Robust Design of Linear Control Laws for Constrained Nonlinear Dynamic Systems ⋆

Boris Houska * **Moritz Diehl** *

* *Electrical Engineering Department (OPTEC & ESAT/SCD)*
Kasteelpark Arenberg 10, 3001 Leuven/Heverlee, Belgium.
boris.houska@esat.kuleuven.be, moritz.diehl@esat.kuleuven.be

Abstract: In this paper we present techniques to solve robust optimal control problems for nonlinear dynamic systems in a conservative approximation. Here, we assume that the nonlinear dynamic system is affected by a time-varying uncertainty whose L-infinity norm is known to be bounded. By employing specialized explicit upper estimates for the nonlinear terms in the dynamics we propose a strategy to design a linear control law which guarantees that given constraints on the states and controls are robustly satisfied when running the system in closed-loop mode. Finally, the mathematical techniques are illustrated by applying them to a tutorial example.

1. INTRODUCTION

In the recent decades, robust optimization problems have received much attention. Especially robust optimization for convex (or concave) problems is a well-established research field for which efficient algorithms exist (cf. Ben-Tal and Nemirovski [1998], El-Ghaoui and Lebret [1997]). Unfortunatly, non-convex robust optimization problems are much more difficult to solve. Although there is a mature theory on semi-infinite optimization available (c.f. Jongen et al. [1998], Rückmann and Stein [2001]) there are only a few special cases in which algorithms can succesfully be applied (c.f. Floudas and Stein [2007]).

When robust control problems are regarded, there exists a huge amount of literature on linear system theory (cf. e.g. Zhou et al. [1996] and the reference therein). As soon as nonlinear dynamic systems are considered much less aproaches exist. Some authors, e.g. Nagy and Braatz [2004, 2007] as well as Diehl et al. [2006], have suggested heuristic techniques for nonlinear robust optimal control. However, in general, these heuristic approaches do not provide a guarantee that a nonlinear system does not violate given hard-constraints in worst-case situations.

The contribution of this paper is that we propose a computationally tractable way of solving robust nonlinear optimal control design problems for time varying uncertainties in a conservative approximation. For this aim, we need to assume that an explicit estimate of the nonlinear terms in the right-hand side function f is given. We demonstrate for a tutorial problem how such an explicit estimate can be constructed illustrating that the results in this paper are not only of theoretical nature but can also be applied in practice.

In Section 2 we concentrate on the problem statement while Section 3 focuses on uncertain linear system with L-infinity bounded uncertainties. In Section 4 the main result of this paper on uncertain nonlinear constrained systems is proven. Finally, we demonstrate the applicability of the proposed strategies in Section 5 by applying them to a small tutorial example. Section 6 concludes.

Notation: Besides mathematical standard notations, we introduce the set $\mathbb{D}^n_{++} \subseteq \mathbb{R}^{n\times n}$ which denotes throughout this paper the set of the diagonal and positive definite matrices in $\mathbb{R}^{n\times n}$.

2. ROBUST NONLINEAR OPTIMAL CONTROL PROBLEMS

In this section we introduce uncertain optimal control problems for dynamic systems of the form

$$\dot{x}(t) = F(x(t), u(t), w(t)) \ , \ x(0) = 0 \ ,$$

where $x : [0,T] \to \mathbb{R}^{n_x}$ denotes the states, $u : [0,T] \to \mathbb{R}^{n_u}$ the control inputs, and $w : [0,T] \to \mathbb{R}^{n_w}$ an unknown time-varying input which can influence the nonlinear right-hand side function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$. Throughout this paper, we assume that our only knowledge about the uncertainty w is that it is contained in an uncertainty set Ω_∞ which is defined as

$$\Omega_\infty \ := \ \{ \, w(\cdot) \mid \text{for all } \tau \in [0,T] : \ \|w(\tau)\|_\infty \ \leq \ 1 \ \} \ .$$

In words, Ω_∞ contains the uncertainties $w(\cdot)$ whose L-infinity norm is bounded by 1.

In this paper, we are interested in designing a feedback law in order to compensate the uncertainties w. Here, we constraint ourselves to the case that the feedback law is linear, i.e. we set $u(t) := K(t)x(t)$ with $K : [0,T] \to \mathbb{R}^{n_u \times n_x}$ denoting the feedback gain. Now, the dynamics of the closed loop system can be summarized as

⋆ Research supported by Research Council KUL: CoE EF/05/006 Optimization in Engineering(OPTEC), OT/03/30, IOF-SCORES4CHEM, GOA/10/009 (MaNet), GOA/10/11, several PhD/postdoc and fellow grants; Flemish Government: FWO: PhD/postdoc grants, projects G.0452.04, G.0499.04, G.0211.05, G.0226.06, G.0321.06, G.0302.07, G.0320.08, G.0558.08, G.0557.08, G.0588.09,G.0377.09, research communities (ICCoS, ANMMM, MLDM); IWT: PhD Grants, Belgian Federal Science Policy Office: IUAP P6/04; EU: ERNSI; FP7-HDMPC, FP7-EMBOCON, Contract Research: AMINAL. Other: Helmholtz-viCERP, EMBOCON, COMET-ACCM.

$$\dot{x}(t) \;=\; f(x(t), K(t), w(t)) \;:=\; F(x(t), K(t)x(t), w(t)) \;.$$

Moreover, we assume that we have $f(0, K, 0) = 0$ for all $K \in \mathbb{R}^{n_u \times n_x}$, i.e. we assume that $x_{\text{ref}}(t) = 0$ is the steady state which we would like to track. The uncertain optimal gain design problem of our interest can now be stated as

$$\begin{aligned} \min_{x(\cdot), K(\cdot)} \quad & \Phi[\, K(\cdot)\,] \\ \text{subject to} \quad & \dot{x}(\tau) \;=\; f(x(\tau), K(\tau), w(\tau)) \\ & x(0) \;=\; 0 \\ & C_i(K(\tau))\, x(\tau) \;\leq\; d_i \quad \text{for all } \tau \in \mathbb{T}_i \end{aligned} \tag{1}$$

with $i \in \{1, \ldots, m\}$. The constraints are assumed to be linear with a given matrix $C : \mathbb{R}^{n_u \times n_x} \to \mathbb{R}^{m \times n_x}$ and a given vector $d \in \mathbb{R}^m$. The sets $\mathbb{T}_i \subseteq [0, T]$ denote the set of times for which the constraints should be satisfied. Here, we can e.g. use $\mathbb{T}_i = [0, T]$ if we want to formulate a path constraint or $\mathbb{T}_i = \{T\}$ if we are interested in a terminal constraint. Note that the above formulation includes the possiblity of formulating both state and control bounds as the controls $u(t) = K(t)x(t)$ are linear in x.

Our aim is now to solve the above optimal control problem guaranteeing that the constraints are satisfied for all possible uncertainties $w \in \Omega_\infty$. Thus, we are interested in the following robust counterpart problem:

$$\min_{u(\cdot)} \; \Phi[u(\cdot)] \quad \text{subject to } V_i[\, t,\, u(\cdot)\,] \;\leq\; d_i \quad \text{for all } t \in \mathbb{T}_i \;.$$

Here, the robust counterpart functional V is defined component-wise by

$$\begin{aligned} V_i[\, t,\, K(\cdot)\,] \;:=\; \max_{x(\cdot), w(\cdot)} \quad & C_i(K(t))x(t) \\ & \text{for all } \tau \in [0, t]: \\ \text{s.t.} \quad & \dot{x}(\tau) \;=\; f(x(\tau), K(\tau), w(\tau)) \\ & x(0) \;=\; 0 \\ & w(\cdot) \in \Omega_\infty \;. \end{aligned} \tag{2}$$

Note that the above problem is difficult to solve as it has a bi-level or min-max structure. For the case that f is linear in x and w, the lower-level maximization problem can be regarded as a convex problem as Ω_∞ is a convex set. This lower-level convex case has in a similar context been discussed in Houska and Diehl [2009, 2010] where Lyapunov differential equations have been employed in order to reformulate the min-max problem into a standard optimal control problem.

However, for the case that f is nonlinear, the problem is much harder to solve as local maxima in the lower level problem can not be excluded. The aim of this paper is to develop a conservative approximation strategy to overestimate the functions V_i planning to solve the robust counterpart problem approximately but with guarantees. For this aim, we will have to go one step back within the next Section 3 where we start with an analysis of linear dynamic systems. Later, in Section 4, we will come back to a discussion of the more difficult nonlinear problem

3. LINEAR DYNAMIC SYSTEMS WITH TIME VARYING UNCERTAINTY

In this section, we introduce the basic concept of robust optimization for linear dynamic systems with infinite dimensional uncertainties. We are interested in a dynamic system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \quad \text{with} \quad x(0) = 0 \;. \tag{3}$$

Here, $x : \mathbb{R} \to \mathbb{R}^{n_x}$ denotes the state while $w : \mathbb{R} \to \mathbb{R}^{n_w}$ is assumed to be a time varying uncertainty. Moreover, $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $B : \mathbb{R} \to \mathbb{R}^{n_x \times n_w}$ are assumed to be given (Lebesgue-) integrable functions.

As outlined in the previous section, we are interested in computing the maximum excitation $V(t)$ of the system at a given time t in a given direction $c \in \mathbb{R}^{n_x}$:

$$\begin{aligned} V(t) \;:=\; \max_{x(\cdot), w(\cdot)} \quad & c^T x(t) \\ & \text{for all } \tau \in [0, t]: \\ \text{s.t.} \quad & \dot{x}(\tau) \;=\; A(\tau)x(\tau) + B(\tau)w(\tau) \\ & x(0) \;=\; 0 \\ & w(\cdot) \in \Omega_\infty \;. \end{aligned} \tag{4}$$

The above maximization problem can be regarded as an infinite dimensional linear program which is convex as the set Ω_∞ is convex. Following the ideas from Ben-Tal and Nemirovski [1998] we suggest to analyze the dual of the above maximization problem in order to compute V via a minimization problem.

In order to construct the dual problem, we need a time varying multiplier $\lambda : [0, T] \to \mathbb{R}^{n_w}$ to account for the constraints of the form $w_i(\tau)^2 \;\leq\; 1$ which have to be satisfied for all times τ and all indices $i \in \{1, \ldots, n_w\}$. Moreover, we express the state function x of the linear dynamic system explicitly as

$$x(t) = \int_0^t H_t(\tau) w(\tau)\, \mathrm{d}\tau \;, \tag{5}$$

with the impulse response function $H_t(\cdot) := G(t, \cdot)B(\cdot)$. Here, $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ denotes the fundamental solution of the linear differential equation (3), which is defined as the solution of the following differential equation:

$$\frac{\partial G(t, \tau)}{\partial t} = A(t)G(t, \tau) \quad \text{with} \quad G(\tau, \tau) = 1 \tag{6}$$

for all $t, \tau \in \mathbb{R}$.

Now, the dual problem for the function V can be written as

$$\begin{aligned} V(t) \;&=\; \inf_{\lambda(\cdot) > 0} \quad \max_{w(\cdot)} \; c^T \left(\int_0^t H_t(\tau) w(\tau)\, \mathrm{d}\tau \right) \\ &\qquad - \sum_{i=1}^{n_w} \int_0^t \lambda_i(\tau) \left(w_i(\tau)^2 - 1 \right) \mathrm{d}\tau \\ &=\; \inf_{\Lambda(\cdot) \succ 0} \quad \int_0^t \frac{c^T H_t(\tau) \Lambda(\tau)^{-1} H_t(\tau)^T c}{4}\, \mathrm{d}\tau \\ &\qquad + \int_0^t \mathrm{Tr}\,[\, \Lambda(\tau)\,]\, \mathrm{d}\tau \;. \end{aligned}$$

Here, we use the short hand

$$\Lambda(\tau) \;:=\; \mathrm{diag}(\lambda(\tau)) \in \mathbb{D}_{++}^{n_w}$$

to denote the diagonal matrix valued function whose entries are the components of the multiplier function λ.

The following Theorem provides a non-relaxed reformulation of the above dual problem such that the associated

value function V can be computed more conveniently. The proof of this Theorem can be found in the Appendix of this paper:

Theorem 1. The function V, which is defined to be the optimal value of the optimization problem (4), can equivalently be expressed as

$$V(t) = \inf_{P(\cdot),\theta(\cdot),R(\cdot)\in\mathbb{D}_{++}^{n_w}} \sqrt{1-\theta(\tau)}\sqrt{c^T P(t) c}$$

$$s.t. \quad \begin{cases} \dot{P}(\tau) &= A(\tau)P(\tau) + P(\tau)A(\tau)^T \\ &\quad +\mathrm{Tr}\left[\, R(\tau)\,\right] P(\tau) \\ &\quad +B(\tau)R^{-1}(\tau)B(\tau)^T \\ P(0) &= 0 \\ \dot{\theta}(\tau) &= -\mathrm{Tr}\left[\, R(\tau)\,\right] \theta(\tau) \\ \theta(0) &= 1 \end{cases} \tag{7}$$

with $P : [0,T] \to \mathbb{R}^{n_x \times n_x}$ and $\theta : [0,T] \to [0,1]$ being auxiliary states.

The main reason why we are interested in the above theorem is that it allows us to guarantee that the reachable states are independent of the choice of w within an ellipsoidal tube. Let us formulate this result in form of the following corollary:

Corollary 2. Let $R : [0,T] \to \mathbb{D}_{++}^{n_w}$ be any given diagonal and positive matrix valued function and $P(t)$ as well as $\theta(t)$ the associated Lyapunov states defined by (7). If we define the matrix

$$Q(t) \;:=\; (1-\theta(t))\, P(t)$$

as well as the ellipsoidal set

$$\mathcal{E}(Q(t)) \;:=\; \left\{\, Q(t)^{\frac{1}{2}}\, v \;\mid\; v^T v \;\leq\; 1 \,\right\}, \tag{8}$$

then we have for all times $t \in [0,T]$ the set inclusion

$$\left\{ \int_0^t H_t(\tau) w(\tau) \mathrm{d}\tau \;\mid\; w(\cdot) \in \Omega_\infty \right\} \;\subseteq\; \mathcal{E}(Q(t)) \,.$$

Proof: This corrolary is a direct consequence of Theorem 1 as this Theorem holds for all directions $c \in \mathbb{R}^{n_x}$ and for all times t. □

Summarizing the above results, the matrix $Q(t)$ can at each time t be interpreted as the coefficients of an outer ellipsoid $\mathcal{E}(Q(t))$ which contains the set of reachable states at the time t under the assumption that the function w is contained in Ω_∞. In addition, we know from Theorem 1 that there exists for every direction $c \in \mathbb{R}^{n_x}$ and every time $t \in [0,T]$ a function $R : [0,T] \to \mathbf{cl}\left(\mathbb{D}_{++}^{n_w}\right)$ such that the associated outer ellipsoid $\mathcal{E}(Q(t))$ touches the set of reachable states in this given direction c at time t.

4. A CONSERVATIVE APPROXIMATION STRATEGY FOR NONLINEAR ROBUST OPTIMAL CONTROL PROBLEMS

In this section, we come back to the discussion of robust counterpart problems for nonlinear dynamic systems. Here, we are interested in a conservative approximation strategy. Unfortunately, we have to require suitable assumptions on the function f in order to develop such a strategy. In this paper, we propose to employ the following assumption:

Assumption 3. We assume that the right-hand side function f is differentiable and that there exists for each component f_i of the function f an explicit nonlinearity estimate $l_i : \mathbb{R}^{n_u \times n_x} \times \mathbb{R}^{n_x \times n_x} \to \mathbb{R}_+$ with

$$|f_i(x,K,w) - A_i x - B_i w| \;\leq\; l_i(K,Q) \tag{9}$$

for all $x \in \mathcal{E}(Q)$ and for all w with $\|w\|_\infty \leq 1$ as well as all possible choices of K and $Q \succeq 0$. Here, we have used the short hands $A_i := \frac{\partial f_i(0,K,0)}{\partial x}$ and $B_i := \frac{\partial f_i(0,K,0)}{\partial w}$.

From a mathematical point of view, the above assumption does not add a main restriction as we do not even require Lipschitz-continuity of the Jacobian of f. However, in practice, it might of course be hard to find suitable functions l_i which satisfy the above property. Nevertheless, once we find such an upper estimate, tractable conservative reformulations of the original non-convex min-max optimal control problem can be found. This is the aim of this section. In order to motivate how we can find such functions l_i, we consider a simple example:

Example 4. Let the function component f_i be convex quadratic in x but linear in w, i.e. we have

$$|f_i(x,K,w) - A_i x - B_i w| \;=\; x^T S_i(K) x$$

for some positive semi-definite matrix $S_i(K)$. In this case, we can employ the function

$$l_i(K,Q) \;:=\; \mathrm{Tr}(\, S_i(K) Q \,)$$

in order to satisfy the above assumption. A less conservative choice would be

$$l_i(K,Q) \;:=\; \lambda_{\max}(\, Q^{\frac{1}{2}}\, S_i(K)\, Q^{\frac{1}{2}} \,)$$

which would involve a computation of a maximum eigenvalue.

Now, we define the matrix valued function

$$\hat{B} : \mathbb{R}^{n_u \times n_x} \times \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^{n_x \times (n_w + n_x)}$$

as

$$\hat{B}(K,Q) = \left(\frac{\partial f_i(0,K,0)}{\partial w} \,,\; \mathrm{diag}\,(\, l(K,Q)\,) \right). \tag{10}$$

Theorem 5. For any $\hat{R} : [0,T] \to \mathbb{D}_{++}^{(n_w+n_x)\times(n_w+n_x)}$ and any $K(\cdot)$ regard the solution of the differential equation

$$\begin{aligned} \dot{P}(\tau) &= A(K(\tau))P(\tau) + P(\tau)A(K(\tau))^T \\ &\quad +\mathrm{Tr}\left[\, \hat{R}(\tau)\,\right] P(\tau) \\ &\quad +\hat{B}(K(\tau),Q(\tau))\hat{R}^{-1}(\tau)\hat{B}(K(\tau),Q(\tau))^T \\ P(0) &= 0 \\ \dot{\theta}(\tau) &= -\mathrm{Tr}\left[\, \hat{R}(\tau)\,\right] \theta(\tau) \\ \theta(0) &= 1 \end{aligned}$$

with $Q(\tau) := [1-\theta(\tau)]\, P(\tau)$. Then for all $t \in [0,T]$ we have the conservative upper bound

$$V_i[\,t,\, K(\cdot)\,] \;\leq\; \sqrt{C_i(K(t))\, Q(t)\, C_i(K(t))^T} \qquad (11)$$

on the worst case functionals V_i which have been defined in (2). Here, we use the notation $A(K) := \frac{\partial f_i(0,K,0)}{\partial x}$.

Proof: The above result is a consequence of the Theorem 1 from the previous section applied to a system of the form

$$\begin{aligned} \dot{x}(\tau) \;&=\; A(K(\tau))x(\tau) + \hat{B}(K(\tau),Q(\tau))\hat{w}(\tau) \\ x(0) \;&=\; 0\,. \end{aligned} \qquad (12)$$

Note that the system (12) is equivalent to the original nonlinear system once we define the auxiliary uncertainty $\hat{w}$ by

$$\hat{w} := \begin{pmatrix} w \\ D(K,Q)\,(f(x,K,w) - A(K)x - B(K)w) \end{pmatrix}.$$

with $D(K,Q) := \mathrm{diag}\,(\,l(K,Q)\,)^{-1}$. Here, $\hat{w}$ summarizes both the physical uncertainties w as well as the influence of the nonlinear terms. Note that due to the construction of $\hat{w}$, we know that $\|\hat{w}(\tau)\|_\infty \leq 1$ for all $\tau \in [0,T]$. Thus, we can transfer the result from Theorem 1 in order to obtain a proof of the inequality (11). □

In the next Section we discuss a tutorial example in order to show how the above Theorem can be applied in practice.

5. A SMALL TUTORIAL EXAMPLE

Let us demonstrate the applicability of the results in this paper by formulating a control design problem for a nonlinear inverted pendulum. The dynamic model is given by

$$\dot{x} \;=\; F(x,K,w) = \begin{pmatrix} x_2 \\ \dfrac{g}{L}\sin(x_1) + \dfrac{u}{L}\cos(x_1) + \dfrac{w}{mL^2} \end{pmatrix} \qquad (13)$$

Here, g is the gravitational constant while m is the mass, L the length, and x_1 the excitation angle of the pendulum. Note that $\dot{x}_1 = x_2$ is denoting the associated angular velocity. Moreover, u is the controllable acceleration of the joint of the pendulum which can be moved in horizontal direction. For $x = 0$, $u = 0$ and $w = 0$ the pendulum has an unstable steady state. Thus, we will need a feedback control to stabilize the inverted pendulum at this point. Note that there is an uncertain torque w acting at the pendulum.

The right-hand side function f for the closed loop system takes the form

$$f(x,K,w) = \begin{pmatrix} x_2 \\ \dfrac{g}{L}\sin(x_1) + \dfrac{Kx}{L}\cos(x_1) + \dfrac{w}{mL^2} \end{pmatrix} \qquad (14)$$

where we employ the linear feedback gain $K \in \mathbb{R}^{1\times 2}$ to be optimized. It is possible to show that the function

$$l(K,Q) = \begin{pmatrix} 0 \\ \dfrac{g}{L} r_1(Q) + \dfrac{r_2(Q)}{L}\sqrt{KQK^T} \end{pmatrix} \qquad (15)$$

with

$$r_1(Q) := \left|\sqrt{Q_{1,1}} - \sin(\sqrt{Q_{1,1}})\right|$$

and

$$r_2(Q) := \left|1 - \cos(\sqrt{Q_{1,1}})\right|$$

is an upper bound function satisfying the condition (9) within Assumption 3 for all $K \in \mathbb{R}^{1\times 2}$ and all $Q \in \mathbb{R}^{2\times 2}$ with $\sqrt{Q_{11}} \leq \frac{\pi}{2}$. Note that the above upper estimate l is locally quite tight in the sense that we have at least

$$l(u,Q) \;\leq\; \mathbf{O}\,(\,\|Q\|^{\frac{3}{2}}\,)\,.$$

However, there are also other estimates possible.

In the following, we assume that the uncertain torque satisfies $w \in \Omega_\infty$. We are interested in minimizing the L_2 norm of the feedback and estimator gains, i.e. $\int_0^T \|K(t)\|_F^2\,\mathrm{d}t$, while guaranteeing that path constraints of the form

$$-d \leq x_1(t) \leq d$$

are satisfied in closed loop mode for all possible uncertainties $w \in \Omega_\infty$ and for all times $t \in [0,T]$.

Using Theorem 5 we can formulate this gain design problem as

$$\inf_{P(\cdot),Q(\cdot),\theta(\cdot),K(\cdot),\hat{R}(\cdot)\in\mathbb{D}^3_{++}} \quad \int_0^T \|K(\tau)\|_F^2\,\mathrm{d}\tau$$

$$\text{s.t.}\quad \begin{cases} & \text{for all } \tau \in [0,T]: \\ d \;&\geq\; \sqrt{Q_{11}(\tau)} \\ \dot{P}(\tau) \;&=\; A(K(\tau))P(\tau) + P(\tau)A(K(\tau))^T \\ &\quad +\mathrm{Tr}\left[\hat{R}(\tau)\right] P(\tau) \\ &\quad +\hat{B}(K(\tau),Q(\tau))\hat{R}^{-1}(\tau)\hat{B}(K(\tau),Q(\tau))^T \\ Q(\tau) \;&=\; P(\tau)\,[1-\theta(\tau)] \\ P(0) \;&=\; 0 \\ \dot{\theta}(\tau) \;&=\; -\mathrm{Tr}\left[\,\hat{R}(\tau)\,\right]\,\theta(\tau) \\ \theta(0) \;&=\; 1\,. \end{cases}$$

Note that the above optimization problem is a standard optimal control problem which can be solved with existing nonlinear optimal control software. Any feasible solution of this problem yields a feedback and an estimator gain which guarantees that the path constraints of the form $-d \leq x_1(t) \leq d$ are robustly satisfied for all possible uncertainties $w \in \Omega_\infty$ when running the nonlinear system in closed loop mode. Note that control bounds of the form $-\overline{v} \leq v \leq \overline{v}$ could be imposed in an analogous way as v is linear in x.

In this paper, the software `ACADO Toolkit` (c.f. Houska et al. [2010]) has been employed in order to solve the above optimal control problem with

$$L = 1\,\mathrm{m}\,,\; m = 1\,\mathrm{kg}\,,\; g = 9,81\,\frac{\mathrm{m}}{2^2}\,,$$

$$T = 5\,\mathrm{s}\,,\;\text{and}\;\; d = \frac{\pi}{8}\,.$$

Figure 1 shows the state x_1 in a worst-case simulation of the closed-loop system using the optimized feedback gain K. Here, the worst case uncertainty $w(t) = 1\,\mathrm{Nm}$ has been found by local maximization. It is guaranteed that

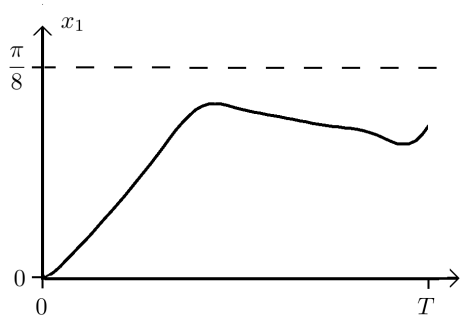

Fig. 1. A closed-loop simulation of the state x_1 for the torque $w(t) = 1\,\mathrm{Nm}$. The dotted line at $d = \frac{\pi}{8}$ is a conservative upper bound on the worst-case excitation of x_1.

x_1 satisfies the constraints of the form $-d \leq x_1(t) \leq d$ independ of the choice of w but this theoretical result does not state how conservative the result might be. However, the constant uncertainty $w(t) = 1\,\mathrm{Nm}$ turns out to be a local maximizer of x_1 for which

$$\max_{t\in[0,5]} x_1(t) \approx 0.33 \geq \frac{1}{1.19}\frac{\pi}{8}$$

is satisfied. Thus, we can state that in this application the level of conservatism was less than 19 %.

6. CONCLUSION

In this paper, we have developed a conservative approximation strategy for robust nonlinear optimal control problems. Here, the main assumption on the right-hand side function f was that we can find an explicit upper bound expression l which over-estimates the nonlinear terms in the differential equation. The approach has been transferred to control design problems and applied to a tutorial example explaining how the proposed strategies can be used in practice.

REFERENCES

A. Ben-Tal and A. Nemirovski. Robust Convex Optimization. *Math. Oper. Res.*, 23:769–805, 1998.

M. Diehl, H.G. Bock, and E. Kostina. An approximation technique for robust nonlinear optimization. *Mathematical Programming*, 107:213–230, 2006.

L. El-Ghaoui and H. Lebret. Robust Solutions to Least-Square Problems to Uncertain Data Matrices. *SIAM Journal on Matrix Analysis*, 18:1035–1064, 1997.

C.A. Floudas and O. Stein. The Adaptative Convexification Algorithm: a Feasible Point Method for Semi-Infinite Programming. *SIAM Journal on Optimization*, 18(4):1187–1208, 2007.

B. Houska and M. Diehl. Robust nonlinear optimal control of dynamic systems with affine uncertainties. In *Proceedings of the 48th Conference on Decision and Control*, Shanghai, China, 2009.

B. Houska and M. Diehl. Nonlinear Robust Optimization of Uncertainty Affine Dynamic Systems under the L-infinity Norm. In *In Proceedings of the IEEE Multi - Conference on Systems and Control*, Yokohama, Japan, 2010.

B. Houska, H.J. Ferreau, and M. Diehl. ACADO Toolkit – An Open Source Framework for Automatic Control and Dynamic Optimization. *Optimal Control Applications and Methods*, (DOI: 10.1002/oca.939), 2011. (in print).

H.T. Jongen, J.J. Rückmann, and O. Stein. Generalized semi-infinite optimization: A first order optimality condition and examples. *Mathematical Programming*, pages 145–158, 1998.

Z.K. Nagy and R.D. Braatz. Open-loop and closed-loop robust optimal control of batch processes using distributional and worst-case analysis. *Journal of Process Control*, 14:411–422, 2004.

Z.K. Nagy and R.D. Braatz. Distributional uncertainty analysis using power series and polynomial chaos expansions. *Journal of Process Control*, 17:229–240, 2007.

J.J. Rückmann and O. Stein. On linear and linearized generalized semi-infinite optimization problems. *Ann. Oper. Res.*, pages 191–208, 2001.

K. Zhou, J.C. Doyle, and K. Glover. *Robust and optimal control*. Prentice Hall, Englewood Cliffs, NJ, 1996.

APPENDIX

In this appendix we provide a proof of Theorem 1. For this aim, we first consider the following Lemma:

Lemma 6. Let $\lambda : [0,t] \to \mathbb{R}^{n_w}_{++}$ be a given positive and (Lebesgue-) integrable function while the short hand $\Lambda := \mathrm{diag}(\lambda) \succ 0$ denotes the associated diagonal matrix valued function. If we define the function $r : [0,t] \to \mathbb{R}_{++}$ and $R : [0,t] \to \mathbb{S}^{n_w}_{++}$ by

$$\forall \tau \in [0,t]: \quad r(\tau) := \frac{\mathrm{Tr}\,[\,\Lambda(\tau)\,]}{\kappa - \int_\tau^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s} \tag{16}$$

with $\kappa > \int_0^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s$ being a sufficiently large constant and

$$\forall \tau \in [0,t]: \quad R(\tau) := \frac{1}{\kappa}\,\Lambda(\tau)\,\exp\left(\int_\tau^t r(s)\,\mathrm{d}s\right) \tag{17}$$

then the following statements are true:

1) The functions r and R are positive and integrable function.

2) The inverse relation

$$\Lambda(\tau) = \kappa\,R(\tau)\,\exp\left(-\int_\tau^t \mathrm{Tr}\,[\,R(s)\,]\,\mathrm{d}s\right) \tag{18}$$

is satisfied for all $\tau \in [0,t]$.

3) The integral over the trace of Λ can equivalently be expressed as

$$\int_0^t \mathrm{Tr}\,[\,\Lambda(\tau)\,]\,\mathrm{d}\tau = \kappa\left[1 - \frac{1}{\exp\left(\int_0^t \mathrm{Tr}\,[\,R(s)\,]\,\mathrm{d}s\right)}\right]. \tag{19}$$

Proof: The positiveness and integrability of the functions r and R follows immediately from their definitions (16) and (17) together with the assumption $\kappa > \int_0^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s$. Let us compute the integral

$$\int_\tau^t r(\tau')\,\mathrm{d}\tau' \overset{(16)}{=} \int_\tau^t \frac{\mathrm{Tr}\,[\,\Lambda(\tau')\,]}{\kappa - \int_{\tau'}^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s}\,\mathrm{d}\tau'$$
$$= -\log\left(1 - \frac{1}{\kappa}\int_\tau^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s\right)\,. \quad (20)$$

for all $\tau \in [0,t]$. In the next step, we solve (20) with respect to the term $\int_\tau^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s$ finding

$$\int_\tau^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s = \kappa\left[1 - \exp\left(-\int_\tau^t r(s)\,\mathrm{d}s\right)\right]\,. \quad (21)$$

It remains to derive from the definition (16) that

$$\forall \tau \in [0,t] : \mathrm{Tr}\,[\,\Lambda(\tau)] \overset{(16)}{=} r(\tau)\left[\kappa - \int_\tau^t \mathrm{Tr}\,[\,\Lambda(s)\,]\,\mathrm{d}s\right]$$
$$\overset{(21)}{=} \kappa\, r(\tau)\,\exp\left(-\int_\tau^t r(s)\mathrm{d}s\right)\,.$$

Comparing this relation with the definition (17) we recognize that we must have $r(\tau) = \mathrm{Tr}\,[\,R(\tau)]$ for all $\tau \in [0,t]$. Thus, the definition (17) implies the relation (18). Finally, we note that the equation (19) follows from (21) for $\tau = 0$ using once more that $r(\tau) = \mathrm{Tr}\,[\,R(\tau)]$. □

The main reason why the above Lemma is useful is that it allows us to perform a variable substitution. I.e. we plan to replace the time-varying multiplier $\lambda(\tau)$ in the optimization problem (7) by the new function R employing the definitions (16) and (17).

The proof of Theorem 1
Using the definition (4) of $V(t)$ we know that there exists a sequence of diagonal and positive definite functions $(\Lambda_n(\cdot))_{n\in\mathbb{N}}$ such that

$$V(t) = \lim_{n\to\infty} \int_0^t \frac{c^T H_t(\tau)\Lambda_n(\tau)^{-1}H_t(\tau)^T c}{4}\,\mathrm{d}\tau$$
$$+ \int_0^t \mathrm{Tr}\,[\,\Lambda_n(\tau)\,]\,\mathrm{d}\tau\,.$$

Thus, we can also construct a sequence $(\kappa_n, R_n(\cdot))_{n\in\mathbb{N}}$ with $\kappa_n > \int_0^t \mathrm{Tr}\,[\,\Lambda_n(s)\,]\,\mathrm{d}s$ such that an application of Lemma 6 yields

$$V(t) =$$
$$\lim_{n\to\infty} \int_0^t \frac{c^T H_t(\tau)R_n(\tau)^{-1}H_t(\tau)^T c\; e^{\int_\tau^t \mathrm{Tr}[\,R_n(s)\,]\,\mathrm{d}s}}{4\kappa_n}\,\mathrm{d}\tau$$
$$+ \kappa_n\left[1 - e^{-\int_0^t \mathrm{Tr}[\,R_n(s)\,]\,\mathrm{d}s}\right]\,.$$

Consequently, we must have

$$V(t) = \inf_{\kappa, R(\cdot)>0} \frac{c^T P(t) c}{4\kappa} + \kappa\left[1 - e^{-\int_0^t \mathrm{Tr}[\,R(s)\,]\,\mathrm{d}s}\right]$$
$$= \inf_{\beta(\cdot)>0} \sqrt{1-\theta(t)}\,\sqrt{c^T P(t) c}\,. \quad (22)$$

Here, we have used that the function

$$P(t) := \int_0^t H_t(\tau)R(\tau)^{-1}H_t(\tau)^T\; e^{\int_\tau^t \mathrm{Tr}[\,R(s)\,]\,\mathrm{d}s}\,\mathrm{d}\tau$$

solves the Lyapunov differential equations in (7) uniquely. Thus, we obtain the statement of the Theorem. □