class of variables (which were observed).

DEFINITION 3. *Let $w = (w_1, w_2)$ be a partition of the variables in $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathbb{w}_1+\mathbb{w}_2}, \mathfrak{B})$. Then w_2 is said to be* observable *from w_1 in $\mathfrak{B}$ if given any two trajectories $(w_1', w_2'), (w_1'', w_2'') \in \mathfrak{B}$ such that $w_1' = w_1''$; then $w_2' = w_2''$.*

A natural situation to use observability is when one looks at the latent variable representation of a behavior. Then one may ask whether the latent variables are

observable from the manifest variables. If this is the case, then we call the latent variable representation *observable*.

As we have already mentioned, every controllable behavior has an image representation. In the case of 1D systems, it can be shown that every controllable behavior has an observable image representation. This is not true for nD systems.

4. QDFs. In [25, 26] a theory was developed for linear (1D) differential systems and quadratic functionals associated with these systems. It was shown that for systems described by *one-variable* polynomial matrices, the appropriate tool to express quadratic functionals are *two-variable* polynomial matrices. In the same vein, in this paper we will use polynomial matrices in $2n$ variables to express quadratic functionals for functions of n variables.

For convenience, let ζ denote $(\zeta_1, \dots, \zeta_n)$, and let η denote $(\eta_1, \dots, \eta_n)$. Let $\mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the $2n$ indeterminates ζ and η. We will consider quadratic forms of the type $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$. Explicitly,

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}}.$$

The sum above ranges over all nonnegative multi-indices $\mathbf{k} = (k_1, k_2, \dots, k_n), \mathbf{l} = (l_1, l_2, \dots, l_n) \in \mathbb{N}^n$, and the sum is assumed to be finite. Moreover, $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$. The polynomial matrix Φ induces a *bilinear differential form* (BLDF), that is, the map

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

defined by

$$L_\Phi(v, w)(\mathbf{x}) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) \right)^T \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}(\mathbf{x}) \right),$$

where $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$ and analogously for $\frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}}$. Note that ζ corresponds to differentiation of terms to the left, and η refers to differentiation of the terms to the right.

If $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$, then Φ induces the QDF

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

defined by

$$Q_\Phi(w) := L_\Phi(w, w).$$

Define the * operator

$$^* : \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta] \to \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$$

by

$$\Phi^*(\zeta, \eta) := \Phi^T(\eta, \zeta).$$

If $\Phi = \Phi^*$, then Φ is called *symmetric*. For the purposes of QDFs induced by polynomial matrices, it suffices to consider the symmetric QDFs since $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$.

We also consider vectors $\Psi \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta])^n$, i.e., $\Psi = (\Psi_1,\dots,\Psi_n)$. Analogous to the QDF induced by Φ, Ψ induces a *vector of quadratic differential forms* (VQDF)

$$Q_\Psi(w) : \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}}) \to (\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}))^n$$

defined by $Q_\Psi = (Q_{\Psi_1},\dots,Q_{\Psi_n})$.

Finally, we define the "div" (divergence) operator that associates with the VQDF induced by Ψ, the scalar QDF:

$$(\text{div } Q_\Psi)(w) := \frac{\partial}{\partial x_1}Q_{\Psi_1}(w) + \cdots + \frac{\partial}{\partial x_n}Q_{\Psi_n}(w).$$

The theory of QDFs has been developed in much detail in [25, 26] for 1D systems. In the next section, we put forward those aspects which are useful in the construction of storage function for distributive systems.

5. Path independence. Consider the integral

$$\int_\Omega Q_\Phi(w)d\mathbf{x}, \tag{6}$$

where Ω is a closed bounded subset of $\mathbb{R}^n$ with a nonempty interior. This integral is said to be independent of the "path" w (or a *path integral*) if the integral depends only on the value of w and its derivatives on the boundary of Ω, denoted by $\partial\Omega$. More precisely, if for any $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ such that $\frac{d^{\mathbf{k}}w_1}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) = \frac{d^{\mathbf{k}}w_2}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})$ for all $\mathbf{x}\in\partial\Omega$ and all $\mathbf{k}\in\mathbb{N}^n$, there holds

$$\int_\Omega Q_\Phi(w_1)d\mathbf{x} \;=\; \int_\Omega Q_\Phi(w_2)d\mathbf{x}.$$

Instead of some $\Omega\subset\mathbb{R}^n$, if we consider the integral (6) over all of $\mathbb{R}^n$, then the integral need not be well defined for all $w\in\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$. We can overcome this by considering it only for w's of compact support. This yields the functional

$$\int Q_\Phi : \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}}) \to \mathbb{R}$$

defined by

$$\int Q_\Phi(w) := \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x},$$

which evaluates the integral over all of $\mathbb{R}^n$.

The following theorem gives several conditions that are equivalent to path independence.

THEOREM 4. *Let $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Then the following statements are equivalent:*

1. *$\int_\Omega Q_\Phi$ is independent of path for all closed bounded subsets Ω of $\mathbb{R}^n$.*
2. *$\int Q_\Phi = 0$.*
3. *$\Phi(-\xi,\xi) = 0$.*
4. *There exist $\Psi_1,\dots,\Psi_n\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that*

$$\Phi(\zeta,\eta) = (\zeta_1+\eta_1)\Psi_1(\zeta,\eta) + \cdots + (\zeta_n+\eta_n)\Psi_n(\zeta,\eta).$$

5. *There exists a* $\Psi \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta])^n$ *such that*

$$\operatorname{div} Q_\Psi = Q_\Phi$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.

At this point we would like to point out an important difference for the cases $n = 1$ and $n > 1$. Although the above theorem holds for all values of n, more can be said in the case when $n = 1$. In the case when $n = 1$, the last condition of the above theorem can be strengthened to state that there exists a unique Ψ such that $\frac{d}{dt}Q_\Psi = Q_\Phi$ (assuming t is the independent variable). This uniqueness of Ψ does not hold when $n > 1$. This will become clear from the subsequent proposition, which will help us in classifying this nonuniqueness. If Ψ_1 and Ψ_2 induce two VQDFs such that

$$Q_\Phi = \operatorname{div} Q_{\Psi_1} = \operatorname{div} Q_{\Psi_2}, \tag{7}$$

then $\Psi = \Psi_1 - \Psi_2$ defines a VQDF such that $\operatorname{div} Q_\Psi(w) = 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Such a VQDF is said to have *null divergence*. Thus it is obvious that given a $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ which defines a path integral and a VQDF induced by $\Psi \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta])^n$ such that $\operatorname{div} Q_\Psi(w) = Q_\Phi(w)$, it is possible to obtain other VQDFs that satisfy this property by adding VQDFs that have null divergence to the already obtained VQDF Ψ. We now characterize those VQDFs that have null divergence.

PROPOSITION 5. *A VQDF induced by* $\Psi = (\Psi_1, \dots, \Psi_n) \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta])^n$ *has null divergence if and only if there exists a family of* n^2 *QDFs induced by* $\Delta_{ij} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, $i = 1, \dots, n$, $j = 1, \dots, n$, *with* $\Delta_{ij} = -\Delta_{ji}$ *such that*

$$\Psi_i = (\zeta_1 + \eta_1)\Delta_{i1} + (\zeta_2 + \eta_2)\Delta_{i2} + \cdots + (\zeta_n + \eta_n)\Delta_{in}.$$

From the above proposition, it is clear that $\Delta_{ii} = 0$. Thus for 1D systems, the QDF induced by Δ_{11} is the zero QDF, and so there exists no nonzero 1D (V)QDFs that have null divergence. Hence the Ψ obtained in Theorem 4 for 1D systems is unique [26, Theorem 3.1]. In fact, $\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta)}{\zeta+\eta}$ in 1D systems. In nD systems with $n > 1$, the Ψ obtained in Theorem 4 is no longer unique since there exist nonzero VQDFs that give rise to null divergences. The above proposition completely classifies the nonuniqueness of these VQDFs. Hence, for every path independent QDF induced by $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, one obtains an equivalence class of VQDFs such that (7) holds. The members of an equivalence class are exactly those that differ by a VQDF that has null divergence.

6. Lossless systems. In this section, we study the notion of path independence generalized to controllable systems $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. We cast this in the context of conservative systems.

Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. Now consider the QDF $Q_\Phi(w)$ for trajectories $w \in \mathfrak{B}$. We consider $Q_\Phi(w)(\mathbf{x})$ (with $\mathbf{x} \in \mathbb{R}^n$) as the rate of supply of some physical quantity (for example, energy) delivered to the system at the point $\mathbf{x}$ (whence positive when the system absorbs supply).

DEFINITION 6. *The system* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ *is said to be* lossless *with respect to the supply rate* Q_Φ *induced by* $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *if* $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} = 0$ *for all* $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.

The interpretation of this condition is that $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x}$ denotes the net amount of supply that the system absorbs integrated over "time" and "space." Whence the system is lossless if this integral is zero: any supply absorbed at some time or place is temporarily stored but eventually recovered perhaps at some other time or place.

A related notion is that of path independence along a behavior. Let Ω be a closed and bounded subset of $\mathbb{R}^n$. The integral $\int_\Omega Q_\Phi(w)d\mathbf{x}$ is said to be independent of path for trajectories $w \in \mathfrak{B}$ if whenever $w_1, w_2 \in \mathfrak{B}$ and $\frac{d^{\mathbf{k}}w_1}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) = \frac{d^{\mathbf{k}}w_2}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ and all $\mathbf{k} \in \mathbb{N}^n$, then

$$\int_\Omega Q_\Phi(w_1)d\mathbf{x} \;=\; \int_\Omega Q_\Phi(w_2)d\mathbf{x}.$$

Define the $\star$ operator mapping from $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\xi]$ to $\mathbb{R}^{\mathtt{w}_2\times\mathtt{w}_1}[\xi]$ by $X^\star(\xi) := X^T(-\xi)$. In other words, if we look at $X(\frac{d}{d\mathbf{x}})$ as a partial differential operator, then $X^\star(\frac{d}{d\mathbf{x}})$ is the (formal) adjoint operator.

The following theorem gives a number of equivalent conditions for a system to be lossless.

THEOREM 7. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. Let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ induce, respectively, a kernel and image representation of $\mathfrak{B}$; i.e., $\mathfrak{B} = \ker(R(\frac{d}{d\mathbf{x}})) = \mathrm{im}(M(\frac{d}{d\mathbf{x}}))$. Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induce a QDF on $\mathfrak{B}$. Then the following conditions are equivalent:*

1. *$\mathfrak{B}$ is lossless with respect to the QDF Q_Φ;*
2. *The QDF induced by Φ is independent of path on $\mathfrak{B}$, i.e., $\int_\Omega Q_\Phi(w)d\mathbf{x}$ is independent of path for all bounded and closed subsets Ω in $\mathbb{R}^n$ with a nonempty interior;*
3. *the QDF corresponding to Φ' is a path integral, where Φ' is given by $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$;*
4. *$\Phi'(-\xi,\xi) = 0$;*
5. *there exists a VQDF Q_Ψ, with $\Psi \in (\mathbb{R}^{m\times m}[\zeta,\eta])^n$, where m is the number of columns of M such that*

$$\mathrm{div}\, Q_\Psi(\ell) = Q_{\Phi'}(\ell) = Q_\Phi(w) \tag{8}$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^m)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

We focus our attention for a moment on the equivalence of conditions 1 and 5 of the above theorem. It states that $\mathfrak{B}$ is lossless with respect to Q_Φ, i.e., that

$$\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} = 0 \tag{9}$$

for all $w \in \mathfrak{B}$ of compact support if and only if $\mathfrak{B}$ admits an image representation $w = M(\frac{d}{d\mathbf{x}})\ell$ and there exists some VQDF Ψ such that

$$\mathrm{div}\; Q_\Psi(\ell) = Q_\Phi(w) \tag{10}$$

for all $w \in \mathfrak{B}$ and ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$.

The equivalence of the global version of losslessness (9) with the local version (10) is a recurrent theme in the theory of dissipative systems. The local version states that there is a function $Q_\Psi(\ell)(\mathbf{x})$ that plays the role of the amount of supply stored at $\mathbf{x} \in \mathbb{R}^n$. Thus (10) says that for lossless systems, it is possible to define a *storage function* Q_Ψ such that the *conservation equation*

$$\mathrm{div}\; Q_\Psi(\ell) = Q_\Phi(w) \tag{11}$$

is satisfied for all w, ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$. Note here that since $\Phi' = \mathrm{div}\;\Psi$, by the Stokes theorem

$$\int_{\partial\Omega}\sum_{i=1}^n (-1)^{i-1}Q_{\Psi_i}(\ell)dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n = \int_\Omega Q_{\Phi'}(\ell)dx_1\wedge\cdots\wedge dx_n$$

(for any $\Omega \subseteq \mathbb{R}^n$ with a reasonable boundary). We can then think of the above as an integral form of the conservation equation (11).

Two important features, both specific to the case when $n > 1$, are worth emphasizing. First is the fact that the storage $Q_\Psi(\ell)$ depends on the latent variable ℓ from the image representation $w = M(\frac{d}{d\mathbf{x}})\ell$. Since $\mathfrak{B} \in \mathfrak{L}^{\mathbf{w}}_{n,\text{cont}}$ may not have an observable image representation, there may not exist a storage function of the form $Q_\Psi(w)$ that depends on the manifest variables $w \in \mathfrak{B}$. Hence the storage in (11) involves "hidden" (i.e., nonobservable) variables. Second, the nonuniqueness of the VQDF Q_Ψ that solves $\operatorname{div} Q_\Psi(\ell) = Q_\Phi(M(\frac{d}{d\mathbf{x}})\ell) = Q_{\Phi'}(\ell)$. Hence, even when the ℓ's have acquired a "physical significance," there will be many possible storage functions. We shall see in the next section that this nonuniqueness is important already in basic physics.

We would like to mention at this point that in many practical examples the independent variables are time and space variables. So, for example, the indexing set would be $\mathbb{R} \times \mathbb{R}^3$. In this case, we will use the notation t, x, y, z to stand for the independent variables (time coordinate and the three space coordinates, respectively), and the partial derivatives with respect to these variables are denoted by $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, respectively. It is important to interpret the storage function Q_Ψ in this context. In the case mentioned above, we denote $\Psi = (\Psi_t, \Psi_x, \Psi_y, \Psi_z)$ and $Q_\Psi = (u, \mathbf{S})$. Here u is the QDF Q_{Ψ_t}, which is the "*internal storage*" and the VQDF $\mathbf{S} := (Q_{\Psi_x}, Q_{\Psi_y}, Q_{\Psi_z})$ is the "*flux*." This interpretation will be useful in the next section. With the above notation, (8) now becomes

$$\frac{\partial}{\partial t} u(\ell) + \nabla \cdot \mathbf{S}(\ell) = Q_\Phi(w),$$

where ∇ is the spatial divergence operator.

7. Maxwell's equations. The prototypical example of a linear shift-invariant differential system is provided by Maxwell's equations in free space:

$$\begin{aligned} \nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} &= 0, \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\ c^2 \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} - \frac{\mathbf{j}}{\epsilon_0} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{12}$$

This describes the relation between the electrical field $\mathbf{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, the magnetic field $\mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, the current density $\mathbf{j} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, and the charge density $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. In the above equations, the constants c and ϵ_0 stand for the speed of light in vacuum and the electric constant, respectively. Hence (12) defines a system $\mathfrak{B}_{ME} \in \mathfrak{L}^{10}_4$. It is well known that $\mathfrak{B}_{ME}$ can be described in terms of the vector potential $\mathbf{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and the scalar potential $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\begin{aligned} \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \\ \rho &= -\epsilon_0 \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \epsilon_0 \nabla^2 \phi, \\ \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{j} &= \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \epsilon_0 c^2 \nabla^2 \mathbf{A} + \epsilon_0 c^2 \nabla(\nabla \cdot \mathbf{A}) + \epsilon_0 \nabla \frac{\partial \phi}{\partial t}. \end{aligned} \tag{13}$$

It is important to note that (13) is an image representation of $\mathfrak{B}_{ME}$. Hence, by Theorem 2, Maxwell's equations define a controllable system. It is also important to note that (13) is an unobservable image representation of $\mathfrak{B}_{ME}$. In fact, there *do not exist* observable image representations of $\mathfrak{B}_{ME}$.

Strictly speaking, the vector potential $\mathbf{A}$ and the scalar potential ϕ are "free" latent variables (i.e., they are allowed to take on any values in the relevant space of trajectories). Note that we can change $\mathbf{A}$ and ϕ to $\mathbf{A}' = \mathbf{A} + \nabla\psi$ and $\phi' = \phi - \frac{\partial\psi}{\partial t}$ (where ψ is some other arbitrary scalar function) without changing the resulting $\mathbf{E}$, $\mathbf{B}$, ρ, and $\mathbf{j}$. These are called *gauge transformations*. Additional conditions may be imposed on $\mathbf{A}$ and ϕ without changing the fact that the image in (13) remains $\mathfrak{B}_{ME}$. For example, the Lorentz condition

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{14}$$

can be imposed on the potentials to obtain symmetry in the representation (13). In this case, the last two terms of the last equation in (13) disappear, thus displaying a symmetry in the equations. Moreover, these new equations then remain invariant under Lorentz transformations of the independent variables. There are other possibilities. The important point is that the gauge transformations and imposition of such conditions like the Lorentz condition do not change the set of $(\mathbf{E}, \mathbf{B}, \mathbf{j}, \rho)$ obtained as solutions to the Maxwell equations. In other words, (13) and (14) together provide a latent variable representation of $\mathfrak{B}_{ME}$. We will not consider such transformations further in this paper.

We are interested in studying the exchange of electrical energy between the environment and the electromagnetic field in free space. This exchange of energy only involves the electrical variables $(\mathbf{E}, \mathbf{j})$. The laws that are described by these variables define, by the elimination theorem, a system $\mathfrak{B}_{\mathrm{E}} \in \mathfrak{L}_4^6$. Consider, therefore, in Maxwell's equations the magnetic field $\mathbf{B}$ and the charge density ρ as latent variables. Then, by eliminating these latent variables, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\nabla \cdot \mathbf{E} + \frac{1}{\epsilon_0}\nabla \cdot \mathbf{j} &= 0, \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2 \nabla \times (\nabla \times \mathbf{E}) + \frac{1}{\epsilon_0}\frac{\partial \mathbf{j}}{\partial t} &= 0. \end{aligned} \tag{15}$$

The above equations give a kernel representation for the behavior $\mathfrak{B}_{\mathrm{E}}$ consisting of all trajectories $(\mathbf{E}, \mathbf{j}) \in \mathfrak{C}^\infty(\mathbb{R}^4, \mathbb{R}^6)$ which are compatible with the solutions of Maxwell's equations. Since $\mathfrak{B}_{\mathrm{ME}}$ is controllable, so is $\mathfrak{B}_{\mathrm{E}}$, and so one can obtain an image representation of it.

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{j} &= \epsilon_0 \frac{\partial}{\partial t}\nabla\phi + \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} + \epsilon_0 c^2 \nabla \times (\nabla \times \mathbf{A}). \end{aligned} \tag{16}$$

Here $\mathbf{A}$ and ϕ are again the vector and scalar potentials, respectively [10].

Consider the QDF $Q_\Phi(\mathbf{E}, \mathbf{j}) = \mathbf{E} \cdot \mathbf{j}$ for all $w \in \mathfrak{B}_{\mathrm{ME}}$. This quantity defines the rate of work done by the field on each unit volume [10].

It is well known that Maxwell's equations define a lossless system. This also follows from Theorem 7. Indeed, by identifying the matrix Φ corresponding to the QDF $Q_\Phi(\mathbf{E}, \mathbf{j}) = \mathbf{E} \cdot \mathbf{j}$ and the M matrix corresponding to the image representation

(13), we can compute $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$. It is easily seen that $\Phi'(-\xi,\xi) = 0$. Losslessness follows from Theorem 7. The QDF induced by Φ' is a path integral on the potentials, which in turn implies that Φ is a path integral on the solutions of Maxwell's equations. By Theorem 7, there exists a VQDF, $\Psi \in (\mathbb{R}^{4\times 4}[\zeta,\eta])^4$, such that $\operatorname{div} Q_\Psi(\phi,\mathbf{A}) = Q_\Phi(\mathbf{E},\mathbf{j}) = \mathbf{E}\cdot\mathbf{j}$. By the terminology defined at the end of last section, we can write the VQDF Q_Ψ as $(-u,-\mathbf{S})$ (the negative signs are purely a matter of convention). Then we have

$$\mathbf{E}\cdot\mathbf{j} = \operatorname{div} Q_\Psi(\phi,\mathbf{A}) = -\frac{\partial u(\phi,\mathbf{A})}{\partial t} - \nabla\cdot\mathbf{S}(\phi,\mathbf{A}).$$

On substituting $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$, we obtain

$$\begin{aligned} u &= \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{\epsilon_0 c^2}{2}\mathbf{B}\cdot\mathbf{B}, \\ \mathbf{S} &= \epsilon_0 c^2 \mathbf{E}\times\mathbf{B}. \end{aligned} \tag{17}$$

This u defines the *energy density* in the field, and $\mathbf{S}$ represents the *energy flux* of the field. The vector $\mathbf{S}$ is known as the "Poynting vector." Thus (8) gives a "conservation law" for Maxwell's equations. It states that the rate at which the field does work on an infinitesimal volume ($Q_\Phi(\mathbf{E},\mathbf{j}) = \mathbf{E}\cdot\mathbf{j}$) is equal to the rate of decrease in the energy density ($-\frac{\partial u}{\partial t}$) and the energy flux ($-\nabla\cdot\mathbf{S}$) that flows into the infinitesimal volume under consideration. Thus (8) states that the total energy is conserved.

We now interpret these results about Maxwell's equations in terms of the theory developed earlier. There are two points that we would like to emphasize.

1. The problem under consideration may be viewed as finding out if the system given by (15) (the behavior $\mathfrak{B}_{\mathrm{E}}$) is lossless with respect to $Q_\Phi(\mathbf{E},\mathbf{j}) = \mathbf{E}\cdot\mathbf{j}$, and if so, finding a storage function for it. Verification of losslessness involves a straightforward calculation. Also, a storage function $(u,\mathbf{S})$ was derived in terms of $\mathbf{E}$ and $\mathbf{B}$ (17). Note that this storage function depends on $\mathbf{E}$ and $\mathbf{B}$. The latter is a latent variable with respect to the electrical quantities $(\mathbf{E},\mathbf{j})$ involved in (15). In fact, $\mathbf{B}$ is not observable from $(\mathbf{E},\mathbf{j})$ in Maxwell's equations. Hence already in this elementary example the storage functions involve hidden variables.
   From Theorem 7 and the example of Maxwell's equations, it is seen that the VQDF acts on some latent variables. These latent variables are related to the latent variables that appear in an image representation of a given controllable behavior. For example, in Maxwell's equations, $\mathbf{B}$ is related to $\mathbf{A}$. One would like the VQDF to act only on the manifest variables. A sufficient condition for the existence of such a VQDF is that the controllable behavior has an observable image representation. In 1D systems, every controllable system has an observable image representation. As a result, in the 1D case, given a QDF induced by Φ which is independent of path on $\mathfrak{B}$, we can actually find a QDF Ψ such that

$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w)$$

   for all $w\in\mathfrak{B}$. In the nD case, a controllable behavior need not necessarily have an observable image representation. So for the nD case, when the QDF induced by Φ is independent of path on $\mathfrak{B}$, it is sufficient for $\mathfrak{B}$ to have

observable potentials for us to find a VQDF Ψ such that

$$\operatorname{div} Q_\Psi(w) = Q_\Phi(w)$$

for all $w \in \mathfrak{B}$.

2. We would also like to make a comment on the nonuniqueness of the VQDF that appears in the conservation equation (8). With reference to Maxwell's equations, we quote from [10], "All we did was to find a *possible* "u" and a *possible* "$\mathbf{S}$." How do we know that juggling the terms around some more we couldn't find another formula for "u" and "$\mathbf{S}$"? ... It's possible. ... There are, in fact, an infinite number of possibilities for u and $\mathbf{S}$, and so far no one has thought of an experiment to tell which one is right!"
We found that this nonuniqueness of the storage function is an intrinsic feature of storage functions for conservative nD systems with $n > 1$. The result in Proposition 5 characterizes the nonuniqueness of the VQDF that goes with a given QDF induced by Φ which is independent of path on all trajectories in $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell)$.

8. Supply, storage, and dissipation. In the previous section, we considered QDFs such that $\int Q_\Phi$ is zero when restricted to some behavior $\mathfrak{B}$: the lossless systems. As we have seen, such QDFs define conservation laws. In this section, we consider QDFs where the integral $\int Q_\Phi$ is nonnegative. In the spirit of [23, 26], we refer to these as dissipative systems. We justify the use of this terminology later.

Our plan is as follows. We first introduce the concepts for general controllable behaviors $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. Subsequently, we analyze the situation $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. We will see that this leads to the problem of factorization of polynomial matrices in several variables. We subsequently return to general controllable behaviors.

DEFINITION 8. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ and $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Consider the QDF Q_Φ induced by Φ. We call* $\mathfrak{B}$ dissipative with respect to Q_Φ *(briefly Φ-dissipative) if*

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0$$

for all $w \in \mathfrak{B}$ with compact support.

The intuitive interpretation is that $Q_\Phi(w)$ is the rate of supply (Q_Φ is called the *supply rate*) absorbed by the system. Dissipativity hence means that the net supply that is absorbed by the system is nonnegative for any trajectory $w \in \mathfrak{B}$ that is of compact support.

Two related notions are those of storage functions and dissipation rate. As we have already seen in the context of lossless systems, the storage function is in general a function of unobservable latent variables, more specifically of the latent variables that appear in an image representation (thus depending on "potentials"). We incorporate this in the definitions.

DEFINITION 9. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$, $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, and $w = M(\frac{d}{d\mathbf{x}})\ell$ be an image representation of $\mathfrak{B}$ with $M \in \mathbb{R}^{\mathtt{w}\times\ell}[\xi]$. Let $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\ell\times\ell}[\zeta, \eta]$ for $k = 1, 2, \ldots, n$. The VQDF Q_Ψ is said to be a* storage function *for $\mathfrak{B}$ with respect to Q_Φ if*

$$\operatorname{div} Q_\Psi(\ell) \leq Q_\Phi(w) \tag{18}$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

$\Delta = \Delta^* \in \mathbb{R}^{\ell\times\ell}[\zeta,\eta]$ *is said to be a* dissipation rate *for* $\mathfrak{B}$ *with respect to* Q_Φ *if*

$$Q_\Delta \geq 0 \text{ and } \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x}$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^\ell)$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.

We define $Q_\Delta \geq 0$ if $Q_\Delta(w(\mathbf{x})) \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^w)$ evaluated at every $\mathbf{x} \in \mathbb{R}^n$. This defines a pointwise positivity condition. Thus $\int_\Omega Q_\Delta(w)d\mathbf{x} \geq 0$ for every $\Omega \subset \mathbb{R}^n$ if $Q_\Delta \geq 0$.

It is easy to see that there is a relation between a storage function for $\mathfrak{B}$ with respect to Q_Φ and a dissipation rate for $\mathfrak{B}$ with respect to Q_Φ, given by

$$Q_\Delta(\ell) = Q_\Phi\left(M\left(\frac{d}{d\mathbf{x}}\right)\ell\right) - \text{div } Q_\Psi(\ell). \tag{19}$$

The definitions of the storage function and the dissipation rate, combined with (19), yield intuitive interpretations. The dissipation rate can be thought of as the rate of supply that is dissipated in the system and the storage function as the rate of supply stored in the system. Intuitively, we could think of the QDF Q_Φ as measuring the power going into the system. In many practical examples, the power is indeed a QDF of some system variables. (For example, $-\mathbf{E}\cdot\mathbf{j}$ is the rate of work done on the system in the case of Maxwell's equations, or, as mentioned earlier, $\mathbf{E}\cdot\mathbf{j}$ is the rate of work done by the field.) Φ-dissipativity would imply that the net power flowing into a system is nonnegative, which in turn implies that the system dissipates energy. Of course, locally the flow of energy could be positive or negative, leading to variations in energy density and fluxes. The energy density and fluxes could be thought of as a storage function for the energy. (Again see the section on Maxwell's equations.) If the system is dissipative, then the rate of change of energy density and fluxes cannot exceed the power delivered into the system. This is captured by the inequality (18) in Definition 9. The excess is precisely what is lost (or dissipated). This interaction between supply, storage, and dissipation is formalized by (19).

When the independent variables are time and space, we can write (19) as

$$\frac{\partial u(\ell)}{\partial t} = Q_\Phi\left(M\left(\frac{d}{d\mathbf{x}}\right)\ell\right) - \nabla\cdot\mathbf{S}(\ell) - Q_\Delta(\ell), \tag{20}$$

where, as before, we use $Q_\Psi = (u,\mathbf{S})$, with u the stored energy and $\mathbf{S}$ the flux. Moreover, $w = M(\frac{d}{d\mathbf{x}})\ell$. Thus (20) states that the change in the stored energy $(\frac{\partial u(\ell)}{\partial t})$ in an infinitesimal volume is exactly equal to the difference between the energy supplied $(Q_\Phi(w))$ into the infinitesimal volume and the energy lost by the infinitesimal volume by means of energy flux flowing out of the volume $(\nabla\cdot\mathbf{S}(\ell))$ and the energy dissipated $(Q_\Delta(\ell))$ within the volume.

The problem we address is the equivalence of (i) dissipativeness of $\mathfrak{B}$ with respect to Q_Φ, (ii) the existence of a storage function, and (iii) the existence of a dissipation rate. Note that this problem also involves the construction of an appropriate image representation. We first consider the case where $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathbb{w}})$. In this case, the definition of the dissipation rate requires that for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^\ell)$

$$\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x} \tag{21}$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$; $M(\frac{d}{d\mathbf{x}})$ a surjective partial differential operator and $Q_\Delta(\ell) \geq 0$ for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^\ell)$. This latter condition is seen to be equivalent to the existence of

a polynomial matrix $D \in \mathbb{R}^{\bullet\times\ell}[\xi]$ such that $\Delta(\zeta,\eta) = D^T(\zeta)D(\eta)$. One direction of the previous claim is trivial. For the other direction, we think of the operator $\Delta(\zeta,\eta)$ as acting on the space of ℓ and its derivatives (the jet space). The operator $\Delta(\zeta,\eta)$ then becomes a symmetric matrix with real entries that acts on this jet space. The condition $Q_\Delta \geq 0$ is a pointwise condition, and so one obtains the matrix $D(\xi)$ in the obvious way. Using Theorem 7, it follows that (21) is equivalent to the factorization equation

$$M^T(-\xi)\Phi(-\xi,\xi)M(\xi) = D^T(-\xi)D(\xi).$$

This equation with $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ given and $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ and $D \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ unknown is discussed in the next section.

9. Factorization of polynomial matrices. In this section, we discuss the following problem. Let $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be a polynomial matrix in n commuting variables, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Can it be factored as

$$\Gamma(\xi) = F^T(-\xi)F(\xi). \tag{22}$$

We are interested in both the case when $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ is itself a polynomial matrix and the case when $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ is a matrix of rational functions.

Note that $\Gamma^\star = \Gamma$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$ are obviously necessary conditions for the existence of a factor $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. The problem is whether these conditions are also sufficient. At this point, it is convenient to discuss the cases when $n = 1$ and $n > 1$ separately.

9.1. The case $n = 1$. In the case when $n = 1$, it is well known that (22) admits a solution $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ if and only if $\Gamma^\star = \Gamma$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. In fact, there even exist square factors $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ that are, moreover, Hurwitz (i.e., with the roots of $\det(F)$ in the closed left half of the complex plane) and square factors that are anti-Hurwitz (i.e., with the roots of $\det(F)$ in the closed right half of the complex plane). These factors are called *spectral factors*. Several algorithms exist for obtaining such factorizations [6, 8, 15, 21].

9.2. The case $n > 1$. We start with the scalar case, i.e., when $\Gamma \in \mathbb{R}[\xi]$. So we need to find $F \in \mathbb{R}^{\bullet\times 1}[\xi]$ or $F \in \mathbb{R}^{\bullet\times 1}(\xi)$ such that $\Gamma(\xi) = F^T(-\xi)F(\xi)$. Substituting $i\omega$ for ξ, the above problem reduces to finding F such that

$$\Gamma(i\omega) = F^\star(i\omega)F(i\omega). \tag{23}$$

If $F(i\omega)$ is decomposed into real and imaginary parts as $F(i\omega) = A(\omega) + iB(\omega)$, then (23) becomes $\Gamma(i\omega) = A^2(\omega) + B^2(\omega)$. Thus the problem reduces to the case of finding a sum of "two" squares which add up to a given positive (or nonnegative) polynomial. This problem has a very venerable history. It is Hilbert's 17th problem that he posed at the International Congress of Mathematicians in 1900. It deals with the representation of positive definite functions as sums of squares [18]. This investigation of positive definite functions began in the year 1888 with the following "negative" result of Hilbert: If $f(\xi) \in \mathbb{R}[\xi]$ is a positive definite polynomial in n variables, then f need not be a sum of squares of polynomials in $\mathbb{R}[\xi]$, except in the case when $n = 1$. Several examples of such positive definite polynomials which cannot be expressed as sum of squares of polynomials are available in the literature; for example, the polynomial

$$\xi_1^2\xi_2^2(\xi_1^2 + \xi_2^2 - 1) + 1$$

is not factorizable as a sum of squares of polynomials [4].

Thus the factorization we were looking for in nD systems (in the form stated above) is not solvable for polynomial factors, not even in the scalar case. However, several results on Hilbert's 17th problem allow us to solve the factorization problem (22) with F as a rational function. Indeed,

- if $\Gamma(\xi) \in \mathbb{R}(\xi)$ and $\Gamma \geq 0$ (i.e., $\Gamma(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$), then there exists some natural number r such that $\Gamma(\xi)$ is the sum of r squares of rational functions in $\mathbb{R}(\xi)$ (shown by Artin [1]);
- there is a sharp upper bound on the number r; it is $r = 2^n$, shown by Pfister [17, 18].

This leads to the following result, which plays a central role in the rest of this paper.

THEOREM 10. *Assume that* $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *satisfies* $\Gamma^\star = \Gamma$ *and* $\Gamma(i\omega) \geq 0$ *for all* $\omega \in \mathbb{R}^n$. *Then there exists an* $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ *such that* $\Gamma(\xi) = F^T(-\xi)F(\xi)$.

Note that even when Γ is a polynomial matrix, the entries of the matrix F are rational functions in n-indeterminates with real coefficients, whereas for the 1D case one can obtain an F with polynomial entries. Several results related to this factorization problem for the two-dimensional (2D) case (with some additional conditions like holomorphicity on certain complex half planes or unit polydiscs) exist in the literature [3, 11, 14, 16]. A different factorization problem involving symmetric nD matrices is shown in [5].

10. Main results. We now return to the problem of existence of a storage function and a dissipation rate for a Φ-dissipative system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. For the sake of clarity, we first consider the case of $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and subsequently the case of a general $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. We start with a proposition that gives a condition on Φ for $\int Q_\Phi$ to be nonnegative.

PROPOSITION 11. *Let* $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then* $(\int Q_\Phi \geq 0)$ *if and only if* $(\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n)$.

Proposition 11 and the factorizability implied by Theorem 10 readily lead to the following theorem.

THEOREM 12. *Let* $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then the following conditions are equivalent:*

1. $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0$ *for all* $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.
2. *There exists a polynomial matrix* $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective and* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *for* $k = 1, 2, \ldots, n$ *such that the VQDF* Q_Ψ *is a storage function, i.e.,*

$$\text{div } Q_\Psi(\ell) \leq Q_\Phi(w)$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.

3. *There exists a polynomial matrix* $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective and a* $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *such that* Q_Δ *is a dissipation rate, i.e.,*

$$Q_\Delta \geq 0 \text{ and } \int_{\mathbb{R}^n} Q_\Delta(\ell) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$$

for all $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$.

4. *There exists a polynomial matrix* $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is surjective, a* $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *for* $k = 1, 2, \ldots, n$, *and a*

$\Delta = \Delta^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *such that*

$$Q_\Delta \geq 0$$
$$\textit{and}$$
$$\operatorname{div} Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell) \tag{24}$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ *and* $w = M(\frac{d}{d\mathbf{x}})\ell$. *Note that this states that the VQDF* Q_Ψ *is a storage function and that* Q_Δ *is a dissipation rate.*

It follows from the proof of Theorem 12 that several Δ's may satisfy the same dissipation condition; i.e., several Δ's may be present, all of them satisfying $Q_\Delta \geq 0$ and $\int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x}$ for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ and satisfying the equation

$$w = M\left(\frac{d}{d\mathbf{x}}\right)\ell$$

for some $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. The first part of this nonuniqueness comes from the factorization of the matrix $\Phi(-\xi,\xi)$. Choosing a particular factorization of the matrix $\Phi(-\xi,\xi)$ still leaves us with a choice for Δ depending on the M we choose. Also note that unlike the 1D case, there is no one-to-one correspondence between storage and dissipation functions. Given a supply QDF and an associated dissipation QDF, one can find several VQDFs that satisfy (24). One should also note the unavoidable emergence of latent variables in the dissipation equation (24) for the nD case. In the 1D case, the dissipation equation can be written in terms of manifest variables alone, whereas in the nD case, this is only possible if the latent variables that appear in the dissipation equation are observable.

The case of an arbitrary $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ is easily reduced to the free case by considering an image representation for $\mathfrak{B}$. This leads to the following theorem, which is the main result of the paper.

THEOREM 13. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ *and* $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. *The following conditions are equivalent:*

1. $\mathfrak{B}$ *is* Φ*-dissipative; i.e.,* $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$ *for all* $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$.
2. *There exists an integer* $\mathtt{l} \in \mathbb{N}$, *a polynomial matrix* $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$ *such that* $M(\frac{d}{d\mathbf{x}})$ *is an image representation of* $\mathfrak{B}$, *a* $\Psi = (\Psi_1,\Psi_2,\ldots,\Psi_n)$ *with* $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ *for* $k = 1,2,\ldots,n$, *and a* $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ *such that*

$$Q_\Delta \geq 0$$
$$\textit{and}$$
$$\operatorname{div} Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell)$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$.

11. Conclusions. In this paper, we dealt with distributed systems described by constant coefficient linear partial differential equations. We started by defining controllability for such systems in terms of patching up of feasible trajectories. We then explained that it is exactly the controllable systems which allow an image representation, i.e., a representation in terms of what in physics is called a potential function. Subsequently, we turned to lossless and dissipative systems.

For lossless systems, we proved the equivalence with the existence of a conservation law involving the storage function. Important features of the storage function are (i) the fact that it depends on latent variables that are in general hidden (i.e.,

nonobservable) and (ii) its nonuniqueness. We have illustrated these features by demonstrating that they are already present in Maxwell's equations.

For dissipative systems, we proved the equivalence with the existence of a storage function and a dissipation rate. The problem of constructing a dissipation rate led to the question of factorizability of certain polynomial matrices in n variables. We reduced this problem to Hilbert's 17th problem, the representation of a nonnegative rational function in n variables as a sum of squares of rational functions.

12. Appendix. We collect the proofs in this appendix.

Proof of Theorem 2. Please refer to [19, Theorem 3]. □

Proof of Theorem 4. First we show the equivalence of the first three statements. Then we will show the equivalence of the last two statements. Finally, we link up these two sets of conditions.

(1) $\Leftrightarrow$ (3) If two trajectories w_1 and w_2 agree along with all their derivatives on the boundary $\partial\Omega$ of some arbitrary closed bounded subset $\Omega \subset \mathbb{R}^n$ (with nonempty interior), then $w = w_1 - w_2$ can be thought of as a trajectory with its support strictly in the interior of Ω. Thus $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$. Since

$$Q_\Phi(w_1) = Q_\Phi(w_2) + Q_\Phi(w) + 2L_\Phi(w_2, w),$$

we conclude that $\int_\Omega Q_\Phi$ is independent of path for all Ω if and only if

$$\int_\Omega Q_\Phi(w)d\mathbf{x} + 2\int_\Omega L_\Phi(w, v)d\mathbf{x} = 0 \tag{25}$$

for any $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$ with support in Ω, any $v \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$, and any closed bounded subset Ω in $\mathbb{R}^n$ with nonempty interior.

In particular, if we choose v in (25) to be zero, we obtain $\int_\Omega Q_\Phi(w)d\mathbf{x} = 0$, which in turn implies $\int Q_\Phi = 0$. Thus (25) yields $\int_\Omega L_\Phi(w, v)d\mathbf{x} = 0$. But

$$L_\Phi(w, v)(\mathbf{x}) = \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}}w}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})\right)^T \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}}v}{d\mathbf{x}^{\mathbf{l}}}(\mathbf{x})\right).$$

Now on integrating each of these terms by parts, we obtain

$$\int_\Omega L_\Phi(w, v)d\mathbf{x} = \int_\Omega w \cdot \left[\sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \left((-1)^{\mathbf{k}} \frac{(d)^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} \left(\frac{d^{\mathbf{l}}v}{d\mathbf{x}^{\mathbf{l}}}\right)\right)\right] = 0$$

for every $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$ with support in the interior of $\Omega \subset \mathbb{R}^n$ and for every such Ω. This can hold if and only if $\sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}}((-1)^{\mathbf{k}} \frac{(d)^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}}(\frac{d^{\mathbf{l}}v}{d\mathbf{x}^{\mathbf{l}}})) = 0$ or, equivalently, $\Phi(-\xi, \xi) = 0$. A simple reversal of arguments shows that (3) $\Rightarrow$ (1).

(2) $\Leftrightarrow$ (3) If $\Phi(-\xi, \xi) = 0$, then using integration by parts it is clear that $\int Q_\Phi = 0$. Conversely, if $\int Q_\Phi = 0$, then, using integration by parts, we obtain the condition $\sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}}((-1)^{\mathbf{k}} \frac{(d)^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}}(\frac{d^{\mathbf{l}}w}{d\mathbf{x}^{\mathbf{l}}})) = 0$ for every $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$. This is only possible if $\Phi(-\xi, \xi) = 0$.

(4) $\Leftrightarrow$ (5) As mentioned earlier,

$$(\text{div } Q_\Psi)(w) = \frac{\partial}{\partial x_1} Q_{\Psi_1}(w) + \cdots + \frac{\partial}{\partial x_n} Q_{\Psi_n}(w).$$

It is easy to see that $\frac{\partial}{\partial x_1} Q_{\Psi_1}(w) = L_{\Psi_1}(\frac{\partial w}{\partial x_1}, w) + L_{\Psi_1}(w, \frac{\partial w}{\partial x_1}) = Q_{\Psi_1'}(w)$, where $\Psi_1' = (\zeta_1 + \eta_1)\Psi_1$. Now the equivalence of (4) and (5) is obvious.

(4) $\Rightarrow$ (3) This is obvious from substitution.

(3) $\Rightarrow$ (4) Given a $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, which contains polynomials in $2n$ variables, we can rewrite this matrix as a polynomial matrix in the $2n$ new variables given by $\gamma_i = \zeta_i + \eta_i$ and $\gamma_{i+n} = \zeta_i - \eta_i$ for $i = 1, \ldots, n$. Now setting $-\zeta = \eta = \xi$ translates to setting $\gamma_i = 0$ for $i = 1, \ldots, n$. Thus if $\Phi(-\xi, \xi) = 0$, then each term in the matrix can be written as polynomials in γ_i, $i = 1, \ldots, n$, without the free term. By the very definition of these γ_i's and the symmetry of the matrices, we have the result. ∎

Proof of Proposition 5. ($\Leftarrow$): Let

$$\Psi_i = (\zeta_1 + \eta_1)\Delta_{i1} + \cdots + (\zeta_n + \eta_n)\Delta_{in}$$

with $\Delta_{ij} = -\Delta_{ji}$ as given in the statement of the proposition. Let Φ be such that $Q_\Phi = \operatorname{div} Q_\Psi$. Then

$$\begin{aligned}
\Phi &= (\zeta_1 + \eta_1)\Psi_1 + \cdots + (\zeta_n + \eta_n)\Psi_n \\
&= \sum_{i=1}^{n} (\zeta_i + \eta_i)\Psi_i \\
&= \sum_{i=1}^{n} (\zeta_i + \eta_i) \sum_{j=1}^{n} (\zeta_j + \eta_j)\Delta_{ij} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (\zeta_i\zeta_j + \zeta_i\eta_j + \eta_i\zeta_j + \eta_i\eta_j)\Delta_{ij} \\
&= 0 \quad \text{since } \Delta_{ij} = -\Delta_{ji}.
\end{aligned}$$

($\Rightarrow$): Let $\Psi \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta])^n$ define a VQDF with null divergence, that is, div $Q_\Psi = 0$. Clearly, it is enough to just consider any (k, l)th entry of the corresponding QDFs since the same arguments would apply to all other entries. For simplicity, let us denote by p_i the (k, l)th entry of the QDF Ψ_i.

As in the proof of Theorem 4, we will again employ a change of variables. Let $\gamma_i = \zeta_i + \eta_i$ and $\gamma_{i+n} = \zeta_i - \eta_i$ for $i = 1, \ldots, n$. Since Ψ defines a null divergence,

$$\sum_{i=1}^{n} \gamma_i p_i = 0. \tag{26}$$

Setting all γ_i's except γ_j to zero in (26), we conclude that p_j can be written as

$$p_j = \sum_{i=1, i\neq j}^{n} f_{ji}\gamma_i$$

for some arbitrary polynomials f_{ji} in the $2n$ variables γ_i. Now reverting back to ζ_i's and η_i's, this precisely means that $\Psi_j = (\zeta_1 + \eta_1)\Delta_{j1} + \cdots + (\zeta_n + \eta_n)\Delta_{jn}$ with the entries of Δ_{ji}'s being the corresponding f_{ji}'s. Note that f_{ii}'s are all zero; that is, $\Delta_{ii} = 0$ for $i = 1, \ldots, n$.

We can rearrange the terms of Δ_{ij}'s to obtain $\Delta_{ij} = -\Delta_{ji}$. This is in some sense similar to the stepping stone algorithm in operations research. We give an outline of the proof here. For a proof of this, we return to the variables γ_i's. Again, it is enough to consider any (k, l)th entry. As before, let p_i be the (k, l)th entry

of Ψ_i. Let $p_i = \sum_{\mathbf{k}} f_{i,\mathbf{k}} \gamma^{\mathbf{k}}$, where γ is a monomial in terms of γ_i's, $i = 1, \ldots, n$. Thus $\gamma^{\mathbf{k}} = \gamma_1^{k_1} \cdots \gamma_n^{k_n}$, where the multi-index $\mathbf{k} = (k_1, \cdots, k_n)$. Note that $f_{i,\mathbf{k}}$ are polynomials in γ_{i+n}'s, $i = 1, \ldots, n$. Let $\mathbf{k_j} = (k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_n)$, and similarly $\mathbf{k_{ij}}$ is a multi-index defined in the obvious way. Fix a multi-index $\mathbf{k}$ and hence a monomial $\gamma^{\mathbf{k}}$. We shall demonstrate how to rearrange the terms for this particular monomial, and similar operations should be carried out for each such monomial. From the condition $\sum_{i=1}^n \gamma_i p_i = 0$, we can conclude that $\sum_{i=1}^n f_{i,\mathbf{k_i}} = 0$ (since they are the coordinates of the monomial $\gamma^{\mathbf{k}}$). Note that some of these $f_{i,\mathbf{k_i}}$'s may be zero. Set the coordinate of $\gamma^{\mathbf{k_{12}}}$ in the corresponding term ((k,l)th term) in Δ_{12} to be $f_{1,\mathbf{k_1}}$. We force the coordinate of $\gamma^{\mathbf{k_{12}}}$ in the (k,l)th term of Δ_{21} to be $-f_{1,\mathbf{k_1}}$ and the coordinate of $\gamma^{\mathbf{k_{23}}}$ in the (k,l)th term of Δ_{23} to be $f_{2,\mathbf{k_2}} + f_{1,\mathbf{k_1}}$ and so on. We perform the above operation for every multi-index $\mathbf{k}$ that might be involved. (These are finite in number.) Then $p_j = \sum_{i=1}^n \gamma_i g_{ji}$, where g_{ji} are the polynomials obtained by the above procedure and are the (k,l)th terms of the corresponding Δ_{ji}'s. We can easily check the entries of $\Delta_{ij} = -\Delta_{ji}$. Thus we have demonstrated how to construct the required Δ_{ij}'s that satisfy the conditions of the proposition. ∎

Proof of Theorem 7. (2) ⇒ (1) Following the lines of the proof of Theorem 4, we can conclude that $\int_\Omega Q_\Phi(w) d\mathbf{x}$ is independent of path on $\mathfrak{B}$ if and only if

$$\int_\Omega Q_\Phi(w) d\mathbf{x} + 2 \int_\Omega L_\Phi(w, v) d\mathbf{x} = 0 \tag{27}$$

for any $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω, any $v \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$, and any closed bounded set $\Omega \subset \mathbb{R}^n$. Again, since $v = 0$ is a trajectory in $\mathfrak{B}$, we can conclude that $\int_\Omega Q_\Phi(w) d\mathbf{x} = 0$ for any $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω. Thus $\mathfrak{B}$ is lossless with respect to the QDF Q_Φ.

(1) ⇒ (4) Since $\int_\Omega Q_\Phi(w) d\mathbf{x} = 0$ for any $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$ with support in Ω, from (27) we can conclude that $\int_\Omega L_\Phi(w, v) d\mathbf{x} = 0$ for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω and all $v \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$. Using integration by parts, it is easy to see that the integral $\int_\Omega (w^T f) d\mathbf{x}$ equals zero for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω if f is in the image of the operator $R^T(-\frac{d}{d\mathbf{x}})$. In $L_\Phi(w, v)$, every v we consider is of the form $v = M(\frac{d}{d\mathbf{x}})\ell$, and so we conclude that $\Phi(-\xi, \xi) M(\xi) = R^T(-\xi) Y(\xi)$ for some Y. Now premultiplying by $M^T(-\xi)$, we obtain $\Phi'(-\xi, \xi) = M^T(-\xi)\Phi(-\xi, \xi) M(\xi) = M^T(-\xi) R^T(-\xi) Y(\xi) = 0$ since $R(\xi) M(\xi) = 0$.

(4) ⇒ (2) From [22], we know that, given a controllable behavior $\mathfrak{B}$, there exists some operator $M_1(\xi)$ such that $\mathfrak{B} = \{w : w = M_1(\frac{d}{d\mathbf{x}})\ell\}$ with the additional property that every $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}})$ can be obtained as the image of some ℓ which is itself compactly supported. It can be shown that if $\Phi'(-\xi, \xi) = M^T(-\xi)\Phi(-\xi, \xi) M(\xi) = 0$ for some image representation of $M(\xi)$, then $M'^T(-\xi)\Phi(-\xi, \xi) M'(\xi) = 0$ for every image representation $M'(\frac{d}{d\mathbf{x}})$ of the controllable behavior $\mathfrak{B}$. In particular, $\Phi_1(\zeta, \eta) = M_1^T(\zeta)\Phi(\zeta, \eta) M_1(\eta)$ has the property that $\Phi_1(-\xi, \xi) = 0$. Using integration by parts, we can now show that $\int_\Omega L_\Phi(w, v) d\mathbf{x} = \int_\Omega L_{\Phi_1}(\ell_1, \ell_2) d\mathbf{x} = 0$ for any $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω, $w = M_1(\frac{d}{d\mathbf{x}})\ell_1$ with ℓ_1 having compact support, and $v = M_1(\frac{d}{d\mathbf{x}})\ell_2$. A similar argument also proves that $\int_\Omega Q_\Phi(w) d\mathbf{x} = 0$ for all $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathbb{w}}) \cap \mathfrak{B}$ with support in Ω. From (27), we then conclude that the QDF induced by Φ is independent of path on $\mathfrak{B}$.

(3) ⇔ (4) This is already shown in Theorem 4.

(4) ⇔ (5) From Theorem 4, it is clear that there exists $\Psi \in (\mathbb{R}^{m \times m}[\zeta, \eta])^n$ such that $\mathrm{div} Q_\Psi(\ell) = Q_{\Phi'}(\ell)$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, since $M(\frac{d}{d\mathbf{x}})$ gives an

image representation of the behavior $\mathfrak{B}$, $Q_\Phi(w) = Q_{\Phi'}(\ell)$ for all w and ℓ related by $w = M(\frac{d}{dx})\ell$. □

Proof of Theorem 10. We will start by considering the scalar case. Let $\Gamma \in \mathbb{R}[\xi]$ with $\Gamma^\star = \Gamma$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. From the above data, we can conclude that the polynomial p defined as $p(\omega) := \Gamma(i\omega)$ is a positive definite (or nonnegative) function. Using the result from Hilbert's 17th problem [1], we can write

$$p(\omega) = \sum_{i=1}^{r} \left(f_i(\omega)\right)^2,$$

where each $f_i \in \mathbb{R}(\omega)$; i.e., f_i's are rational functions in ω. Now consider each of these f_i's. We can write them as

$$f_i(\omega) = f_{i+}(\omega) + f_{i-}(\omega),$$

where

$$f_{i+}(\omega) = \frac{f_i(\omega) + f_i(-\omega)}{2},$$
$$f_{i-}(\omega) = \frac{f_i(\omega) - f_i(-\omega)}{2}.$$

Thus we separate the even and odd parts of the functions f_i. Now

$$\begin{aligned} p(\omega) &= \sum_{i=1}^{r} \left(f_{i+}(\omega) + f_{i-}(\omega)\right)^2 \\ &= \sum_{i=1}^{r} \left[\left(f_{i+}(\omega)\right)^2 + \left(f_{i-}(\omega)\right)^2 + 2f_{i+}(\omega)f_{i-}(\omega)\right], \end{aligned}$$

and

$$\begin{aligned} p(-\omega) &= \sum_{i=1}^{r} \left(f_{i+}(-\omega) + f_{i-}(-\omega)\right)^2 \\ &= \sum_{i=1}^{r} \left[\left(f_{i+}(\omega)\right)^2 + \left(f_{i-}(\omega)\right)^2 - 2f_{i+}(\omega)f_{i-}(\omega)\right]. \end{aligned}$$

Since $p(\omega) = p(-\omega)$ (from the initial assumption on Γ), we can conclude $\sum_{i=1}^{r} f_{i+}(\omega)f_{i-}(\omega) = 0$, and so

$$\begin{aligned} p(\omega) &= \sum_{i=1}^{r} \left[\left(f_{i+}(\omega)\right)^2 + \left(f_{i-}(\omega)\right)^2\right] \\ &= \sum_{i=1}^{r} \left[f_{i+}(\omega) + if_{i-}(\omega)\right]\left[f_{i+}(\omega) - if_{i-}(\omega)\right] \\ &= \sum_{i=1}^{r} \left[f_{i+}(\omega) + if_{i-}(\omega)\right]\left[f_{i+}(-\omega) + if_{i-}(-\omega)\right]. \end{aligned}$$

Thus note that p has now been written down as the sum of r terms, each of which is a product of a function evaluated at ω and $-\omega$. Construct an $r \times 1$ matrix E

whose $(i,1)$th entry is $(f_{i+} + if_{i-})$. Then $\Gamma(i\omega) = p(\omega) = E^T(-\omega)E(\omega)$, which in turn implies that $\Gamma(\xi) = F^T(-\xi)F(\xi)$, where F is obtained from E by the obvious substitution; i.e., the $(i,1)$th entry of F is $(f_{i+} + f_{i-})$.

We now tackle the case where $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ with the properties $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Let $\Gamma(\xi) = [a_{ij}(\xi)]$; i.e., let the polynomial in the (i,j)th coordinate of Γ be denoted by a_{ij}. Clearly, all diagonal elements a_{ii} are even polynomials in ξ since $\Gamma^T(-\xi) = \Gamma(\xi)$. Let $I = \{i : a_{ii}(\xi) = 0\}$. Consider the submatrix obtained from Γ by choosing elements a_{ij}, where both $i, j \in I$. Let us call this matrix $H(\xi) = [a_{ij}(\xi)]_{i,j\in I}$. Observe that $\Gamma(i\omega)$ (and hence $H(i\omega)$) are Hermetian matrices for any $\omega \in \mathbb{R}^n$. Since the trace of $H(i\omega)$ is zero, we can conclude that $H(i\omega)$ has a negative real eigenvalue (provided $H(i\omega)$ has nonzero entries). In this case, $H(i\omega) \not\geq 0$, and hence $\Gamma(i\omega) \not\geq 0$. However, this contradicts the assumption that $\Gamma(i\omega) \geq 0$. Hence we conclude that $H(\xi) = 0$.

We now construct a 2×2 submatrix of $\Gamma(\xi)$, denoted by $H_1(\xi)$, by considering the four elements $a_{ij}(\xi) \in \Gamma(\xi)$, where $i \in I$ and $j \notin I$ are fixed. Clearly, the determinant of $H_1(\xi)$ is given by $-a_{ij}(\xi)a_{ji}(\xi) = -a_{ij}(\xi)a_{ij}(-\xi)$, and this is an even polynomial in ξ. Substituting $i\omega$ for ξ, we obtain $H_1(i\omega)$—a 2×2 Hermetian matrix with a negative determinant. This implies a negative real eigenvalue, and hence $H_1(i\omega) \not\geq 0$. But since $\Gamma(i\omega) \geq 0$, we conclude that $a_{ij}(\xi) = a_{ji}(\xi) = 0$. From the above discussion, we conclude that the rows and columns corresponding to $i \in I$ have only zero entries.

We now define and prove an algorithmic step. In this step, we take as an input a $\mathtt{w}\times\mathtt{w}$ matrix Γ that has the following properties: (i) $\Gamma^\star = \Gamma$ and (ii) $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. The outputs of this algorithmic step are two matrices, which we call E and Q. E has the same number of columns as Γ. The matrix Q has the same properties as Γ but is a $(\mathtt{w}-1)\times(\mathtt{w}-1)$ matrix. Let

$$\Gamma = \begin{bmatrix} a_{11} & \gamma \\ \gamma^\star & \Gamma_{\mathtt{w}-1} \end{bmatrix}.$$

If $a_{11} \neq 0$, then construct $F_1(\xi)$ such that $a_{11}(\xi) = F_1^T(-\xi)F_1(\xi)$. Define the matrix $E = \begin{bmatrix} F_1 & F_1 a_{11}^{-1}\gamma \end{bmatrix}$. Clearly,

$$\begin{bmatrix} F_1^\star \\ \gamma^\star(a_{11}^{-1})^\star F_1^\star \end{bmatrix} \begin{bmatrix} F_1 & F_1 a_{11}^{-1}\gamma \end{bmatrix} = \begin{bmatrix} a_{11} & \gamma \\ \gamma^\star & \gamma^\star(a_{11}^{-1})^\star\gamma \end{bmatrix}.$$

Note that $(a_{11}^{-1})^\star = a_{11}^{-1}$. On the other hand, if $a_{11} = 0$, from the discussion above we know $\gamma = 0$. In this case, define the matrix $E = \begin{bmatrix} 0_{1\times 1} & 0_{1\times\{\mathtt{w}-1\}} \end{bmatrix}$. Define the $(\mathtt{w}-1)\times(\mathtt{w}-1)$ matrix

$$Q = \begin{cases} \Gamma_{\mathtt{w}-1} - \gamma^\star a_{11}^{-1}\gamma & \text{when} \quad a_{11}, \neq 0 \\ \Gamma_{\mathtt{w}-1} & \text{when} \quad a_{11} = 0. \end{cases}$$

(When $a_{11} \neq 0$, then the above matrix Q is known in the literature as the Schur complement of a_{11} in Γ; see [7, 9].) Clearly, the matrix Q is such that $Q^T(-\xi) = Q(\xi)$. Moreover, we claim that $Q(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. This is clear for the case when $a_{11} = 0$. For the case when $a_{11} \neq 0$, we have $Q = \Gamma_{\mathtt{w}-1} - \gamma^\star a_{11}^{-1}\gamma$. Suppose $Q(i\omega) \not\geq 0$ for some $\omega \in \mathbb{R}^n$. This implies that there exists a vector a such that $a^T Q(i\omega)a < 0$. Consider the vector

$$v = \begin{bmatrix} -(a_{11}^{-1}(i\omega)\gamma(i\omega)a \\ a \end{bmatrix}.$$

Then $v^T\Gamma(i\omega)v = a^TQ(i\omega)a < 0$, and this contradicts $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Hence we have a $(\mathtt{w}-1)\times(\mathtt{w}-1)$ matrix Q such that $Q^T(-\xi) = Q(\xi)$ and $Q(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$.

We now give the algorithm to construct F such that $\Gamma = F^T(-\xi)F(\xi)$.

1. Set $Q_0 = \Gamma$. Set $i = 0$.
2. Invoke the algorithmic step defined above with input Q_i to obtain outputs E and Q. Set $E_{i+1} = E$ and $Q_{i+1} = Q$.
3. If $Q_{i+1} \neq 0$, increment i and go back to step 2.
4. Construct

$$F = \begin{bmatrix} E_1 & \\ 0_{\bullet\times 1} & E_2 \\ 0_{\bullet\times 2} & E_3 \\ \vdots & \vdots \\ 0_{\bullet\times i} & E_i \end{bmatrix}.$$

It is now easy to check that $\Gamma = F^T(-\xi)F(\xi)$. □

Proof of Proposition 11. ($\Leftarrow$) Since $w \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$, we can take the multidimensional Fourier transform. Let $\hat{w}$ denote the multidimensional Fourier transform of w. Then, using Parseval's theorem, we have

$$\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{w}(-i\omega)\Phi(-i\omega, i\omega)\hat{w}(i\omega)d\omega, \tag{28}$$

where $\omega \in \mathbb{R}^n$. Then ($\Leftarrow$) is clear.

($\Rightarrow$) Consider a compactly supported function $u(\mathbf{x})$, where the variables can be separated, i.e., $u(\mathbf{x}) = v_1(x_1)v_2(x_2)\cdots v_n(x_n)$. Let $\int_{\mathbb{R}^n} Q_\Phi(u)d\mathbf{x} = E$. Suppose $\Phi(-i\omega, i\omega) \not\geq 0$ for all $\omega \in \mathbb{R}^n$. Let $a \in \mathbb{C}^q$ and $\omega_0 = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ be such that $a^T\Phi(-i\omega_0, i\omega_0)a < 0$. Without loss of generality, we will assume that $\omega_1 \neq 0$. Then we can choose functions $u_N(\mathbf{x})$ given by $u_N(\mathbf{x}) = v_{1,N}(x_1)v_{2,N}(x_2)\cdots v_{n,N}(x_n)$, where

$$v_{1,N}(x_1) = \begin{cases} e^{i<\omega_1, x_1>}a, & \mid x_1 \mid \leq \frac{2\pi N}{\omega_1}, \\ v_1(x_1 + \frac{2\pi N}{\omega_1}), & x_1 < -\frac{2\pi N}{\omega_1}, \\ v_1(x_1 - \frac{2\pi N}{\omega_1}), & x_1 > \frac{2\pi N}{\omega_1}, \end{cases} \tag{29}$$

and for $i = 2, \ldots, n$ and $\omega_i \neq 0$,

$$v_{i,N}(x_i) = \begin{cases} e^{i<\omega_i, x_i>}[1, 1, \ldots, 1]^T, & \mid x_i \mid \leq \frac{2\pi N}{\omega_i}, \\ v_i(x_i + \frac{2\pi N}{\omega_i}), & x_i < -\frac{2\pi N}{\omega_i}, \\ v_i(x_i - \frac{2\pi N}{\omega_i}), & x_i > \frac{2\pi N}{\omega_i}, \end{cases} \tag{30}$$

and for $i = 2, \ldots, n$ with $\omega_i = 0$, we have

$$v_{i,N}(x_i) = v_i(x_i). \tag{31}$$

Then, on evaluating $\int_{\mathbb{R}^n} Q_\Phi(u_N)d\mathbf{x}$, one gets a negative term that depends on N and another term that is independent of N. Then, by choosing N large enough, this integral can be made negative and hence we obtain a contradiction to $\int Q_\Phi \geq 0$. □

Proof of Theorem 12. Note that (4) is a statement that combines (2) and (3). So we first we show the equivalence of the first three statements. Then we show how statement (4) is equivalent.

(1) $\Rightarrow$ (3) Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. By Proposition 11, we know that $\int Q_\Phi \geq 0$ implies that $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Thus $\Phi(-\xi,\xi)$ satisfies all the conditions on Γ in Theorem 10, and hence $\Phi(-\xi,\xi) = F^T(-\xi)F(\xi)$ for some $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$.

Let us now consider the matrix $F \in \mathbb{R}^{r\times\mathtt{w}}(\xi)$. (Here we assume that the number of rows of F is some arbitrary (finite) number r.) It is easy to show that $F = DM^{-1}$, where $D \in \mathbb{R}^{r\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. Note that both D and M are now matrices with polynomial entries. Moreover, we can choose M such that it is a diagonal matrix (and hence the operator $M(\frac{d}{d\mathbf{x}})$ is surjective). We can now define latent variables ℓ related to the manifest variables w by the equation $w = M(\frac{d}{d\mathbf{x}})\ell$. Now consider $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ given by the matrix $\Delta(\zeta,\eta) = D^T(\zeta)D(\eta)$. Obviously, $Q_\Delta \geq 0$. Now for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ we have $M(\frac{d}{d\mathbf{x}})\ell = w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$. Hence it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} &= \int_{\mathbb{R}^n} (w)^T \left(\Phi\left(-\frac{d}{d\mathbf{x}},\frac{d}{d\mathbf{x}}\right)w\right)d\mathbf{x}\\ &= \int_{\mathbb{R}^n} \left(M\left(\frac{d}{d\mathbf{x}}\right)\ell\right)^T \left(\Phi\left(-\frac{d}{d\mathbf{x}},\frac{d}{d\mathbf{x}}\right)M\left(\frac{d}{d\mathbf{x}}\right)\ell\right)d\mathbf{x}\\ &= \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x}.\end{aligned}$$

Thus the matrix $\Delta(\zeta,\eta) = D^T(\zeta)D(\eta)$ induces a QDF, which defines a dissipation function for the given Φ.

(3) $\Rightarrow$ (2) Let $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induce a QDF on the latent variables ℓ such that Q_Δ is a dissipation function associated to the given Φ. Moreover, let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ induce a surjective operator, such that $w = M(\frac{d}{d\mathbf{x}})\ell$. Consider the system defined by $w - M(\frac{d}{d\mathbf{x}})\ell = 0$ in the variables (w,ℓ) with a QDF induced by

$$\begin{bmatrix} \Phi & 0 \\ 0 & -\Delta \end{bmatrix} \in \mathbb{R}^{(2\mathtt{w})\times(2\mathtt{w})}[\zeta,\eta].$$

Note that the system given by $w - M(\frac{d}{d\mathbf{x}})\ell = 0$ is a controllable system. In fact, this system has an observable image representation given by

$$\begin{bmatrix} w \\ \ell \end{bmatrix} = \begin{bmatrix} M(\frac{d}{d\mathbf{x}}) \\ I \end{bmatrix} \ell$$

(for details see [19]). The QDF defined above gives

$$\int_{\mathbb{R}^n} [Q_\Phi(w) - Q_\Delta(\ell)]\, d\mathbf{x} = 0$$

on every compactly supported trajectory in the full behavior involving both w and ℓ, and so, applying Theorem 7, we obtain a QDF $\Phi'(\zeta,\eta) = M^T(\zeta)\Phi(\zeta,\eta)M(\eta) - \Delta(\zeta,\eta)$ which acts on the variables ℓ. Since $\Phi'(-\xi,\xi) = 0$, we can find a corresponding VQDF Q_Ψ acting on the latent variables ℓ such that $Q_\Phi(w) - Q_\Delta(\ell) = \operatorname{div} Q_\Psi(\ell)$. This VQDF can now, in turn, be interpreted as a storage function for the original Φ since $\operatorname{div} Q_\Psi(\ell) \leq Q_\Phi(w)$ for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$.

(2) $\Rightarrow$ (1) Let Φ admit a storage function. So $\operatorname{div} Q_\Psi(\ell) \leq Q_\Phi(w)$ for all $\ell \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ and $w = M(\frac{d}{d\mathbf{x}})\ell$. On integrating this inequality over all of $\mathbb{R}^n$, we get $0 \leq \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x}$. Now consider any $w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$. Since $M(\frac{d}{d\mathbf{x}})$ is surjective, there exist some $\ell_0 \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ such that $w = M(\frac{d}{d\mathbf{x}})\ell_0$. Choose a sequence of $\ell_i \in$

$\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ such that $\lim_{i\to\infty} \ell_i = \ell_0$. Let $w_i = M(\frac{d}{d\mathbf{x}})\ell_i$. Clearly, $w_i \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Hence, by continuity, we conclude that $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$. Thus $\int Q_\Phi \geq 0$.

(1) $\Rightarrow$ (4) Given a Φ, the steps in (1) $\Rightarrow$ (3) above explain how Δ can be obtained. The proof of (3) $\Rightarrow$ (2) explains how Ψ can be obtained. Observe that the latent variables involved in the construction of Ψ are the latent variables ℓ. These are the latent variables associated with the image representation of the controllable behavior given by $w - M(\frac{d}{d\mathbf{x}})\ell = 0$. Thus we obtain the equation

$$\operatorname{div} Q_\Psi(\ell) = Q_\Phi(w) - Q_\Delta(\ell).$$

Note that the same set of latent variables is associated to storage and dissipation.

(4) $\Rightarrow$ (1) This is obvious from (2) $\Rightarrow$ (1). ∎

Proof of Theorem 13. (1) $\Rightarrow$ (2) First, we convert the problem into a problem on a free behavior by using the image representation of the behavior. Let $M' \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$ define an operator $M'(\frac{d}{d\mathbf{x}})$ that defines an image representation of the given behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. Define $\Phi' := M'^T(\zeta)\Phi(\zeta,\eta)M'(\eta)$, and look at the action of the QDF induced by Φ' on the full space of latent variables ℓ' that appear in the image representation of $\mathfrak{B}$ (i.e., $\mathfrak{B} = \{w \mid w = M'(\frac{d}{d\mathbf{x}})\ell'\}$). Clearly, $\int_{\mathbb{R}^n} Q_{\Phi'}(\ell') \geq 0$ for all $\ell' \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$. By Theorem 12, we can find $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ for $k = 1, 2, \ldots, n$ and $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ such that

$$\operatorname{div} Q_\Psi(\ell) = Q_{\Phi'}(\ell') - Q_\Delta(\ell)$$

with ℓ' and ℓ related by $\ell' = M(\frac{d}{d\mathbf{x}})\ell$ and $M \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\xi]$. Observe that by definition $Q_\Phi(w) = Q_{\Phi'}(\ell')$, where $w = M'(\frac{d}{d\mathbf{x}})\ell'$. Hence we obtain the required result with w and ℓ related by $w = M'(\frac{d}{d\mathbf{x}})M(\frac{d}{d\mathbf{x}})\ell$, which is also an image representation of $\mathfrak{B}$, since $M(\frac{d}{d\mathbf{x}})$ is surjective (Theorem 12).

(2) $\Rightarrow$ (1) Consider any $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Then $w = M(\frac{d}{d\mathbf{x}})\ell$ for some $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell)$. Clearly, if $\ell \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$, then

$$0 = \int_{\mathbb{R}^n} \operatorname{div} Q_\Psi(\ell)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} - \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x}.$$

Thus $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x} \geq 0$. If $\ell \notin \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$, then one can find a sequence of $\ell_i \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ that converges to ℓ. Then, by continuity, we obtain $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. ∎

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