Splitting algorithms for nonlinear RLC circuits ⋆

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Abstract: Splitting algorithms are well established in large-scale convex optimization problems reformulated as zero finding of monotone operators. We explore such algorithms in solving nonlinear circuits represented as port interconnections of monotone or mixed-monotone elements. For circuits made of linear capacitors and inductors with nonlinear resistive elements, we propose that the splitting of the circuit between its LTI lossless passive component and its static resistive component is attractive both for physical and algorithmic reasons. We illustrate this splitting on the well-known FitzHugh-Nagumo circuit model, a paradigmatic example of a spiking neuron.

Keywords: Neuromorphic engineering, Circuit theory, Computational neuroscience, Distributed optimization for large-scale systems, Dynamic systems.

1. INTRODUCTION

Neuromorphic electronics and engineering have attracted the attention of scientists from various disciplines in recent years. This new field promises a paradigm shift, with technologies that are *orders of magnitude* more energy efficient than the current large-scale digital systems (Mead, 1990). In addition to efficiency, these systems are believed to be significantly better at handling ill-conditioned inputs and dealing with component failure (Furber, 2016). The fields of spiking neural networks, neuromorphic computing, and computational neuroscience, create new demand and interest in the simulation, analysis, and design of large-scale electrical circuits.

Despite the promising qualities of the neuromorphic framework, there lacks a "neuromorphic theory" for the systematic design and algorithmic analysis of neuromorphic systems. In fact, even the task of simulating large-scale neural networks with biophysically relevant models is problematic. This underlines the need for a computationally tractable and efficient simulation framework.

The current methodology for the simulation of these networks is *forward simulation in time* of the governing nonlinear ordinary differential equations. However, this approach suffers from several shortcomings. First, because of the spiking nature of neurons, the ODE models are stiff, which challenges numerical integration. Second, the numerical integration of ODE models rarely scales with the dimension of the large-scale network. Finally, and most importantly, this method offers no robustness analysis with respect to the variability and uncertainty of circuit components.

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Circuit theory is the modeling framework originally proposed by Hodgkin and Huxley for biophysical neural networks. Conductance-based models are nowadays modeled as massive parallel interconnections of memristive components (ion channels and synaptic interactions).

In this paper, we wish to explore the circuit representation of neural networks for analysis and simulation. A key insight of study (Chaffey and Sepulchre, 2023) is departing from an ODE-governed state-space representation to an operator-governed input-output (I/O) representation. In this framework, the behavior of the circuit is expressed as the trajectory that satisfies the *zero inclusion problem* posed by the dynamics of the circuit elements and their interconnection as the kernel of the sum of monotone operators. Hence, provided an input to the circuit, the output is obtained by solving this zero inclusion problem. To do so, the problem is transformed to a *fixed-point iteration* (FPI) and solved using proximal methods. This alternative representation provides advantages compared to the ODE-governed representation, with scalability at its core (Ryu and Yin, 2022). Moreover, the circuit topology provides the splitting decomposition of the numerical algorithm.

Electrical *element extraction* is a classical method (Anderson and Vongpanitlerd, 2013), which has recently been used in the context of circuit simulation by (Chaffey et al., 2023), using operator-theoretic I/O representation. In this approach, the behavior of the circuit is written as a sum of operators (accounting for element behavior) and a skew-symmetric operator (accounting for the interconnection). Such decomposition is amenable to modern proximal splitting algorithms such as Condat-Vu or Chambolle–Pock (Condat et al., 2023) to determine the *output trajectory*.

The *Resistance extraction* method of (Anderson and Vongpanitlerd, 2013) decomposes the circuit into its lossless and resistive parts. We argue that this decomposition is appealing both for physical and algorithmic reasons. From a modeling viewpoint, it aligns with the energy-based modeling framework reviewed in (Jeltsema and Scherpen, 2009) where the lossless part of the circuit corresponds to its kinetic energy and co-energy, whereas the resistive part is the gradient of the resistive content and co-contents. For networks with linear capacitors and inductors, such decomposition expresses the system (in operator representation) as the sum of a

(i) lossless (skew) LTI operator,
(ii) a (resistive) gradient operator which is the difference of two cyclically monotone operators.

Cyclic monotonicity is a strengthening of monotonicity that guarantees that an operator is the subgradient of a convex functional. From an algorithmic viewpoint, this decomposition enables a representation of the LTI operator in the frequency domain and a representation of the static nonlinear operator in the time-domain, enabling a splitting into two diagonal representations.

The rest of this paper is structured as follows. Section 2 describes the proposed splitting of this work in detail and discusses an algorithm that is capable of solving such a problem. Section 3 will explain how using both the time and frequency domains can allow the exploitation of the LTI structure of the lossless part. Section 4 showcases this idea with the simulation of the FitzHugh-Nagumo model of the spiking neuron.

2. ENERGY-BASED SPLITTING OF NONLINEAR CIRCUITS

In this section, the *energy-based* splitting of the circuit dynamics will be introduced and briefly discussed. The main idea behind this splitting is similar to famous energy-based formulations such as the Lagrangian and Hamiltonian frameworks.

2.1 Preliminaries

The signal space $\mathcal{H}$ is a Hilbert space defined with the inner product $\langle \cdot, \cdot \rangle$. $\mathcal{L}^2[0, \infty)$ is the space of square-integrable signals which is also a Hilbert space.

Definition 1. Operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is *monotone* if

$$\langle u_1 - u_2, A(u_1) - A(u_2) \rangle \geq 0 \tag{1}$$

for u_1 and u_2 belonging to the operator's domain. Operator A will be maximally monotone if its graph is not properly contained in the graph of any alternative monotone operator.

Definition 2. Operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is *n-cyclic monotone* if, for every $(u_1, u_2, ..., u_{n+1}) \in \mathcal{H}^{n+1}$ and every $(y_1, y_2, ..., y_n) \in \mathcal{H}^n$ where $y_i = A(u_i)$

$$\sum_{i=1}^{n} \langle u_{i+1} - u_i, y_i \rangle \leq 0 \tag{2}$$

If A is n-cyclically monotone for every integer $n \geq 2$, then A is *cyclically monotone*.

2.2 Energy-based Splitting

The dynamics of a circuit $\mathscr{N}$ can be written in an operator-governed framework as

$$A \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} = 0 \tag{3}$$

where $i(t), v(t) \in \mathcal{L}^2$ and A is a general nonlinear operator. We wish to split this operator into the form

$$S \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} + DCM \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} = 0 \tag{4}$$

where S is a lossless (and thus skew-symmetric) LTI operator and DCM represents a nonlinear operator that can be written as the difference of two cyclic monotone operators.

The splitting above is motivated by spiking biophysical neural networks. Neurons with spiking behavior consist of both passive and active components that translate to monotone and anti-monotone elements (Ribar and Sepulchre, 2019). This mixed nature of the existing elements in spiking systems was acknowledged and discussed in detail in the framework of mixed feedback systems (Sepulchre, 2022). Thus, the representation of Eq. (4) can be further expanded to

$$S \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} + C_1 \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} - C_2 \begin{pmatrix} i(t) \\ v(t) \end{pmatrix} = 0 \tag{5}$$

where C_1 and C_2 are cyclic monotone operators. In this decomposition, C_1 groups the elements of positive conductance whereas C_2 groups the elements of negative conductance.

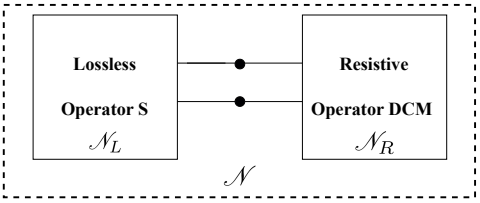


Fig. 1. Decomposition of the circuit network into a lossless part and a resistive part

Figure 1 illustrates this splitting in terms of resistance extraction where the network $\mathscr{N}$ is divided into a lossless subcircuit $\mathscr{N}_L$ and a resistive subcircuit $\mathscr{N}_R$.

In addition to physical interpretability, the separation of the monotone part from the anti-monotone part facilitates the algorithmic computation of the response by using the ideas of concave-convex programming (Yuille and Rangarajan, 2003).

2.3 Connection with existing energy-based formulations

The energy-based splitting considered in this paper can be systematically obtained from an electrical circuit using the *resistance extraction* method. Additionally, there is a tight connection between this representation and the energy-based representation of electrical systems.

There exists an extensive literature on how to represent electrical (and mechanical) systems using the underlying

energy functions. To do so, it is essential to define the total co-energy (co-Hamiltonian) as

$$H^*(i,v) = \mathcal{T}^*(i) + \mathcal{V}^*(v) \tag{6}$$

where $\mathcal{T}^*(i)$ and $\mathcal{V}^*(v)$ are the inductive co-energy and capacitive co-energy (Jeltsema and Scherpen, 2009). It is now possible to define the potential function of the resistive elements as the sum of the content and co-content

$$\mathcal{R}(i,v) = \mathcal{D}(i) + \mathcal{D}^*(v) \tag{7}$$

Forming the equations of motion via these potential functions using the Lagrangian and co-Lagrangian formulation leads to

$$\begin{cases} \frac{d}{dt}\nabla_i H^*(i,v) + \nabla_i(v^T N i) = -\nabla_i \mathcal{R}(i,v) \\ \frac{d}{dt}\nabla_v H^*(i,v) - \nabla_v(v^T N i) = -\nabla_v \mathcal{R}(i,v) \end{cases} \tag{8}$$

where N is the turn-ratio matrix and describes the interconnection of elements.

This is identical to the format of Eq. (4) where the LHS accounts for lossless components and interconnections (and thus, skew-symmetric) and the RHS highlights the effect of the resistive elements. The cyclic monotonicity of the resistive parts of Eq.(4) ensures that it can be written as the gradient of the difference of two convex functions which is concurrent with the representation of Eq. (8).

2.4 Difference of Monotone Douglas-Rachford

Monotonicity is central to solving the *zero inclusion problems* of the form of Eq. (3) since it ensures the transition of the problem to a nonexpansive (or more commonly averaged) fixed point iteration (Ryu and Yin, 2022). In addition, if the operator defining the zero inclusion problem can be decomposed to the summation of two monotone operators, it is possible to use various splitting algorithms such as Forward-Backward and Douglas-Rachford algorithms. However, solving zero inclusion problem of Eq. (5) with the difference of monotone structure is challenging and is considered a relatively new topic. Study (Chuang et al., 2022) offers an algorithm to solve the problem of the form

$$S(x) + A_1(x) - B_1(x) \in 0 \tag{9}$$

where A_1, S and B_1 are monotone operators. In this setup, the difference of monotone Douglas-Rachford (DMDR) algorithm can be defined as

Algorithm 1 DMDR

1: **for** $j = 1, 2, \ldots, max - iteration$ **do**
2: $\quad x^{j+1} = J_{\alpha S}\left(z^j\right)$
3: $\quad z^{j+1} = z^j - x^{j+1} + J_{\alpha A_1}\left(2x^{j+1} - z^j + \alpha B_1\left(x^{j+1}\right)\right)$
4: $\quad j = j + 1$
5: **end for**

where $J_{\alpha A}$ represents the *resolvent operator* of operator A and α is the step size. z defines the auxiliary variable that is used within the structure of the algorithm and the superscripts denote the iteration count of the algorithm. A similar algorithm was used recently to numerically solve the limit cycle of the Van der Pol oscillator (Das et al., 2022).

3. TIME-FREQUENCY SPLITTING OF NONLINEAR CIRCUITS

In the splitting of the form of Eq. (5), the lossless operator is dynamic but LTI whereas the resistive operator is static but nonlinear. The former is diagonal in the frequency domain whereas the latter is diagonal in the time domain. Those properties can be exploited by transforming the signals from one domain to the other via FFT.

This can be incorporated into the first step of the DMDR algorithm (line 2). More rigorously, this step can be rewritten as

$$\begin{gathered} (\mathbb{I} + \alpha S)\begin{pmatrix} x_1^{j+1}(t) \\ x_2^{j+1}(t) \end{pmatrix} = \\ (\mathbb{I} + \alpha \begin{bmatrix} P(D) & F^* \\ -F & Q(D) \end{bmatrix})\begin{pmatrix} x_1^{j+1}(t) \\ x_2^{j+1}(t) \end{pmatrix} = \begin{pmatrix} z_1^{j+1}(t) \\ z_2^{j+1}(t) \end{pmatrix} \end{gathered} \tag{10}$$

where superscripts represent iteration counts. The signals x and z are broken into two signals since the original problem consists of both the voltage and current. In fact x_1 corresponds to current and x_2 corresponds to voltage. The lossless operator S is also expanded to show its structure. P and Q are related to the inductors and capacitors and are functions of the derivative operator. F incorporates the interconnection. Taking the FFT of both sides

$$\underbrace{(\mathbb{I} + \alpha \begin{bmatrix} \hat{P}(j\omega) & F^* \\ -F & \hat{Q}(j\omega) \end{bmatrix})\begin{pmatrix} X_1(j\omega) \\ X_2(j\omega) \end{pmatrix}}_{\text{Frequency domain}} = \begin{pmatrix} Z_1(j\omega) \\ Z_2(j\omega) \end{pmatrix} \tag{11}$$

where $\hat{P}$ and $\hat{Q}$ are now diagonal matrices. By solving for $X_1(j\omega)$ and $X_2(j\omega)$, the new iteration of the signal X is obtained in the frequency domain and all that is required is to take the iFFT of the signals to return to the time domain. This should be done for every iteration. The next step of the DMDR algorithm (line 3) can be carried out in the time domain. By switching between these two domains, the computational cost of the algorithm decreases significantly, and many numerical issues are avoided.

4. FITZHUGH-NAGUMO MODEL

The FitzHugh-Nagumo model can be written as

$$\begin{aligned} C\dot{v} &= I_g(v) - i \\ L\dot{i} &= -Ri + v \end{aligned} \tag{12}$$

where $I_g(v) = v - \frac{v^3}{3}$. This can be rewritten as

$$\begin{bmatrix} CD & +\mathbb{I} \\ -\mathbb{I} & LD \end{bmatrix}\begin{bmatrix} v(t) \\ i(t) \end{bmatrix} - \begin{bmatrix} I_g(v(t))) \\ -Ri(t) \end{bmatrix} = 0 \tag{13}$$

where $D : \mathcal{L}^2 \to \mathcal{L}^2$ defines the derivative operator (s in the Laplace domain and $j\omega$ in the frequency domain) and $\mathbb{I} : \mathcal{L}^2 \to \mathcal{L}^2$ is the identity operator. The first operator is a lossless operator that incorporates the capacitor, inductor, and interconnection of elements. The second operator is in fact the difference of two cyclically monotone operators and thus, we can rewrite Eq. (13) as

$$\begin{bmatrix} CD & +\mathbb{I} \\ -\mathbb{I} & LD \end{bmatrix}\begin{bmatrix} v(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} \frac{(\cdot)^3}{3} & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} v(t) \\ i(t) \end{bmatrix} - \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = 0 \tag{14}$$

which is exactly in the format of Eq. (5) and (8). By using the DMDR algorithm, and using both the time and frequency domains simultaneously, as discussed in Section 3, it is possible to obtain the response of this system. Figure 2 illustrates the steady-state response of the FitzHugh-Nagumo circuit in the space of T-periodic square-integrable signals ($\mathcal{L}^2_T$) with parameters $L = 20$, $C = R = 1$. To proceed with the computations, the signals $v(t)$ and $i(t)$ were both discretized to vectors with 556 elements. The step size (α) for the DMDR algorithm was set to 0.1 and the sampling frequency of the FFT was chosen to be 10 Hz.

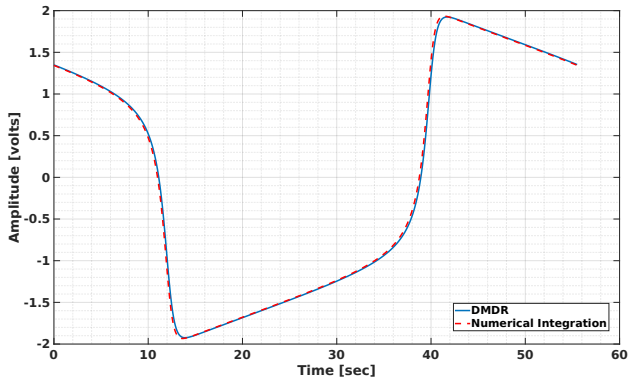


Fig. 2. Simulation of one period of the Fitzhugh-Nagumo circuit using DMDR and numerical integration for verification

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