Nonlinear Robust Optimization of Uncertainty Affine Dynamic Systems under the L-infinity norm

Boris Houska and Moritz Diehl

***Abstract*— In this paper, we discuss robust optimal control techniques for dynamic systems which are affine in the uncertainty. Here, the uncertainty is assumed to be time-dependent but bounded by an L-infinity norm. We are interested in finding a tight upper bound for the worst case excitation of the inequality state constraints requiring to solve a parameterized lower-level maximization problem. In this paper, we suggest to replace this lower level maximization problem by an equivalent minimization problem using a special version of modified Lyapunov equations. This new reformulation offers advantages for robust optimal control problems where the uncertainty is time-dependent, i.e. infinite dimensional, while the inequality state constraints need to be robustly regarded on the whole time horizon.**

I. INTRODUCTION

In this paper, we are interested in robust optimal control of uncertain systems that are affine in the uncertain states and disturbances but possibly nonlinear in the remaining states and the control input. Here, the main challenge is to robustly regard inequality state constraints. In the recent decade robust nonlinear optimal control problems have received quite some attention, e.g. in form of robust counterpart formulations for convex problems in [3], [1], over semi-infinite optimization approaches [10], to approximation techniques and polynomial chaos expansions for nonlinear optimization problems [5], [8].

Note that in a previous work [6], the authors of this paper suggested a reformulation technique for nonlinear systems with affine uncertainties that are bounded by an L_2-norm. The framework of this approach will briefly be reviewed in Section II.

The contribution of this paper is presented in Section III, where the maximum excitation of an uncertain linear system is represented as a special minimization problem using a set of modified Lyapunov equations. Here, the main challenge is that the uncertainty is bounded by an L_∞-norm, which is different from the L_2 norm case as these two function norms are not equivalent. In Section IV it is outlined why the new representation can be useful for approximate robust optimal control and it is also discussed in which cases we track the robust counterpart exactly. Finally, we test the new formulation numerically in Section V and conclude the paper in Section VI.

B. Houska and M. Diehl are with the Optimization in Engineering Center (OPTEC), K. U. Leuven, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium. boris.houska@esat.kuleuven.be, moritz.diehl@esat.kuleuven.be

II. NONLINEAR ROBUST OPTIMAL CONTROL PROBLEMS FOR AFFINE UNCERTAINTIES

In this section we introduce the notation for uncertain optimal control problems of the form:

$$\boxed{\begin{array}{l}\min\limits_{x(\cdot),q(\cdot),T} \quad J[q(\cdot),T] \\ \text{s.t.} \quad \forall t \in \mathbb{T}: \\ \quad \begin{array}{rcl} \dot{x}(t) & = & A(q(t))x(t) + B(q(t))w(t) + r(q(t)) \\ 0 & \geq & C(q(t))x(t) - d(q(t)) \\ x(0) & = & r_0(q(0)) + B_0 w_0 \ , \\ q & \in & \mathcal{Q} \end{array}\end{array}} \quad . \quad (1)$$

Here, we use the definition $\mathbb{T} := [0,T]$. In this formulation, $x : \mathbb{T} \to \mathbb{R}^{n_x}$ denotes the states, which can be influenced by an unknown time-dependent disturbance $w : \mathbb{T} \to \mathbb{R}^{n_w}$ as well as by an unknown parameter $w_0 \in \mathbb{R}^{n_{w_0}}$. Note that the functions $A : \mathbb{R}^{n_q} \to \mathbb{R}^{n_x \times n_x}$, $B : \mathbb{R}^{n_q} \to \mathbb{R}^{n_x \times n_w}$, $C : \mathbb{R}^{n_q} \to \mathbb{R}^{n_x \times n_c}$, $r : \mathbb{R}^{n_q} \to \mathbb{R}^{n_x}$, and $d : \mathbb{R}^{n_q} \to \mathbb{R}^{n_c}$ are general nonlinear functions in q. However, these functions are in our framework not allowed to depend on x or w, i.e. the disturbance w enters the optimization problem in an affine way.

Here it should be explained that $q : \mathbb{R} \to \mathbb{R}^{n_q}$ is the nonlinear behavior of the system. This behavior is assumed to lie in a given set $\mathcal{Q}$ of measurable functions on the time interval $\mathbb{T}$. In the context of optimal control problems, $\mathcal{Q}$ has typically the form

$$\mathcal{Q} = \left\{ q : \mathbb{T} \to \mathbb{R}^{n_q} \;\middle|\; \begin{array}{l} \forall t \in \mathbb{T}: \\ \quad \begin{array}{rcl} 0 & = & F(t,\dot{q},q) \\ 0 & = & G(q(0),q(T)) = 0 \\ 0 & \geq & H(q(t)) \end{array} \end{array} \right\} \quad . \quad (2)$$

Note that $\mathcal{Q}$ is not allowed to depend on $\omega := (w(\cdot), w_0)$. In our formulation, the objective functional J is assumed to be independent of x. A dependence of J on x can always be eliminated by an introduction of slack parameters.

In this paper we are interested in the case that the uncertainties $\omega := (w(\cdot), w_0)$ are known to lie in a given bounded uncertainty set $\mathcal{B}$. To formulate the corresponding robust counterpart formulation of the optimization problem (1) we follow the classical approach [2], i.e. we consider the

optimization problem

$$\boxed{\begin{array}{lcll} \min\limits_{q(\cdot),T} & & J[q(\cdot),T] & \\ \text{s.t.} & & & \\ \forall t \in \mathbb{T}: & 0 & \geq & y^{\max}(\,t\,;\,q(\cdot),T) \\ & q & \in & \mathcal{Q} \end{array}} \quad , \tag{3}$$

where $y_i^{\max}(\,t\,;\,q(\cdot),T)$ is for each $t \in \mathbb{T}$ and each $i \in I := \{1,\ldots,n_c\}$ defined to be the optimal value of the following parameterized sub-maximization problem

$$\boxed{\begin{array}{l} \max\limits_{x(\cdot),w_0,w(\cdot)} \quad C_i(q(t))x(t) - d_i(q(t)) \\ \text{s.t.} \quad \forall \tau \in [0,t]: \\ \quad \dot{x}(\tau) \;=\; A(q(\tau))x(\tau) + B(q(\tau))w(\tau) + r(q(\tau)) \\ \quad x(0) \;=\; r_0(q(0)) + B_0 w_0 \\ \quad\;\; \omega \;\in\; \mathcal{B} \end{array}} \tag{4}$$

with C_i denoting the $i-th$ row of the matrix valued function C. Now, we encounter several difficulties: first, each time we need to evaluate the constraints of the optimization problem (3), we have to solve a maximization problem of the form (4) globally. Even if the set $\mathcal{B}$ is convex, this maximization can be expensive. Moreover, the functions $y_i^{\max}$ will in general not be differentiable. Second, we have a general time dependent disturbance, i.e. the uncertainty is not in a finite dimensional space. And third, the inequality constraints in the problem (3) need to be satisfied for all times $t \in \mathbb{T}$. In this sense, we have also an infinite number of constraints.

In this paper, we will not be able to overcome all these difficulties, but we rather concentrate on a special case: we will focus on uncertainties that are bounded by an L_∞ type norm of the form

$$\mathcal{B} := (\ \omega \mid \|w_0\| \leq 1 \ \wedge\ \forall \tau \in \mathbb{T}:\ \|w(\tau)\| \leq 1\ )\ . \tag{5}$$

We speak here of an L_∞ type norm in the sense that we require $\|w(\tau)\| \leq 1$ for all $\tau \in \mathbb{T}$. The choice of the vector norm $\|\cdot\|$ is not as crucial as in finite dimensional spaces all norms are equivalent. In the following we will typically assume that $\|\cdot\|$ denotes the Euclidean norm. Note that this framework is different from the case that ω is bounded by an L_2-type norm for which a robust optimization approach has been suggested in [6]. Here, the main difference is due to the fact that the L_∞ norm and the L_2-norm are not equivalent.

III. LINEAR UNCERTAIN SYSTEMS

Let us analyze the linear system for the state $x : \mathbb{R} \to \mathbb{R}^{n_x}$ first. We write this system in the form

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \quad \text{with} \quad x(0) = B_0 w_0 \tag{6}$$

with $A : \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ and $B : \mathbb{R} \to \mathbb{R}^{n_x \times n_w}$ being (Lebesgue-) integrable functions , and $B_0 : \mathbb{R} \to \mathbb{R}^{n_x \times n_{w_0}}$. Here $w : \mathbb{R} \to \mathbb{R}^{n_w}$ as well as $w_0 \in \mathbb{R}^{n_{w_0}}$ denote uncertainties, which are only known to be bounded by $\mathcal{B}$, as defined in Equation (5).

We are interested in an approximation of the maximum excitation $V(t)$ of the system at a time t in a given direction $c \in \mathbb{R}^{n_x}$:

$$V(t) \;:=\; \max_{x(\cdot),w(\cdot),w_0} c^T x(t) \quad \text{s.t.} \begin{cases} \forall t \in \mathbb{R}: \\ \dot{x}(t) = A(t)x(t) + B(t)w(t) \\ x(0) = B_0 w_0 \\ w(t)^T w(t) \leq 1 \\ w_0^T w_0 \leq 1 \end{cases} . \tag{7}$$

The value function V can also be expressed via the dual problem:

Lemma 3.1: The value function V defined by (7) can equivalently be expressed as[1]

$$\begin{aligned} V(t) \;=\; & \inf_{\lambda(\cdot)>0} \int_0^t \frac{c^T G(t,\tau)B(\tau)B(\tau)^T G(t,\tau)^T c}{4\lambda(\tau)}\,\mathrm{d}\tau \\ & + \int_0^t \lambda(\tau)\,\mathrm{d}\tau \\ & + \inf_{\lambda_0>0} \ \frac{c^T G(t,0)B_0 B_0^T G(t,0)^T c}{4\lambda_0} + \lambda_0 \ , \end{aligned} \tag{8}$$

where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n_x \times n_x}$ is the fundamental solution associated with the linear differential equation (6), which is defined as the solution of the following differential equation:

$$\frac{\partial G(t,\tau)}{\partial t} = A(t)G(t,\tau) \quad \text{with} \quad G(\tau,\tau) = 1 \tag{9}$$

for all $t, \tau \in \mathbb{R}$.

Proof: Writing x explicitly as

$$x(t) = G(t,0)B_0 w_0 + \int_0^t G(t,\tau)B(\tau)w(\tau)\,\mathrm{d}\tau$$

the dual of the optimization problem (7) is given by

$$\begin{aligned} V(t) \;:=\; & \inf_{\lambda(\cdot)>0} \max_{w(\cdot)} c^T \int_0^t G(t,\tau)B(\tau)w(\tau)\,\mathrm{d}\tau \\ & - \int_0^t \lambda(\tau)w(\tau)^T w(\tau)\,\mathrm{d}\tau + \int_0^t \lambda(\tau)\,\mathrm{d}\tau \\ & + \inf_{\lambda_0>0} \max_{w_0} \ c^T G(t,0)B_0 w_0 - \lambda_0 w_0^T w_0 + \lambda_0\,. \end{aligned}$$

As the maximum is obtained at

$$w_0^* := \frac{B_0^T G(t,0)^T c}{2\lambda_0} \quad \text{and} \quad w^*(\tau) := \frac{B(\tau)^T G(t,\tau)^T c}{2\lambda(\tau)}$$

(for all $\tau \in \mathbb{R}$) we obtain (8). □

Note that the above Lemma can be used to generate a variety of conservative approximations of the worst case excitation $V(t)$ of the system (6). For example, a very simple

[1]Note that under mild reachibility assumptions of the linear system, the infinimum could also be replaced by a minimum.

approximation is obtained by just minimizing over a constant function $\lambda(\tau) = \lambda_0$ for all $\tau \in \mathbb{R}$. In this case we find

$$V(t) \quad \leq \quad \inf_{\lambda_0 > 0} \quad \frac{1}{4\lambda_0} c^T P(t) c + (1+t)\lambda_0 \,, \qquad (10)$$

where

$$\begin{aligned} P(t) \quad &:= \quad G(t,0) B_0 B_0^T G(t,0)^T \\ &\quad + \int_0^t G(t,\tau) B(\tau) B(\tau)^T G(t,\tau)^T \, \mathrm{d}\tau \end{aligned}$$

is the solution of a Lyapunov differential equation:

$$\begin{aligned} \dot{P}(t) \quad &= \quad A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T \\ P(0) \quad &= \quad B_0 B_0^T \,. \end{aligned} \qquad (11)$$

Solving (10) yields the conservative approximation

$$V(t) \quad \leq \quad \sqrt{1+t} \sqrt{c^T P(t) c} \,. \qquad (12)$$

Unfortunately, this approximation is very weak: just assume that the system is stable. In this case we expect that $V(t)$ is bounded for all $t \in \mathbb{R}$. However, due to the factor $\sqrt{1+t}$, the above upper bound tends to infinity for large t.

Another possibility is to simplify the infimum in equation (8) explicitly finding

$$\begin{aligned} V(t) \quad &= \quad \sqrt{c^T G(t,0) B_0 B_0^T G(t,0)^T c} \\ &+ \quad \int_0^t \sqrt{c^T G(t,\tau) B(\tau) B(\tau)^T G(t,\tau)^T c} \, \mathrm{d}\tau \,. \end{aligned} \qquad (13)$$

However, this expression for V is unfortunately not very attractive from a numerical point of view: if we need to compute V for all t in a given interval $[0,T]$, we would need to evaluate the adjoint direction $c^T G(\cdot,\cdot)$ of the fundamental solution G everywhere on its 2-dimensional domain $[0,T] \times [0,T]$. As G is rather expensive to compute in our optimization context, the expression (13) does not lead to an appropriate formulation of the robust counterpart (3).

In order to overcome these limitations, we start with a rather technical Lemma:

Lemma 3.2: Let $\lambda : [0,t] \to \mathbb{R}_{++}$ be a given positive and (Lebesgue-) integrable function. If we define the function $\beta : [0,t] \to \mathbb{R}_{++}$ by

$$\forall \tau \in [0,t] : \; \beta(\tau) := \frac{\lambda(\tau)}{\kappa - \int_\tau^t \lambda(s) \, \mathrm{d}s} \qquad (14)$$

with $\kappa > \int_0^t \lambda(s) \mathrm{d}s$ being a sufficiently large constant, then the following statements are true:

1) The function β is a positive and integrable function.
2) The inverse relation

$$\lambda(\tau) = \kappa \beta(\tau) \, \exp\left(- \int_\tau^t \beta(s) \, \mathrm{d}s \right) \qquad (15)$$

is satisfied for all $\tau \in [0,t]$.
3) The integral over λ can equivalently be expressed as

$$\int_0^t \lambda(\tau) \, \mathrm{d}\tau = \kappa \left[1 - \exp\left(- \int_0^t \beta(s) \mathrm{d}s \right) \right] \,. \qquad (16)$$

Proof: The positiveness and integrability of β follows immediately from the definition (14) together with the assumption $\kappa > \int_0^t \lambda(s) \mathrm{d}s$. Let us compute the integral

$$\begin{aligned} \int_\tau^t \beta(\tau') \, \mathrm{d}\tau' \quad &\overset{(14)}{=} \quad \int_\tau^t \frac{\lambda(\tau')}{\kappa - \int_{\tau'}^t \lambda(s) \, \mathrm{d}s} \, \mathrm{d}\tau' \\ &= \quad - \log \left(1 - \frac{1}{\kappa} \int_\tau^t \lambda(s) \, \mathrm{d}s \right) \,. \end{aligned} \qquad (17)$$

for all $\tau \in [0,t]$. In the next step, we solve (17) with respect to the term $\int_\tau^t \lambda(s) \, \mathrm{d}s$ finding

$$\int_\tau^t \lambda(s) \, \mathrm{d}s = \kappa \left[1 - \exp\left(- \int_\tau^t \beta(s) \mathrm{d}s \right) \right] \,. \qquad (18)$$

Note that this relation yields the equation (16) for $\tau = 0$. It remains to derive from the definition (14) that

$$\begin{aligned} \forall \tau \in [0,t] : \lambda(\tau) \quad &\overset{(14)}{=} \quad \beta(\tau) \left[\kappa - \int_\tau^t \lambda(s) \, \mathrm{d}s \right] \\ &\overset{(18)}{=} \quad \kappa \beta(\tau) \exp\left(- \int_\tau^t \beta(s) \mathrm{d}s \right) \,, \end{aligned}$$

which completes the proof of the Lemma. □

Theorem 3.1: The value function V defined by (7) can equivalently be expressed as

$$V(t) = \inf_{\beta(\cdot) > 0} \sqrt{\theta(\tau)} \sqrt{c^T P_0(t) c} + \sqrt{1 - \theta(\tau)} \sqrt{c^T P_1(t) c} \,,$$

where the functions $P_0, P_1 : [0,t] \to \mathbb{R}^{n \times n}$ and the function $\theta : [0,t] \to \mathbb{R}$ are implicitly depending on a positive function $\beta : [0,t] \to \mathbb{R}_{++}$ and defined to be the solutions of the following modified Lyapunov differential equations:

$$\begin{aligned} \dot{P}_0(\tau) \quad &= \quad A(\tau)P_0(\tau) + P_0(\tau)A(\tau)^T + \beta(\tau)P_0(\tau) \\ \dot{P}_1(\tau) \quad &= \quad A(\tau)P_1(\tau) + P_1(\tau)A(\tau)^T \\ &\quad + \beta(\tau)P_1(\tau) + \tfrac{1}{\beta(\tau)} B(\tau)B(\tau)^T \\ \dot{\theta}(\tau) \quad &= \quad -\beta(\tau)\theta(\tau) \\ P_0(0) \quad &= \quad B_0 B_0^T \,, \; P_1(0) \; = \; 0 \,, \; \theta(0) \; = \; 1 \end{aligned} \qquad (19)$$

for all $\tau \in [0,t]$.

Proof: Let us define the value $V_0(t)$ by

$$\begin{aligned} V_0(t) \quad &:= \quad \inf_{\lambda_0 > 0} \quad \frac{1}{4\lambda_0} c^T G(t,0) B_0 B_0^T G(t,0)^T c + \lambda_0 \\ &= \quad \sqrt{c^T G(t,0) B_0 B_0^T G(t,0)^T c} \,. \end{aligned}$$

As a consequence of Lemma 3.1 we know that there exists a sequence of positive functions $(\lambda_n(\cdot))_{n \in \mathbb{N}}$ such that

$$\begin{aligned} &V(t) - V_0(t) \; = \\ &\lim_{n \to \infty} \int_0^t \tfrac{c^T G(t,\tau) B(\tau) B(\tau)^T G(t,\tau)^T c}{4\lambda_n(\tau)} \mathrm{d}\tau + \int_0^t \lambda_n(\tau) \, \mathrm{d}\tau \,. \end{aligned}$$

Thus, we can also construct a sequence $(\kappa_n, \beta_n(\cdot))_{n \in \mathbb{N}}$ with

$$\kappa_n > \int_0^t \lambda_n(s) \, \mathrm{d}s \quad \text{and} \quad \beta_n(\tau) := \frac{\lambda_n(\tau)}{\kappa_n - \int_\tau^t \lambda_n(s) \, \mathrm{d}s}$$

such that an application of Lemma 3.2 yields

$$V(t) - V_0(t) \;=$$

$$\lim_{n\to\infty} \int_0^t \frac{c^T G(t,\tau)B(\tau)B(\tau)^T G(t,\tau)^T c \;\exp\left(\int_\tau^t \beta_n(s)\,\mathrm{d}s\right)}{4\kappa_n \beta_n(\tau)}\,\mathrm{d}\tau$$
$$+\kappa_n \left[1 - e^{-\int_0^t \beta_n(s)\mathrm{d}s}\right] .$$

Consequently, we must have

$$V(t) - V_0(t) \;=$$

$$\inf_{\kappa,\beta(\cdot)>0} \int_0^t \frac{c^T G(t,\tau)B(\tau)B(\tau)^T G(t,\tau)^T c \;\exp\left(\int_\tau^t \beta(s)\,\mathrm{d}s\right)}{4\,\kappa\,\beta(\tau)}\,\mathrm{d}\tau$$
$$+\kappa \left[1 - \exp\left(-\int_0^t \beta(s)\mathrm{d}s\right)\right]$$
$$= \inf_{\kappa,\beta(\cdot)>0} \tfrac{1}{4\kappa} c^T P_1(t) c + \kappa \left[1 - \exp\left(-\int_0^t \beta(s)\mathrm{d}s\right)\right]$$
$$= \inf_{\beta(\cdot)>0} \sqrt{1-\theta(t)}\sqrt{c^T P_1(t) c}\,.$$

Here, we have used that

$$P_1(t) \;:=\; \int_0^t \frac{G(t,\tau)B(\tau)B(\tau)^T G(t,\tau)^T \;\exp\left(\int_\tau^t \beta(s)\,\mathrm{d}s\right)}{\beta(\tau)}\,\mathrm{d}\tau$$

solves together with

$$P_0(t) \;:=\; \theta(t)^{-1} G(t,0) B_0 B_0^T G(t,0)^T$$

the associated Lyapunov differential equations (19). Thus, we obtain the statement of the Theorem. □

Corollary 3.1: We have the conservative approximation

$$V(t) \;\leq\; \inf_{\beta(\cdot)>0} \sqrt{c^T P(t) c}\,, \tag{20}$$

where $P : [0,t] \to \mathbb{R}^{n\times n}$ is the solution of the differential equation:

$$\begin{aligned} \dot{P}(\tau) &= A(\tau)P(\tau) + P(\tau)A(\tau)^T + \beta(\tau)P(\tau) \\ &\quad + \frac{1}{\beta(\tau)} B(\tau)B(\tau)^T \qquad (21) \\ P(0) &= B_0 B_0^T \qquad (22) \end{aligned}$$

for all $\tau \in [0,t]$.

Proof: Note that the fundamental inequality

$$\sqrt{a}\,\sqrt{\theta} + \sqrt{b}\,\sqrt{1-\theta} \;\leq\; \sqrt{a+b} \tag{23}$$

holds for all $a, b \in \mathbb{R}_+$ and all $\theta \in [0,1]$. Using this inequality with $\theta = e^{-\int_0^t \beta(s)\,\mathrm{d}s}$ we find with Theorem 3.1 that

$$V(t) \;\leq\; \inf_{\beta(\cdot)>0} \sqrt{c^T P_0(t) c + c^T P_1(t) c}\,. \tag{24}$$

As the function $P := P_0 + P_1$ satisfies the differential equation (21), the proof is complete. □

Remark 3.1: The result of the above corollary seems to be known in the literature, although it is not stated in this form. A Lyapunov equation of the form (21) has for the first time been proposed in [9], where bounding ellipsoids for uncertain linear systems are discussed. Later, e.g. in [4], this idea has further been developed where also the possibility of optimizing β is mentioned. However, as far as the authors are aware, only conservative worst case estimates have been proposed so far. Thus, Theorem 3.1 and especially the introduction of the state θ are an original contribution of this paper. Note that also the proof of Theorem 3.1 and the associated Corollary 3.1 are quite different from the argumentation techniques in [9], [4].

IV. APPROXIMATE ROBUST OPTIMAL CONTROL

In this section we transfer the result of Theorem 3.1 in order to replace the exact robust counterpart formulation (3) by a conservative approximation. For this aim, we use the notation

$$\xi \;:=\; (\,P_0,\, P_1,\, \theta,\, \beta\,)$$

to collect the auxiliary functions which are needed in Theorem 3.1. Now, we consider a robust counterpart formulation of the form

$$\boxed{\begin{array}{lll} \multicolumn{3}{l}{\min\limits_{q(\cdot),\gamma(\cdot),\xi(\cdot),T} \quad J[q(\cdot),T]} \\ \text{s.t.} & \forall t \in \mathbb{T}\,,\ \forall i \in I : & \\ 0 \;\geq\; & C_i(q(t))\gamma(t) - d_i(q(t)) & \\ & +\sqrt{\theta(\tau)}\sqrt{C_i(q(t))^T P_0(t) C_i(q(t))} & \\ & +\sqrt{1-\theta(\tau)}\sqrt{C_i(q(t))^T P_1(t) C_i(q(t))} & \\ q \;\in\; \mathcal{Q} & (\gamma,q) \;\in\; \mathcal{R} & (\xi,q) \;\in\; \mathcal{L} \end{array}} \tag{25}$$

where $\mathcal{L}$ denotes the set of functions (ξ, q) which satisfy

$$\begin{aligned} \dot{P}_0(\tau) &= A(q(\tau))P_0(\tau) + P_0(\tau)A(q(\tau))^T \\ &\quad +\beta(\tau)P_0(\tau) \\ \dot{P}_1(\tau) &= A(q(\tau))P_1(\tau) + P_1(\tau)A(q(\tau))^T \\ &\quad +\beta(\tau)P_1(\tau) + \tfrac{1}{\beta(\tau)} B(q(\tau))B(q(\tau))^T \\ \dot{\theta}(\tau) &= -\beta(\tau)\theta(\tau) \quad \beta(\tau) \;>\; 0 \\ P_0(0) &= B_0 B_0^T \quad P_1(0) \;=\; 0 \quad \text{and} \quad \theta(0) \;=\; 1 \end{aligned}$$

for all $\tau \in \mathbb{T}$. Moreover, $\mathcal{R}$ is defined to be the set of all functions (γ, q) which satisfy the reference condition

$$\begin{aligned} \dot{\gamma}(t) &= A(q(t))\gamma(t) + r(q(t)) \\ \gamma(0) &= r_0(q(0))\,. \end{aligned}$$

Remark 4.1: The above robust counterpart formulation is in general only a conservative replacement for the optimization problem (3) as we would need to add a set of Lyapunov differential equations for each active constraint otherwise. However, if we are in the situation that in the optimal solution at one time only one of the robustified constraints is active, then we know a posteriori from Theorem 3.1 that our formulation is exact without having known beforehand which of the constraints is active at which time.

Despite the fact that the above formulation is only a conservative approximation for the case that more than one constraint is active, we might still consider to use it hoping that the function β gives us sufficient degree of freedom to reduce the induced level of conservatism.

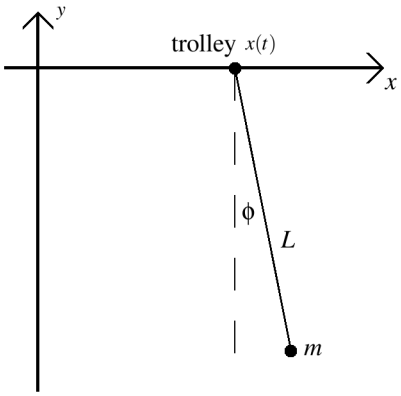

Fig. 1. A sketch of the crane.

V. A NUMERICAL TEST EXAMPLE

In order to assess the numerical applicability of the robust optimization approach as suggested above, we consider in this section a simple crane model (cf. Figure 1). If the crane's line with length L is very long with respect to the horizontal excitation $L\sin(\phi)$ of the mass $m = 1.0\,\text{kg}$, which is affected by an unknown force F, the dynamics of the excitation angle of the line of the crane can be described by the following differential equation:

$$\dot{\chi}(t) \;=\; A(t)\chi(t) + B(t)F(t) + r(t) \tag{26}$$

with

$$A(t) := \begin{pmatrix} 0 & 1 \\ -\frac{g}{L(t)} & -\left(b + 2\frac{\dot{L}(t)}{L(t)}\right) \end{pmatrix},\; B(t) := \begin{pmatrix} 0 \\ \frac{1}{mL(t)} \end{pmatrix},$$
$$r(t) := \begin{pmatrix} 0 \\ -\frac{\ddot{x}}{L} - \frac{\dot{L}\dot{x}}{L^2} \end{pmatrix} \quad \text{and} \quad \chi := \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}.$$

Here, the crane is only considered in a plane $\mathbb{R}^2$, in which the mounting point of the cable is at the time t located at the position $(x(t), 0)^T \in \mathbb{R}^2$ while the mass has the position $(x(t) + L(t)\sin(\phi(t)), -L(t)\cos(\phi(t)))^T \in \mathbb{R}^2$. In this notation, $g = 9.81\,\frac{\text{m}}{\text{s}^2}$ is the gravitational constant and $b = 0.05\,\frac{1}{\text{s}}$ a friction coefficient. Note that the above model is only valid for small excitations ϕ where the dynamics can be linearized in the states ϕ and $\dot{\phi}$.

The external force F, acting at the mass in horizontal direction, is assumed to be unknown in our example. The optimal control problem we would like to solve now assumes that we have the control $u := \left(\ddot{x}, \ddot{L}\right)$ as a degree of freedom. More precisely, we define the feasible behavior

$$q := \left(z^T\,,\, u^T\right)^T := \left(x, L, \dot{x}, \dot{L}, u^T\right)^T \in \mathcal{Q}$$

of the dynamic system by

$$\mathcal{Q} := \left\{ q : [0,T] \to \mathbb{R}^6 \;\middle|\; \begin{array}{l} \forall t \in [0,T]: \\ \quad \dot{z}(t) \;=\; (\dot{x}(t), \dot{L}(t), u(t)^T)^T \\ \quad z(0) \;=\; z_0 \\ \quad z(T) \;=\; z_T \\ \quad u_{\min} \;\geq\; u(t) \;\geq\; u_{\max} \end{array} \right\},$$

where we use the following values for our example

$$\begin{aligned} z_0 &:= \left(0\,\text{m},\, 5\,\text{m},\, 0\,\frac{\text{m}}{\text{s}},\, 0\,\frac{\text{m}}{\text{s}}\right)^T \\ z_T &:= \left(1\,\text{m},\, 5\,\text{m},\, 0\,\frac{\text{m}}{\text{s}},\, 0\,\frac{\text{m}}{\text{s}}\right)^T \\ u_{\min} &:= \left(-1.0\frac{\text{m}}{\text{s}^2}, -0.5\frac{\text{m}}{\text{s}^2}\right)^T \\ u_{\max} &:= \left(1.0\frac{\text{m}}{\text{s}^2}, 0.5\frac{\text{m}}{\text{s}^2}\right)^T. \end{aligned}$$

We are first interested in a minimum time optimal control problem for the nominal case with $F = 0$:

$$\boxed{\begin{array}{ll} \min\limits_{q(\cdot),\gamma(\cdot),T} & T \\ \text{s.t.} & \\ \forall t \in [0,T]: & \frac{\text{d}}{\text{d}t}\gamma(t) = A(q(t))\gamma(t) + r(q(t)) \\ \forall t \in [0,T]: & \phi_{\min} \leq \phi(t) \leq \phi_{\max} \\ & \gamma(0) = \gamma_0 \\ & q \in \mathcal{Q} \end{array}}, \tag{27}$$

where we chose the following numerical values:

$$\begin{aligned} \gamma_0 &:= \left(0\text{ rad},\, 0\,\frac{\text{rad}}{\text{s}}\right)^T \\ \phi_{\max} &:= -\phi_{\min} := 0.125\,\text{rad}. \end{aligned}$$

It can easily be checked that the problem (27) has a trivial solution: we simply choose $u_1(t) = 1\,\frac{\text{m}}{\text{s}^2}$ in order to accelerate as fast as possible for $t \in [0\,\text{s}, 1\,\text{s}]$ and brake the trolley with $u_1(t) = -1\,\frac{\text{m}}{\text{s}^2}$ for $t \in [1\,\text{s}, 2\,\text{s}]$. Choosing constant line length $L(t) = 5\text{m}$, the bounds for ϕ are not violated during this maneuver. Thus, this trivial bang-bang solution is already feasible and consequently globally optimal with $T_{\min} = 2\,\text{s}$.

However, the problem becomes much less trivial if robustness aspects are taken into account. We assume that the force F is unknown but bounded by

$$\forall t \in [0,T]: \quad \|F(t)\|_2^2 \;\leq\; 0.1\,\text{N}^2\,, \tag{28}$$

while the initial value $\chi(0) = \gamma_0$ is exactly known. Now, we solve the associated robust counterpart problem (25), where the objective functional is not exactly T but replaced by a Mayer term of the form

$$J \;:=\; T + \text{Trace}\left(\,\Gamma\, P_1(T)\, \Gamma^T\,\right) \tag{29}$$

with Γ being a heuristic and very small constant scaling matrix which has only been introduced for numerical regularization purposes.

In this paper, the software ACADO-Toolkit [7] was used to solve the nonlinear optimal control problem. A locally optimal result is shown in Figure 2. Note that the solution for u_1 and u_2 has still a bang-bang structure. However, the control u_1 switches more than once from the upper to the lower bound. Here, we use a piecewise constant control discretization consisting of 80 equidistant pieces. The optimal value for the time T is $2.23\,\text{s}$. The difference to the

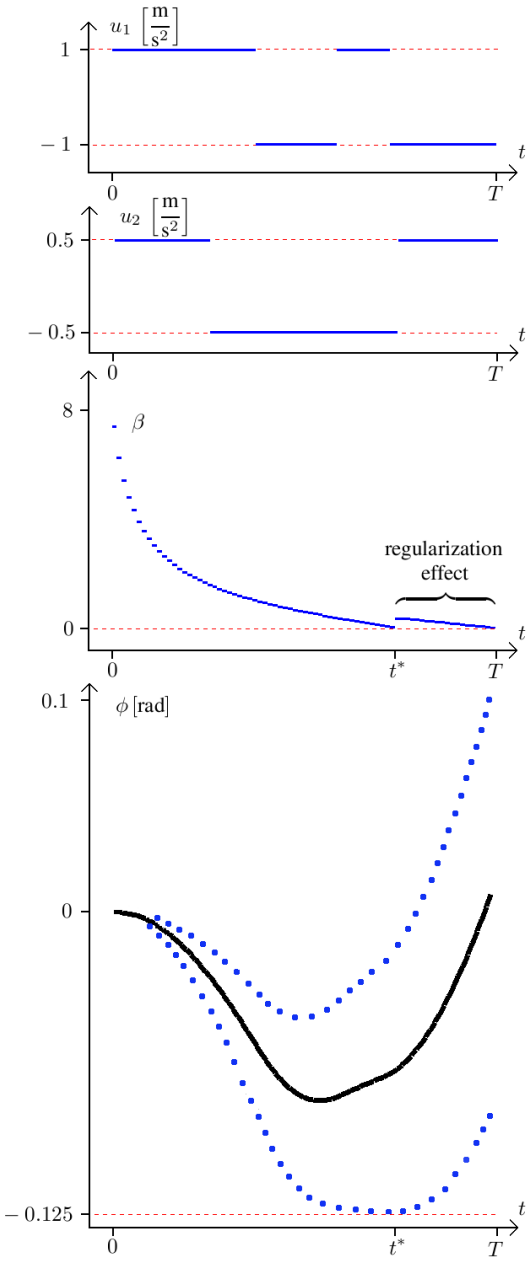

Fig. 2. The optimal robustified solution for the control inputs u_1 and u_2 as well as the function β and the solution for the nominal excitation ϕ together with the associated uncertainty region indicated by the dotted upper and lower robustness margins.

nominally optimal solution $T_{\min} = 2\,\text{s}$ can be interpreted as the price to be paid for robustness.

Furthermore, Figure 2 shows the nominal solution for the state ϕ as well as the associated dotted upper and lower margins of the uncertainty region. The constraint

$$\phi(t) - \sqrt{e_\phi^T P_1(t) e_\phi} \;\geq\; \phi_{\min} \tag{30}$$

is active at the time $t^* \approx 1.60\text{s}$. As this is the only robustified constraint which becomes active, we know that our robust counterpart formulation is exact in this example.

It remains to discuss that the optimal solution for the auxiliary control function β is monotonically decreasing on the interval $[0, t^*]$ reaching its minimum $\beta(t^*) \approx 0.03$ before a discontinuity occurs. This discontinuity at t^* is not due to our discretization but it turns out to be sensitive to the regularization term. This regularization effect occurs as the function values $\beta(t)$ for $t \in [t^*, T]$ enter the optimization problem via the regularization term in the objective only. Finally, we remark that the square-root factor $\sqrt{1-\theta(t^*)} \approx 0.959$, which occurs in the formulation (25), is at the time t^* close to 1 - but not negligibly close.

VI. CONCLUSION

In this paper we have discussed robust counterpart formulations for optimal control problems with affine uncertainty. Under the assumption that the uncertainty is bounded by an L_∞-type norm, we have proposed a way to reformulate the associated worst case leading us to a set of modified Lyapunov differential equations as summarized in Theorem 3.1 and Corollary 3.1. We have used this result to transform robust optimal control problems into standard optimal control problems stacking a set of modified Lyapunov differential equations to the original ODE in order to compute robustness margins for the inequality state constraints. This approach is guaranteed to be exact for the case that exactly one of the robustified constraints is active while an in-depth analysis of the case that more than one of the robustified constraints is active might be a subject of future research.

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