Efficient Calculation of the MMSE-GDFE Decoding Order in Non-Ideal DSL Multiple-Access Channels

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***Abstract*—The minimum mean squared error (MMSE)-generalized decision feedback equalizer (GDFE) provides a high multi-user crosstalk cancellation performance in upstream digital subscriber lines (DSL) scenarios. The weighted sum rate maximization approach can be used to find the points located on the boundary of the corresponding rate region (RR). The performance of the MMSE-GDFE depends on the decoding order of the users. Unfortunately, unlike in ideal multiple access channels (MAC), the optimal decoding order is not determined by the order of weight factors in practical DSL MACs. To reach the boundary of the RR, a tonal exhaustive search (TES) with computational complexity $\mathcal{O}(N^2N!)$ is required for each tone, where N is the number of users. In this paper, we investigate the main deviations of the DSL receiver from the ideal MAC receiver and model their effects as a set of update factors on the weight factors. The order of the updated weight factors is then used to determine the decoding order of the users in an iterative fashion. The computational complexity of our updated weight factor ordering (UWFO) algorithm is $\mathcal{O}(N^4)$ which is significantly smaller than that of TES. Moreover, our simulation results show that the algorithm achieves bit rates very close to the boundary of the RR leading to approximately the same performance as TES. UWFO can be applied to wireless MACs and zero-forcing GDFEs as well.**

I. Introduction

Digital subscriber line (DSL) systems use telephone networks to provide broadband services to customers. When more than one DSL user is active in the same telephone cable bundle, each user receives some interfering signal from the other users due to the electromagnetic induction between the neighboring twisted wire pairs inside the cable bundle. This interfering signal is called crosstalk. Crosstalk can reduce the reach and bit rate of DSL services significantly.

When it is possible to jointly decode the received signals for all users, the DSL transmission channel in upstream (US) direction can be modeled as a multiple-access channel (MAC). Multiuser detectors or vectoring techniques can be used to effectively cancel crosstalk in DSL MACs. These include the linear zero-forcing (ZF) equalizer [1], linear minimum mean squared error (MMSE) equalizer [2], ZF generalized decision feedback equalizer (GDFE) [3], and MMSE-GDFE [4]. It has been shown that the MMSE-GDFE achieves noticeably higher bit-rates than all other available receiver structures in the presence of spatially colored noise [5].

The ZF-GDFE and MMSE-GDFE decode the received signal in a per-user fashion. When decoding the signal of a particular user, the interference from the users that are already decoded is removed. Therefore, the signal-to-the-noise-power ratio (SNR) improves for the users that are decoded later. By changing the order of decoding, users experience different SNRs and their bit rates could vary significantly. Therefore, determining the optimal decoding order is a crucial factor in the performance of the ZF- and MMSE-GDFE receivers. There are a total of $N!$ decoding orders with N users. To reach the boundary of the rate region (RR), tonal exhaustive search (TES) over all $N!$ decoding orders is required to find the optimal order of decoding. This means that the computational complexities of the ZF- and MMSE-GDFE receivers grow exponentially with N.

In [6] finding the optimal decoding order for the ZF-GDFE is considered and two algorithms, namely, successive order search (SOS) and modified greedy search (MGS), have been proposed. The computational complexities of SOS and MGS are $\mathcal{O}\left(LN^4\right)$ and $\mathcal{O}\left(N^3\right)$, respectively, where L is the number of iterations required by SOS for convergence. Simulation results show that SOS provides solutions close to the boundary of the RR, while MGS provides solutions that can be slightly away from the boundary of the RR.

In this paper, we propose a new algorithm, namely, modified weight factor ordering (MWFO), for determining the decoding order in MMSE-GDFE receivers. We consider the weighted sum rate (WSR) maximization approach to decouple the problem into independent per-tone problems. In this case, the

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optimal decoding order for the MMSE-GDFE receiver structure in the ideal MAC is known and is obtained by the order of weight factors (referred to weight factor ordering (WFO) in this paper) [7]. Unfortunately, WFO is not optimal for practical DSL MACs due to the presence of the non-zero so-called SNR gap and a bound on the per-tone QAM constellation size. By careful investigation of these two aspects, we model their effects by a set of correction factors on the weight factors. Then, the decoding order can be calculated according to the order of the updated weight factors in an iterative fashion. The resulting algorithm is called updated WFO (UWFO). Although, we discuss the MMSE-GDEF, our algorithm can be readily used for ZF-GDFE as well. The computational complexity of our algorithm is $\mathcal{O}(N^4)$. Our simulation results show that UWFO can provide solutions close to the boundary of the RR of the MMSE-GDFE. On the other hand, UWFO can be applied to ZF-GDFE and/or wireless MACs as well.

This paper is organized as follows. We describe the DSL MAC for both ideal and practical receivers in Sec. II. The proposed algorithm is explained in Sec. III. Simulation results are presented in Sec. IV, and finally the paper is concluded in Sec. V.

II. System Description

We consider a discrete multi-tone (DMT) transmission with K tones. We assume that the signals of N coordinated users are jointly decoded using a MMSE-GDFE receiver. The transmission over tone k can be modeled by:

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_k + \boldsymbol{z}_k, \tag{1}$$

where $\boldsymbol{x}_k$, $\boldsymbol{y}_k$, and $\boldsymbol{z}_k$ are the $N \times 1$ vectors of the transmitted signal, received signal, and noise, respectively, and $\boldsymbol{H}_k$ is the channel transfer matrix on tone k. The received noise $\boldsymbol{z}_k$ is the sum of the background noise and crosstalk signals from the uncoordinated users. The noise covariance matrix on tone k is defined by

$$\boldsymbol{Z}_k = \mathrm{E}\left\{\boldsymbol{z}_k \boldsymbol{z}_k^{\mathrm{H}}\right\}, \tag{2}$$

where $\mathrm{E}\{\,\cdot\,\}$ and $[\,\cdot\,]^{\mathrm{H}}$ denote the expected value and transpose conjugate operations, respectively.

The bit-rate of user n is obtained by the sum of the bit-loadings over the tones, i.e.,

$$R_n = f_{\mathrm{s}} \sum_{k=1}^{K} b_k^{(n)}, \tag{3}$$

where f_{s} is the DMT symbol rate, $b_k^{(n)}$ denotes the bit loading on tone k for the n-th user, and K is the total number of tones. Each user n has a minimum target rate $R_{\min}^{(n)}$. We assume that the power spectral density (PSD) of the users is fixed and is equal to the nominal PSD mask. The optimization parameters are the decoding orders on the tones. That is we should find the decoding order on all tones so that

$$R_n \geq R_{\min}^{(n)} \text{ for } n = 1 \text{ to } N. \tag{4}$$

It has been shown that this problem can be solved by the WSR maximization approach provided that the RR is *strictly* convex[1] [8], [9], i.e.,

$$\text{maximize} \sum_{n=1}^{N} w_n R_n = f_{\mathrm{s}} \sum_{n=1}^{N} \sum_{k=1}^{K} w_n b_k^{(n)}, \tag{5}$$

where w_n is the weight factor for user n, which should be adapted to satisfy (4). By changing the order of the summations over K and N, the problem is decoupled into K independent per-tone problems. Therefore, we should identify the decoding order to solve the following per-tone WSR maximization problem, for each tone k:

$$\text{maximize} \sum_{n=1}^{N} w_n b_k^{(n)}. \tag{6}$$

This problem is different in the ideal MAC and practical DSL MAC. In the following subsections, we discuss the problem for these two channels.

A. Ideal MAC

The sum capacity of the ideal MAC on tone k is given by

$$C_k = \log_2 \left(\det\left(\boldsymbol{Z}_k + \boldsymbol{H}_k \boldsymbol{S}_k \boldsymbol{H}_k^{\mathrm{H}} \right) / \det\left(\boldsymbol{Z}_k\right) \right), \tag{7}$$

where $\boldsymbol{S}_k \equiv \mathrm{diag}\left(s_k^{(1)}, s_k^{(2)}, \ldots, s_k^{(N)}\right)$ and $s_k^{(n)}$ is the transmit power of user n on tone k. Consider a MMSE-GDFE receiver structure and assume the users are sorted in the order of decoding. The number of bits that can be loaded for user n is obtained by

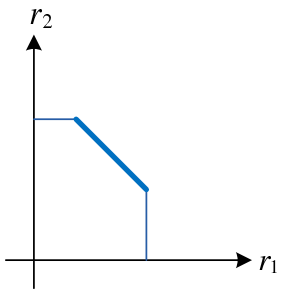
$$b_k^{(n)} = \log_2 \left(1 + \mathrm{SNR}_k^{(n)}\right), \tag{8}$$

where

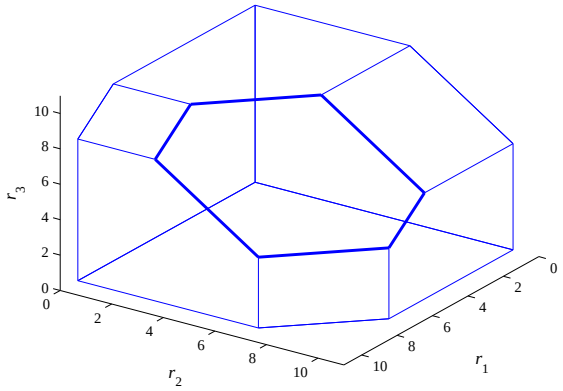
$$\mathrm{SNR}_k^{(n)} \equiv s_k^{(n)} [\boldsymbol{h}_k^{(n)}]^{\mathrm{H}} \left(\boldsymbol{Z}_k + \boldsymbol{H}_k \bar{\boldsymbol{S}}_k^{(n)} \boldsymbol{H}_k^{\mathrm{H}} \right)^{-1} \boldsymbol{h}_k^{(n)}, \tag{9}$$

$\boldsymbol{h}_k^{(n)}$ is the n-th column of $\boldsymbol{H}_k$, and $\bar{\boldsymbol{S}}_k^{(n)} = \mathrm{diag}(\overbrace{0, \ldots, 0}^{n \text{ times}}, s_k^{(n+1)}, s_k^{(n+2)}, \ldots, s_k^{(N)})$. It can be shown that $C_k = \sum_{n=1}^{N} b_k^{(n)}$. That is the MMSE-GDFE receiver achieves the capacity of the MAC channel. Therefore, for a fixed decoding order, we obtain a point on the boundary of the RR. Considering all feasible decoding orders, we obtain $N!$ points on the boundary of the RR. All of these points achieve the maximum sum rate and all are located on a facet with the normal vector $\frac{1}{\sqrt{N}}(\overbrace{1, 1, \ldots 1}^{N \text{ times}})$. More importantly, the MAC RR is the convex hull of these points and their projections on the coordinate hyperplanes [10]. Therefore, they fully characterize the MAC RR. Examples for 2 and 3 user MAC channels are plotted in Fig. 1.

[1] Sometimes the MAC RR is not strictly convex. In these cases, the iterative facet dividing algorithm (IFDA) [8] can be used to solve the problem. IFDA is based on a WSR maximization approach. Therefore, UWFO is applicable to MACs with non-strictly convex RRs as well.



(a) RR for a two-user MAC



(b) RR for a three-user MAC

Fig. 1. Ideal MAC RRs.

In the ideal MAC, the decoding order, which maximizes the WSR, is determined by the order of weight factors, where the user with the smallest weight factor is decoded first [7]. Therefore, there is no need to check all $N!$ decoding orders for the ideal MAC.

B. MAC with Practical DSL Receivers

For a practical DSL system, the bit-loadings do not follow the Shannon capacity formula (8). Instead the bit-loading for the n-th decoded user is obtained by

$$\tilde{b}_k^{(n)} = \begin{cases} \min\left\{\left\lfloor \bar{b}_k^{(n)} \right\rfloor, b_{\max}\right\}; & \bar{b}_k^{(n)} \geq b_{\min}, \\ 0; & \bar{b}_k^{(n)} < b_{\min}, \end{cases} \tag{10}$$

where $\lfloor \cdot \rfloor$ denotes the floor function, $b_{\min} = 2$ and $b_{\max} = 15$ are the minimum and maximum number of bits that can be loaded to each tone and

$$\bar{b}_k^{(n)} = \log_2\left(1 + \tfrac{1}{\Gamma}\mathrm{SNR}_k^{(n)}\right), \tag{11}$$

where $\Gamma > 0$ dB is the SNR gap and $\mathrm{SNR}_k^{(n)}$ is defined in (9). Once again, we obtain $N!$ different sets of bit-loadings by changing the decoding order. However, due to the effect of the SNR gap and the bit-loading bounding to $b_{\max}$, the bit-loadings sum is not the same for these sets and the optimal decoding order cannot be determined by the order of the weight factors. Therefore, we need to test all decoding orders to find the one that maximizes the per-tone WSR in (6). Although this approach results in the highest achievable bit rates, the computational complexity of this approach is high due to the $N!$ different sets of bit-loadings which should be checked for each tone.

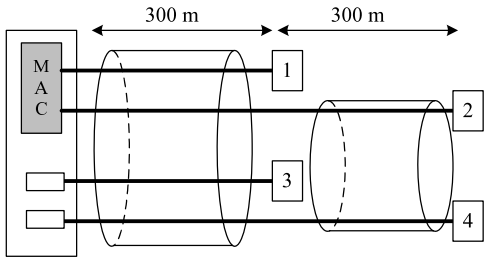


Fig. 2. A VDSL2 scenario consisting of two coordinated loops and two uncoordinated loops.

Figure 2 shows a scenario in which two coordinated VDSL2 users are transmitting in the presence of two similar uncoordinated users. We assume that all users are transmitting at the nominal PSD mask. If the MMSE-GDFE decodes the shorter loop first and then the longer loop on all tones, we obtain 75.7 Mbps for user 1, 52.1 Mbps for user 2, and 127.8 Mbps for the sum rate[2]. If we switch the order of decoding on all tones, we obtain 117.3 Mbps for user 1, 21.5 Mbps for user 2, and 138.8 Mbps for the sum rate. Finally, if we optimize the decoding order for sum rate maximization (sum rate maximal point is obtained by setting $w_1 = w_2 = 1$), we obtain 109.9 Mbps for user 1, 30.7 Mbps for user 2, and 140.6 Mbps for the sum rate. As it can be realized, unlike the ideal case, the sum rate of the users is not the same for different decoding orders. Moreover, the optimal decoding order cannot be identified by the order of the weight factors[3].

III. Algorithm Description

A. Two-User Case

The bit-loading formula for the practical case differs from the ideal case formula in two main aspects: 1) the non-zero SNR gap, and 2) bit-loading bounding to $b_{\max}$. The effect of other differences, namely, the flooring operation and the bit-loading lower-bounding to $b_{\min}$, are not as strong as those of the first two mentioned differences. We study potential effects of these differences on the optimal decoding order compared to the ideal case. To study the effect of the SNR gap, we ignore all other differences. We have

$$\begin{aligned} \tilde{b}_k^{(n)} &= \log_2\left(1 + \tfrac{1}{\Gamma}\mathrm{SNR}_k^{(n)}\right) \\ &\approx \log_2\left(\tfrac{1}{\Gamma} + \tfrac{1}{\Gamma}\mathrm{SNR}_k^{(n)}\right) \\ &= \log_2\left(1 + \mathrm{SNR}_k^{(n)}\right) - \log_2\Gamma \\ &= b_k^{(n)} - \log_2\Gamma, \end{aligned} \tag{12}$$

[2]The simulation parameters are listed in Table I.

[3]Since $w_1 = w_2$, both decoding orders lead to the optimal solution in ideal MAC but not in non-ideal MAC.

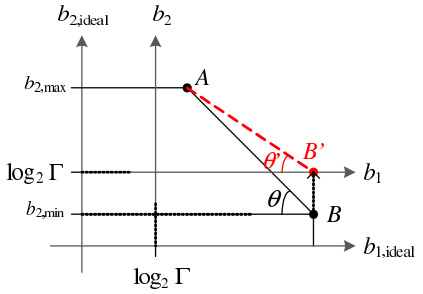


Fig. 3. Effect of non-zero SNR gap on the achievable points.

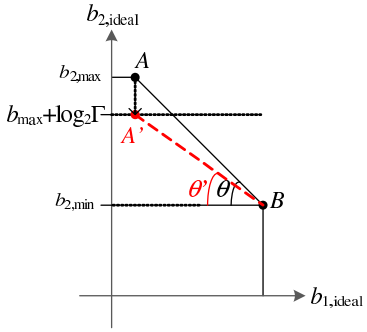


Fig. 4. Effect of bit-loading bounding to $b_{\max}$ on the achievable points.

where the approximation is valid when $\text{SNR}_k^{(n)} \gg \Gamma$. That is, the bit-loadings are approximately obtained by a constant shift of the bit-loadings in the ideal case. Since, $\log_2\Gamma$ is independent of the decoding order, the obtained points corresponding to different decoding orders are still located on a facet provided that $\text{SNR}_k^{(n)} \gg \Gamma$ for all users and decoding orders. The new facet is parallel to the ideal case facet but translated towards the origin by $\log_2\Gamma$ in all dimensions. Therefore, when $\text{SNR}_k^{(n)} > \Gamma$ or $b_k^{(n)} > \log_2\Gamma$ for all n, the optimal decoding order can still be determined based on the order of the weight factors. However, when $\text{SNR}_k^{(n)} \leq \Gamma$ the approximation in (12) is not tight and $\tilde{b}_k^{(n)} \approx 0 > b_k^{(n)} - \log_2\Gamma$.

Let $b_{k,\max}^{(2)}$ denote the number of bits that can be loaded to user 2 in the ideal MAC channel when the crosstalk from user 1 is removed (i.e., user 2 is the last decoded user) and $b_{k,\min}^{(2)}$ is the number of bits that can be loaded to user 2 in the ideal MAC channel when the crosstalk from user 1 is present (i.e., user 2 is the first decoded user). Figure 3 shows the effect of the non-zero SNR gap on the RR for a two-user MAC where for simplicity $b_k^{(n)}$, $\tilde{b}_k^{(n)}$, $b_{k,\min}^{(n)}$, and $b_{k,\max}^{(n)}$ are shown by $b_{n,\text{ideal}}$, b_n, $b_{n,\min}$, and $b_{n,\max}$, respectively. In this figure, two coordinate systems are plotted. One coordinate system shows the achievable RR for the ideal MAC receiver. The second coordinate system can be used to *approximately* calculate the achievable bit-rates for a MMSE-GDFE receiver with non-zero SNR gap. The second coordinate system is simply the translation of the first coordinate system by $\log_2\Gamma$ in all dimensions. Two operating points A and B are indicated for the ideal case. Point A is achieved when the first user is decoded first and point B is achieved when the second user is decoded first. The slope of $\overline{AB}$ (i.e. $\tan\theta$) indicates for which weight factors point A is optimum and for which weight factors point B is optimum. More specifically, A is the optimal operating point when $w_1 < \tan\theta \times w_2$ and otherwise B is the optimal operating point. Note that since the normal vector to $\overline{AB}$ is $\frac{1}{\sqrt{2}}(1,1)$, we have $\theta = 45°$ and $\tan\theta = 1$. Therefore the optimal decoding order can be identified by the order of weight factors.

Now consider point B for which $\text{SNR}_k^{(2)} < \Gamma$. In a system with non-zero SNR gap, when $\text{SNR}_k^{(n)} < \Gamma$ for a particular decoding order, the point will be mapped along the n-th dimension to $\tilde{b}_k^{(n)} \approx 0$ (or equivalently $b_k^{(n)} \approx \log_2\Gamma$). This is illustrated in Fig. 3. As it can be seen, when user 2 is decoded first, the resulting point (i.e. point B) is mapped to B' located on $\tilde{b}_k^{(n)} \approx 0$. This changes the slope of $\overline{AB}$ to $\tan\theta'$, where $\tan\theta' < \tan\theta$. Now, A is the optimal operating point when $w_1 < \tan\theta' \times w_2$ and otherwise B' is the optimal operating point. If we define $w_1' = w_1$ and $w_2' = \tan\theta' \times w_2$, then the optimal decoding order can be obtained by the order of w_1' and w_2'. From Fig. 3:

$$\tan\theta' = \frac{b_{k,\max}^{(2)} - \log_2\Gamma}{b_{k,\max}^{(2)} - b_{k,\min}^{(2)}}. \tag{13}$$

The effect of the bit-loading bounding to $b_{\max}$ can be studied in a similar way. Figure 4 gives an example for which the bit-loading of user 2 is truncated to $b_{\max}$ when it is decoded last. Note that due to the effect of the SNR gap, the bit-loadings are truncated to $b_{\max} + \log_2\Gamma$ for the ideal case.

As it can be seen, this time the location of point A has changed. Assuming that no other changes are made, we can follow the same steps to show that the weight factor should be updated by the factor:

$$\tan\theta' = \frac{b_{\max} + \log_2\Gamma - b_{k,\min}^{(2)}}{b_{k,\max}^{(2)} - b_{k,\min}^{(2)}}. \tag{14}$$

Note that in a practical system both effects should be taken into account simultaneously. Let $\text{SNR}_{k,\max}^{(2)}$ ($\text{SNR}_{k,\min}^{(2)}$) denote the SNR of user 2 on tone k when user 2 is the last (first) decoded user. Similarly, let $\tilde{b}_{k,\max}^{(2)}$ ($\tilde{b}_{k,\min}^{(2)}$) denote the bit-loadings for user 2 on tone k when user 2 is the last (first) decoded user. The bit-loadings $\tilde{b}_{k,\max}^{(2)}$ and $\tilde{b}_{k,\min}^{(2)}$ are obtained from $\text{SNR}_{k,\max}^{(2)}$ and $\text{SNR}_{k,\min}^{(2)}$ using (10) and (11). In fact, the exact location of points A' and B' along the horizontal axis, taking into account all practical effects, are obtained by $\tilde{b}_{k,\max}^{(2)}$ and $\tilde{b}_{k,\min}^{(2)}$. Thus instead of using (13) or (14) we may update w_2 as

$$w_2' = \frac{\tilde{b}_{k,\max}^{(2)} - \tilde{b}_{k,\min}^{(2)}}{b_{k,\max}^{(2)} - b_{k,\min}^{(2)}} \times w_2. \tag{15}$$

In order to take into account the effect of a practical DSL receiver on the bit-loadings of the first user, w_1 should be

updated as $w_1' = \frac{\tilde{b}_{k,\max}^{(1)} - \tilde{b}_{k,\min}^{(1)}}{b_{k,\max}^{(1)} - b_{k,\min}^{(1)}} \times w_1$ as well. Then if $w_1' > w_2'$, user 1 should be decoded first and otherwise user 2 should be decoded first.

B. Generalization for Larger Number of Users

Here, we generalize the algorithm to $N > 2$ using an iterative procedure. In each iteration of the algorithm, the weight factors are updated and the not-yet-decoded user with the minimum updated weight factor is selected as the user to be decoded next. Note that by each iteration, $\mathrm{SNR}_{k,\min}^{(n)}$, $b_{k,\min}^{(n)}$, and $\tilde{b}_{k,\min}^{(n)}$ change for the not-yet-decoded users, because the crosstalk originating from the last decoded user is removed using the GDFE. Consider the i-th iteration of the algorithm and let U_i denote the set of users that are not-yet-decoded at the beginning of this iteration. For all users $n \in U_i$, we update the weight factors as follows:

$$w_n' = \begin{cases} \frac{\tilde{b}_{k,\max}^{(n)} - \tilde{b}_{k,\min}^{(n)}}{b_{k,\max}^{(n)} - b_{k,\min}^{(n)}} \times w_n; & b_{k,\max}^{(n)} \neq b_{k,\min}^{(n)}, \\ 0; & b_{k,\max}^{(n)} = b_{k,\min}^{(n)}, \end{cases} \tag{16}$$

where $\tilde{b}_{k,\max}^{(n)}$ and $b_{k,\max}^{(n)}$ depend on $\mathrm{SNR}_{k,\max}^{(n)}$ and are calculated using (10) and (8), respectively. Similarly $\tilde{b}_{k,\min}^{(n)}$ and $b_{k,\min}^{(n)}$ depend on $\mathrm{SNR}_{k,\min}^{(n)}$ through the same formulas. To calculate $\mathrm{SNR}_{k,\max}^{(n)}$ we assume that user n is the last user to be decoded and the crosstalk from all other users is removed. Therefore

$$\mathrm{SNR}_{k,\max}^{(n)} \equiv s_k^{(n)} [\boldsymbol{h}_k^{(n)}]^{\mathrm{H}} \boldsymbol{Z}_k^{-1} \boldsymbol{h}_k^{(n)}. \tag{17}$$

On the other hand, to calculate $\mathrm{SNR}_{k,\min}^{(n)}$ we assume that user n is the first user to be decoded among the not-yet-decoded users. Thus

$$\mathrm{SNR}_{k,\min}^{(n)} = s_k^{(n)} [\boldsymbol{h}_k^{(n)}]^{\mathrm{H}} \left(\boldsymbol{Z}_k + \boldsymbol{H}_k \bar{\boldsymbol{S}}_{k,U_i}^{(n)} \boldsymbol{H}_k^{\mathrm{H}} \right)^{-1} \boldsymbol{h}_k^{(n)}, \tag{18}$$

where $\bar{\boldsymbol{S}}_{k,U_i}^{(n)} = \mathrm{diag}\left\{ s_{k,U_i}^{(1)}, s_{k,U_i}^{(2)}, \ldots, s_{k,U_i}^{(N)} \right\}$, and

$$s_{k,U_i}^{(m)} = \begin{cases} s_k^{(m)}; & m \in U_i, \\ 0; & m \notin U_i. \end{cases} \tag{19}$$

Note that $\mathrm{SNR}_{k,\min}^{(n)}$ depends on U_i and should be updated in each iteration.

The next user to be decoded, referred to as μ_i, is the user with the smallest weight factor w_n'. The user is removed from U_i to obtain U_{i+1} for the next iteration and the algorithm continues until all users are decoded . As it can be seen in (16), if $b_{k,\max}^{(n)} = b_{k,\min}^{(n)}$ we set $w_n' = 0$, so that user n is the next user to be decoded. The reason is that when $b_{k,\max}^{(n)} = b_{k,\min}^{(n)}$, we have $\mathrm{SNR}_{k,\max}^{(n)} = \mathrm{SNR}_{k,\min}^{(n)}$ and $\tilde{b}_{k,\max}^{(n)} = \tilde{b}_{k,\min}^{(n)}$. That is the decoding order does not change the rate of user n. However, by decoding user n before the other not-yet-decoded users, we may increase their rate.

The algorithm is called updated weight factor ordering (UWFO) and is summarized in Algorithm 1. The algorithm can be used to detect the decoding order for the ZF-GDFE as well. The only difference is that the SNR should be calculated assuming a ZF equalizer instead of a MMSE equalizer. Upon termination of the algorithm, the decoding decoded is indicated by μ_1 to μ_N and the bit loading for user n is stored in $\tilde{b}_{k,\min}^{(n)}$.

Algorithm 1: UWFO for Determining the Decoding Order on Tone k.

```
/* Initialization                                              */
U_1 ← {1, 2, ..., N};
for n = 1 to N do
    SNR_{k,max}^{(n)} ← s_k^{(n)} [h_k^{(n)}]^H Z_k^{-1} h_k^{(n)};
    b_{k,max}^{(n)} ← log_2 (1 + SNR_{k,max}^{(n)});
    Calculate b̃_{k,max}^{(n)} using (10);
/* Main                                                        */
for i = 1 to N do
    for all n ∈ U_i do
        SNR_{k,min}^{(n)} ← s_k^{(n)} [h_k^{(n)}]^H ×
            (Z_k + H_k S̄_{k,U_i}^{(n)} H_k^H)^{-1} h_k^{(n)};
        b_{k,min}^{(n)} ← log_2 (1 + SNR_{k,min}^{(n)});
        Calculate b̃_{k,min}^{(n)} using (10);
        Calculate w_n' using (16);
    μ_i ← index min_{n∈U_i} w_n';
    U_{i+1} ← U_i / {μ_i};
```

C. Computational Complexity

The main part of the algorithm consists of two nested loops, the outer loop and the inner loop. The number of iterations for each of these loops is at most N. In the inner loop, we take the inverse of $\boldsymbol{X}_i \equiv \boldsymbol{Z}_k + \boldsymbol{H}_k \bar{\boldsymbol{S}}_{k,U_i}^{(n)} \boldsymbol{H}_k^{\mathrm{H}}$ which is $\mathcal{O}(N^3)$. However, this operation can be done in $\mathcal{O}(N^2)$ operations using X_{i-1}^{-1} and the matrix inversion lemma as follows:

$$\boldsymbol{X}_i^{-1} = \left(\boldsymbol{X}_{i-1} + \boldsymbol{h}_k^{(\mu_{i-1})} s_k^{(\mu_{i-1})} \left[\boldsymbol{h}_k^{(\mu_{i-1})} \right]^{\mathrm{H}} \right)^{-1} = \boldsymbol{X}_{i-1}^{-1} - \boldsymbol{X}_{i-1}^{-1} \boldsymbol{h}_k^{(\mu_{i-1})} \left(\frac{-1}{s_k^{(\mu_{i-1})}} + \left[\boldsymbol{h}_k^{(\mu_{i-1})} \right]^{\mathrm{H}} X_{i-1}^{-1} \boldsymbol{h}_k^{(\mu_{i-1})} \right)^{-1} \left[\boldsymbol{h}_k^{(\mu_{i-1})} \right]^{\mathrm{H}} X_{i-1}^{-1}.$$

The rest of the operations in the inner loop are $\mathcal{O}(N^2)$. Therefore, the overall computational complexity of the algorithm is $\mathcal{O}\left(N^4\right)$. Note that the matrix inversion lemma can be used to reduce the computational complexity of TES as well, i.e., from $\mathcal{O}(N^3 N!)$ to $\mathcal{O}(N^2 N!)$.

IV. Simulation Results

We have plotted the achievable rate tuples and the RR for the TES, WFO, and UWFO algorithms in Fig. 5. As it can be seen, UWFO can reach the same RR as that achieved by TES. However, WFO misses a considerable portion of the RR.

TABLE I
SIMULATION PARAMETERS

Parameter	Value
Bandplan and PSD mask	US VDSL2E17 B7-9 bandplan (3-5.1, 7.05-12, 14-17.67 MHz) [11], [12]
Cable type	26 AWG [13]
Noise	White noise, -140 dBm/Hz
Symbol rate, f_s	4 kHz
$b_{\max}$	15
$b_{\min}$	2
SNR Gap	9.45 dB

TABLE II
SIMULATION RESULTS FOR A SIX-USER VDSL2 SCENARIO

Index	Loop Length (m)	w_n	Achievable Rates (Mbps)		
			TES	WFO	UWFO
1	250	0.34998	127.63	127.16	127.62
2	350	0.1966	56.876	55.948	56.744
3	250	0.25108	121.54	120.64	121.6
4	350	0.61604	115.6	115.6	115.6
5	250	0.47329	130.78	130.53	130.78
6	250	0.35166	129.74	129.4	129.73
WSR (Mbps)			265.1	264.29	265.09

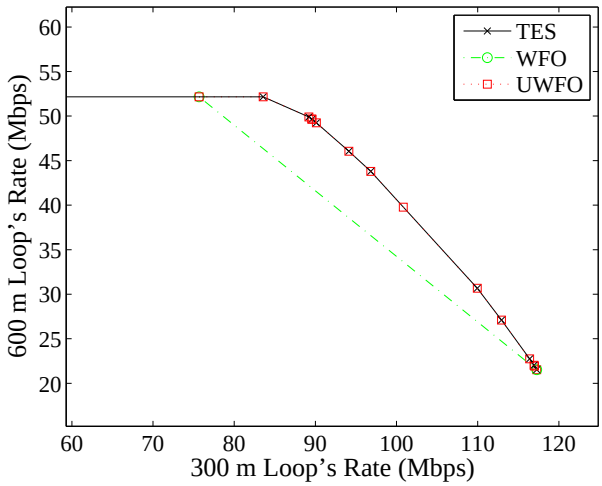


Fig. 5. Achievable RR by TES, WFO, and UWFO for the two-user scenario in Fig. 2.

The computational complexity reduction by UWFO compared to TES is significant when N is large. In Table II we have presented simulation results for a scenario consisting of 6 coordinated VDSL2 users in presence of 2 uncoordinated VDSL2 users. The lengths of the uncoordinated loops are 250 m and 350 m. The weight factors are chosen randomly and are given in the third column of the table. The lengths of the coordinated loops and their achievable rates for TES, WFO, and UWFO are also listed in the table. As it can be seen, the achievable WSR by UWFO is again very close to that of TES. In this example, WFO also performs close to TES, however, it should be noted that WFO could reach only $N!$ potentially sub-optimal rate tuples and may miss some portions of the RR in the general case [6] as for the two-user scenario simulated in Fig. 5.

V. CONCLUSION

We have proposed a new algorithm, namely, UWFO, for determining the decoding order in the MMSE-GDFE receiver structure for non-ideal DSL MACs. The computational complexity of UWFO is $\mathcal{O}\left(N^4\right)$ which is significantly smaller than $\mathcal{O}\left(N^2N!\right)$, the computational complexity of the optimal algorithm, i.e., TES. The computational complexity of our algorithm is also smaller than SOS with $\mathcal{O}\left(LN^4\right)$ but larger than MGS with $\mathcal{O}\left(N^3\right)$. Simulation results show that UWFO closely approximates the RR of the MMSE-GDFE. For comparison, SOS is capable of closely approximating the RR of the ZF-GDFE, but MGS provides solutions which are slightly away from the boundary of the ZF-GDFE's RR. UWFO provides higher bit rates than SOS and MGS because MMSE-GDFE outperforms ZF-GDFE in presence of spatially colored noise. However, these algorithms can potentially be adapted to the MMSE-GDFE structure as well.

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