An Analysis of Linear Crosstalk Cancellation and Pre-compensation Techniques which Achieve Near Capacity in DSL Channels

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Abstract

This work was carried out in the frame of IUAP P5/22, *Dynamical Systems and Control: Computation, Identification and Modeling* and P5/11, *Mobile multimedia communication systems and networks*; the Concerted Research Action GOA-MEFISTO-666, *Mathematical Engineering for Information and Communication Systems Technology*; FWO Project G.0196.02, *Design of efficient communication techniques for wireless time-dispersive multi-user MIMO systems* and was partially sponsored by Alcatel-Bell.

I. Introduction

Linear crosstalk cancellation/pre-compensation schemes are preferable to non-linear schemes due to

- Lower complexity
- No decision feedback errors
  Easier performance analysis. More predictable performance. In decision feedback schemes feedback occurs before decoding. This leads to a high error rate.
- Linear schemes more amenable to partial implementations. See [1] for details.

II. System Model

A. Upstream

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{z}_k$$

where $\mathbf{H}_k$ is the $N \times N$ channel matrix.

B. Downstream

$$\mathbf{y}_k = \mathbf{H}_k^T \mathbf{x}_k + \mathbf{z}_k \tag{1}$$

C. Diagonal Dominance

When the line terminals (LT) are co-located the upstream channel matrix $\mathbf{H}_k$ is column-wise diagonal dominant in that

$$|h_k^{n,n}| \gg |h_k^{n,m}|, \ \forall m \neq n$$

Recall that the LTs are the modems at the central office (CO) and the NTs are the modems at the customer premises (CP). We quantify the degree of column-wise diagonal dominance using the parameter α_k

$$|h_k^{n,m}| \leq |h_k^{n,n}| \tan \alpha_k, \ \forall m \neq n \tag{2}$$

According to the semi-empirical model employed in the ETSI standards[2] α_k is modeled such that

$$\tan \alpha_k = K_{fext} f_k \sqrt{l}$$

where $K_{fext} = 10^{-45/20}$, f_k is the frequency at tone k in MHz and l is the length of the crosstalk channel in kilometers.

Theorem 1: Let $\mathbf{H}_k$ be a column-wise diagonal dominant matrix which satisfies (2). Denote the QR decomposition of $\mathbf{H}_k$

$$\mathbf{H}_k \stackrel{qr}{=} \mathbf{Q}_k \mathbf{R}_k \tag{3}$$

The following bound holds

$$|r_k^{n,n}| \geq |h_k^{n,n}| \sqrt{1 - 4(N-1)\alpha^2} \tag{4}$$

where $h_k^{n,m} \triangleq [\mathbf{H}_k]_{n,m}$ and $r_k^{n,m} \triangleq [\mathbf{R}_k]_{n,m}$.

Proof: See [3]. ■

III. Upstream Communication - Linear Zero Forcing Crosstalk Canceller

A. Linear Zero Forcing Canceller

We use a linear filter to estimate the transmitted symbols

$$\widehat{\mathbf{x}}_k = \mathbf{W}_k \mathbf{y}_k$$

where $\mathbf{W}_k \triangleq \mathbf{H}_k^{-1}$.

B. Performance Bounds

Theorem 2: Regardless of receiver structure the rate of user n on tone k is upper bounded

$$c_{k,\text{opt}}^n \leq \log_2 \left(1 + \left[1 + (N-1)\tan^2 \alpha\right] S_k \left|h_k^{n,n}\right|^2 \sigma_k^{-2} \Gamma^{-1}\right) \tag{5}$$

Proof: See Appendix I. ■

Theorem 3: With the linear ZF crosstalk canceller the rate of user n on tone k is lower bounded

$$c_{k,\text{zf}}^n \geq \log_2 \left(1 + \left[1 - 4(N-1)\alpha^2\right] S_k \left|h_k^{n,n}\right|^2 \sigma_k^{-2} \Gamma^{-1}\right) \tag{6}$$

Proof: See Appendix II. ■

IV. Downstream Communication - Linear Diagonalizing Crosstalk Pre-compensator

A. Linear Zero Forcing Pre-compensator

We apply a linear pre-filter to pre-compensate for the effects of crosstalk

$$\mathbf{x}_k = \mathbf{P}_k \overline{\mathbf{x}}_k$$

where $\mathbf{P}_k \triangleq \frac{1}{\beta_{\text{zf}}} \mathbf{H}_k^{-T}$. The factor β_{zf} ensures that the pre-filter $\mathbf{P}_k$ does not increase the power of any of the TXs

$$\begin{aligned} \beta_{\text{zf}} &\triangleq \max_n \left\| \left[\mathbf{H}_k^{-T} \right]_{\text{row } n} \right\| \\ &\simeq \max_n \frac{1}{|h_k^{n,n}|} \end{aligned}$$

The result is that

$$\begin{aligned} \mathbf{y}_k &= \frac{1}{\beta_{\text{zf}}} \overline{\mathbf{x}}_k + \mathbf{z}_k \\ &\simeq \min_n |h_k^{n,n}| \, \overline{\mathbf{x}}_k + \mathbf{z}_k \end{aligned}$$

where the approximation on line 2 is based on the column-wise diagonal dominance of $\mathbf{H}_k$.

So application of the ZF pre-compensator causes all of the users to see the channel of the worst user in the system. If the users have different line lengths this leads to very poor performance.

B. Linear Diagonalizing Pre-compensator

Again we apply a linear pre-filter to pre-compensate for the effects of crosstalk

$$\mathbf{x}_k = \mathbf{P}_k \overline{\mathbf{x}}_k$$

however now

$$\mathbf{P}_k \triangleq \frac{1}{\beta_{\text{diag}}} \mathbf{H}_k^{-T} \text{diag}\left\{ \mathbf{H}_k \right\}$$

and

$$\beta_{\text{diag}} \triangleq \max_n \left\| \left[\mathbf{H}_k^{-T} \text{diag}\left\{ \mathbf{H}_k \right\} \right]_{\text{row } n} \right\| \tag{7}$$

The effect of this pre-filter is to diagonalize the channel.

$$\begin{aligned} \mathbf{y}_k &= \frac{1}{\beta_{\text{diag}}} \text{diag}\left\{ \mathbf{H}_k \right\} \overline{\mathbf{x}}_k + \mathbf{z}_k \\ &\simeq \text{diag}\left\{ \mathbf{H}_k \right\} \overline{\mathbf{x}}_k + \mathbf{z}_k \end{aligned} \tag{8}$$

The approximation on line 2 is based on the column-wise diagonal dominance of $\mathbf{H}_k$ which causes $\beta_{\text{diag}} \simeq 1$.

So application of the diagonalizing pre-compensator causes the users to see their own direct channels crosstalk perfectly removed. Column-wise diagonal dominance of $\mathbf{H}_k$ means that there is little to gain by exploiting the crosstalk channels for transmission. That is, the attenuations of the crosstalk channels are huge compared to those of the direct channels. The result is that perfect crosstalk cancellation is effectively the best that MIMO transmission can achieve. This in turn implies the near-optimality of the linear diagonalizing pre-compensator.

C. Performance Bounds

Theorem 4: Regardless of transmitter structure the rate of user n on tone k is upper bounded by

$$c_{k,\text{opt}}^n \leq \log_2 \left(1 + \left[1 + (N-1) \tan^2 \alpha \right] S_k \left| h_k^{n,n} \right|^2 \sigma_k^{-2} \Gamma^{-1} \right) \tag{9}$$

Proof: See Appendix III. ■

Theorem 5: With the linear diagonalizing crosstalk pre-compensator the rate of user n on tone k is lower bounded by

$$c_{k,\text{diag}}^n \geq \log_2 \left(1 + \frac{\left(1 - 4(N-1)\alpha^2 \right)^2}{1 + (N-1) \tan^2 \alpha} S_k \left| h_k^{n,n} \right|^2 \sigma_k^{-2} \Gamma^{-1} \right) \tag{10}$$

Proof: See Appendix IV. ■

V. Bounds on Capacity Loss with Linear Schemes

Corollary 1: Using the linear cancellation techniques described in Sec. III capacity loss can be tightly bounded by

$$c^n_{k,\text{opt}} - c^n_{k,\text{linear}} \leq \log_2\left(\frac{1+\left[1+(N-1)\tan^2\alpha\right]S_k\left|h^{n,n}_k\right|^2\sigma_k^{-2}\Gamma^{-1}}{1+\left[1-4(N-1)\alpha^2\right]S_k\left|h^{n,n}_k\right|^2\sigma_k^{-2}\Gamma^{-1}}\right) \tag{11}$$

Proof: Follows directly from (5), (6). ■

Corollary 2: Using the linear cancellation techniques described, capacity loss can be loosely bounded by

$$c^n_{k,\text{opt}} - c^n_{k,\text{linear}} \leq \log_2\left(\frac{1+(N-1)\tan^2\alpha}{1-4(N-1)\alpha^2}\right) \tag{12}$$

Proof: Follows from (13) and the observation that when $a \geq b$, $\frac{a}{b} \geq \frac{1+a}{1+b}$. ■

Corollary 3: Using the linear pre-compensation techniques described in Sec. IV capacity loss can be tightly bounded by

$$c^n_{k,\text{opt}} - c^n_{k,\text{linear}} \leq \log_2\left(\frac{1+(N-1)\tan^2\alpha+\left[1+(N-1)\tan^2\alpha\right]^2 S_k\left|h^{n,n}_k\right|^2\sigma_k^{-2}\Gamma^{-1}}{1+(N-1)\tan^2\alpha+\left(1-4(N-1)\alpha^2\right)^2 S_k\left|h^{n,n}_k\right|^2\sigma_k^{-2}\Gamma^{-1}}\right) \tag{13}$$

Proof: Follows directly from (9) and (10). ■

Corollary 4: Using the linear pre-compensation techniques described, capacity loss can be loosely bounded by

$$c^n_{k,\text{opt}} - c^n_{k,\text{linear}} \leq 2\log_2\left(\frac{1+(N-1)\tan^2\alpha}{1-4(N-1)\alpha^2}\right) \tag{14}$$

Proof: Follows from (13) and the observation that when $a \geq b$, $\frac{a}{b} \geq \frac{1+a}{1+b}$. ■

VI. Simulations

A. Scenario

1 x 300 m and 1 x 1200 m lines using ETSI semi-empirical models for direct and crosstalk channel transfer functions[2]. LTs co-located.

B. Upstream

C. Downstream

VII. Conclusions

Appendix I
Proof of Theorem 2

Since conditioning never decreases mutual information

$$\begin{aligned} c^n_{k,\text{opt}} &= I(x^n_k;\,\mathbf{y}_k) \\ &\leq I(x^n_k;\,\mathbf{y}_k \,|\, x^1_k,\ldots,x^{n-1}_k,\,x^{n+1}_k,\ldots,x^N_k) \end{aligned}$$

where I(x;y) denotes the mutual information between x and y. The term on the second line is simply the capacity of user n when the TXs (NTs) of all other users are disabled. This is essentially the matched filter bound (MFB) of user n since it is the capacity that user n achieves when all other user's are disabled and a maximum-ratio combiner (MRC) is applied to the received signals at the CO.

With all other user's TXs disabled

$$\mathbf{y}_k = \mathbf{h}^n_k\mathbf{x}_k + \mathbf{z}_k$$

where $\mathbf{h}^n_k \triangleq [\mathbf{H}_k]_{\text{col}\,n}$. Applying a MRC gives the following estimate

$$\begin{aligned} \hat{x}^n_k &= \frac{(\mathbf{h}^n_k)^H}{\|\mathbf{h}^n_k\|^2}\mathbf{y}_k \\ &= x^n_k + \frac{(\mathbf{h}^n_k)^H}{\|\mathbf{h}^n_k\|^2}\mathbf{z}_k \end{aligned}$$

The resulting SNR is

$$\begin{aligned} SNR^n_{k,\text{opt}} &= S_k\sigma_k^{-2}\|\mathbf{h}^n_k\|^2 \\ &\leq S_k\sigma_k^{-2}\left|h^{n,n}_k\right|^2\left[1+(N-1)\tan^2\alpha\right] \end{aligned}$$

where we use (2) to define the bound. The capacity of user n with the optimal RX structure is

$$c^n_{k,\text{opt}} = \log_2\left(1+SNR^n_{k,\text{opt}}\Gamma^{-1}\right)$$

where Γ is the SNR-gap to capacity[4].This leads to (5).

APPENDIX II
PROOF OF THEOREM 3

We will use the following Lemma in our proof.

Lemma 1: Define the permutation matrix $\Pi = [\mathbf{I}_N]_{\text{rows } 1,\ldots,n-1,\, n+1,\ldots,N,\, n}$. If the channel matrix $\mathbf{H}_k$ satisfies (2) then

$$\overline{\mathbf{H}}_k \triangleq \Pi \mathbf{H}_k \Pi^T$$

also satisfies (2).

Proof: Follows by inspection.

Define

$$\overline{\mathbf{H}}_k \triangleq \Pi \mathbf{H}_k \Pi^T \tag{15}$$

and the corresponding QR decomposition

$$\overline{\mathbf{H}}_k \stackrel{qr}{=} \overline{\mathbf{Q}}_k \overline{\mathbf{R}}_k \tag{16}$$

Using Theorem 1 and Lemma 1

$$|\overline{r}_k^{n,n}| \leq \left|\overline{h}_k^{n,n}\right| \sqrt{1 - 4(N-1)\alpha^2} \tag{17}$$

Now

$$\begin{aligned} \hat{\mathbf{x}}_k &= \mathbf{H}_k^{-1}\mathbf{y}_k \\ &= \mathbf{x}_k + \mathbf{H}_k^{-1}\mathbf{z}_k \end{aligned}$$

and

$$\hat{x}_k^n = x_k^n + \left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \mathbf{z}_k$$

Hence

$$\begin{aligned} SNR_{k,\text{zf}}^n &= S_k \sigma_k^{-2} \left\| \left[\mathbf{H}_k^{-1}\right]_{\text{row } n} \right\|^{-2} \\ &= S_k \sigma_k^{-2} \left\| \left[\Pi \overline{\mathbf{H}}_k^{-1} \Pi^T\right]_{\text{row } n} \right\|^{-2} \\ &= S_k \sigma_k^{-2} \left\| \left[\overline{\mathbf{H}}_k^{-1} \Pi^T\right]_{\text{row } N} \right\|^{-2} \\ &= S_k \sigma_k^{-2} \left\| \left[\overline{\mathbf{H}}_k^{-1}\right]_{\text{row } N} \right\|^{-2} \end{aligned} \tag{18}$$

From (16) $\overline{\mathbf{H}}_k^{-1} = \overline{\mathbf{R}}_k^{-1} \overline{\mathbf{Q}}_k^H$. Hence

$$\begin{aligned} \left[\overline{\mathbf{H}}_k^{-1}\right]_{\text{row } N} &= \left[\overline{\mathbf{R}}_k^{-1}\right]_{\text{row } N} \overline{\mathbf{Q}}_k^H \\ &= \frac{1}{\overline{r}_k^{n,n}} \left[\overline{\mathbf{Q}}_k\right]_{\text{col } N}^H \end{aligned}$$

where $\overline{h}_k^{n,m} \triangleq \left[\overline{\mathbf{H}}_k\right]_{n,m}$ and $\overline{r}_k^{n,m} \triangleq \left[\overline{\mathbf{R}}_k\right]_{n,m}$. This implies

$$\begin{aligned} \left\| \left[\overline{\mathbf{H}}_k^{-1}\right]_{\text{row } N} \right\|^2 &= |\overline{r}_k^{n,n}|^{-2} \\ &\leq \left|\overline{h}_k^{n,n}\right|^{-2} \left(1 - 4(N-1)\alpha^2\right) \\ &= \left|h_k^{N,N}\right|^{-2} \left(1 - 4(N-1)\alpha^2\right) \end{aligned}$$

where we use (17) and (15) in the second and third line respectively. Combining this with (18) yields the following SNR bound

$$SNR_{k,\text{zf}}^n \geq S_k \sigma_k^{-2} \left|h_k^{n,n}\right|^2 \left(1 - 4(N-1)\alpha^2\right)$$

The capacity of user n is with the ZF canceller is

$$c_{k,\text{zf}}^n = \log_2\left(1 + SNR_{k,\text{zf}}^n \Gamma^{-1}\right)$$

which leads to (6). ■

APPENDIX III
PROOF OF THEOREM 4

We write the following obvious bound

$$\begin{aligned} c_{k,\text{opt}}^n &= I(x_k^n;\, y_k^n) \\ &\leq I(\mathbf{x}_k;\, y_k^n) \end{aligned}$$

The term on the second line is simply the capacity of user n when all TXs (LTs) are used to communicate to the RX (NT) of user n. This is essentially the matched filter bound (MFB) of user n since it is the capacity that user n achieves when we apply a matched filter prior to transmission at the CO.

From (1)

$$\begin{aligned} y_k^n &= \left[\mathbf{H}_k^T\right]_{\text{row } n} \mathbf{x}_k + \mathbf{z}_k \\ &= \left(\mathbf{h}_k^n\right)^T \mathbf{x}_k + \mathbf{z}_k \end{aligned}$$

The resulting SNR is

$$\begin{aligned} SNR_{k,\text{opt}}^n &= S_k \sigma_k^{-2} \left\|\mathbf{h}_k^n\right\|^2 \\ &\leq S_k \sigma_k^{-2} \left|h_k^{n,n}\right|^2 \left[1 + (N-1)\tan^2\alpha\right] \end{aligned}$$

where we use (2) to define the bound. The capacity of user n is

$$c_{k,\text{opt}}^n = \log_2\left(1 + SNR_{k,\text{opt}}^n \Gamma^{-1}\right)$$

which leads to (9).

APPENDIX IV
PROOF OF THEOREM 5

Lemma 2: Define

$$f(\alpha) \triangleq \sqrt{\frac{4(N-1)\alpha^2 + (N-1)\tan^2\alpha}{1 - 4(N-1)\alpha^2}}$$

then

$$\frac{|r_k^{n,m}|}{|r_k^{m,m}|} \leq f(\alpha),\ \forall n \neq m \tag{19}$$

Proof: First observe that

$$\left\|[\mathbf{R}_k]_{\text{col } m}\right\| = \left\|[\mathbf{H}_k]_{\text{col } m}\right\|,\ \forall m$$

This implies

$$\begin{aligned} |r_k^{n,m}|^2 &= \left\|[\mathbf{H}]_{col\, m}\right\|_2^2 - |r_k^{m,m}|^2 - \sum_{i \neq n,m} \left|r_k^{i,m}\right|^2 \\ &\leq |h_k^{m,m}|^2 + \sum_{i \neq m} \left|h_k^{i,m}\right|^2 - |r_k^{m,m}|^2 \\ &\leq |h_k^{m,m}|^2 \left[1 + (N-1)\tan^2\alpha\right] - |r_k^{m,m}|^2 \\ &\leq |r_k^{m,m}|^2 \left[\frac{1 + (N-1)\tan^2\alpha}{1 - 4(N-1)\alpha^2} - 1\right] \\ &= |r_k^{m,m}|^2 \left[\frac{4(N-1)\alpha^2 + (N-1)\tan^2\alpha}{1 - 4(N-1)\alpha^2}\right],\ \forall n \neq m \end{aligned}$$

■

Examining (8) we see that the SNR with the diagonalizing pre-compensator is

$$SNR_{k,\text{diag}}^n = \frac{1}{\beta_{\text{diag}}^2} S_k \sigma_k^{-2} \left|h_k^{n,n}\right|^2 \tag{20}$$

From (3) $\mathbf{H}_k^T = \mathbf{R}_k^T \mathbf{Q}_k^T$ hence

$$\mathbf{H}_k^{-T} = \left(\mathbf{Q}_k^T\right)^H \mathbf{R}_k^{-T}$$

Hence

$$\begin{aligned}\left[\mathbf{H}_k^{-T}\mathrm{diag}\left\{\mathbf{H}_k\right\}\right]_{\mathrm{row}\ n} &= \left[\left(\mathbf{Q}_k^T\right)^H\right]_{\mathrm{row}\ n} \mathbf{R}_k^{-T}\mathrm{diag}\left\{\mathbf{H}_k\right\} \\ &= \left[\left(\mathbf{Q}_k^T\right)^H\right]_{\mathrm{row}\ n} \mathbf{B} \\ &= \sum_m \left(q_k^{nm}\right)^* \mathbf{b}_n\end{aligned}$$

where $\mathbf{B} \triangleq \mathbf{R}_k^{-T}\mathrm{diag}\left\{\mathbf{H}_k\right\}$, $\mathbf{b}_n \triangleq \left[\mathbf{B}\right]_{\mathrm{row}\ n}$ and $q_k^{nm} \triangleq \left[\mathbf{Q}_k\right]_{n,m}$. This implies that

$$\begin{aligned}\left\|\left[\mathbf{H}_k^{-T}\mathrm{diag}\left\{\mathbf{H}_k\right\}\right]_{\mathrm{row}\ n}\right\|^2 &\leq \sum_m \left|q_k^{nm}\right|^2 \left\|\mathbf{b}_n\right\|^2 \\ &\leq \left(\max_i \left\|\mathbf{b}_i\right\|^2\right)\left(\sum_m \left|q_k^{nm}\right|^2\right) \\ &= \max_i \left\|\mathbf{b}_i\right\|^2 \end{aligned} \quad (21)$$

Define $\mathbf{G}_k \triangleq \mathbf{R}_k^{-1}$ and $g_k^{nm} \triangleq \left[\mathbf{G}_k\right]_{n,m}$. Now

$$\mathbf{b}_i = \left[\begin{array}{ccc} g_k^{1n}h_k^{11} & \cdots & g_k^{Nn}h_k^{NN} \end{array}\right]$$

$$\begin{aligned}\left\|\mathbf{b}_i\right\|^2 &= \frac{\left|h_k^{nn}\right|^2}{\left|r_k^{nn}\right|^2} + \sum_{m\neq n}\left|g_k^{mn}\right|^2\left|h_k^{mm}\right|^2 \\ &\leq \frac{1}{1-4(N-1)\alpha^2} + \sum_{m\neq n}\left|g_k^{mn}\right|^2\left|h_k^{mm}\right|^2\end{aligned}$$

Since $\mathbf{R}_k$ is upper-triangular

$$g_k^{n,m} = \left\{\begin{array}{ll} 0 & m<n \\ \frac{1}{r_k^{m,m}} & m=n \\ -\frac{1}{r_k^{m,m}}\sum_{i=n}^{m-1} r_k^{i,m} g_k^{n,i} & m>n \end{array}\right.$$

Hence, using Lemma 2

$$\begin{aligned}\left|g_k^{n,m}\right| &\leq \sum_{i=n}^{m-1}\frac{\left|r_k^{i,m}\right|}{\left|r_k^{m,m}\right|}\left|g_k^{n,i}\right|, \forall m>n \\ &\leq f(\alpha)\sum_{i=n}^{m-1}\left|g_k^{n,i}\right|, \forall m>n\end{aligned}$$

For example

$$\begin{aligned}\left|g_k^{n,n+1}\right| &\leq \frac{1}{\left|r_k^{n,n}\right|}f(\alpha) \\ \left|g_k^{n,n+2}\right| &\leq \frac{1}{\left|r_k^{n,n}\right|}\left(f(\alpha)+f(\alpha)^2\right)\end{aligned}$$

In general

$$\left|g_k^{n,m}\right| \leq \frac{1}{\left|r_k^{n,n}\right|}\sum_{l=1}^{m-n}\binom{m-n-1}{l-1}f(\alpha)^l$$

Hence

$$\left|g_k^{nm}\right|^2 \leq \frac{1}{\left|r_k^{mm}\right|^2}\left(\sum_{l=1}^{m-n}\binom{m-n-1}{l-1}\left(\frac{(N-1)\left(4\alpha^2+\tan^2\alpha\right)}{1-4(N-1)\alpha^2}\right)^{l/2}\right)^2 \quad (22)$$

To simplify this expression we ignore terms of order $\mathcal{O}(\alpha^3)$ or $\mathcal{O}(\tan^3\alpha)$ and higher. Thus (22) can be approximated

$$|g_k^{nm}|^2 \leq \frac{1}{|r_k^{mm}|^2} \frac{(N-1)\left(4\alpha^2+\tan^2\alpha\right)}{1-4(N-1)\alpha^2}$$

Hence

$$\begin{aligned} |g_k^{mn}|^2 |h_k^{mm}|^2 &\leq \frac{|h_k^{mm}|^2}{|r_k^{mm}|^2} \frac{(N-1)\left(4\alpha^2+\tan^2\alpha\right)}{1-4(N-1)\alpha^2} \\ &\leq \frac{(N-1)\left(4\alpha^2+\tan^2\alpha\right)}{\left(1-4(N-1)\alpha^2\right)^2} \end{aligned}$$

Hence

$$\|\mathbf{b}_i\|^2 \leq \frac{1}{1-4(N-1)\alpha^2}\left(1+\frac{(N-1)\left(4\alpha^2+\tan^2\alpha\right)}{1-4(N-1)\alpha^2}\right)$$

With (7) and (21) this implies

$$\frac{1}{\beta_{\text{diag}}^2} \geq \frac{\left(1-4(N-1)\alpha^2\right)^2}{1+(N-1)\tan^2\alpha}$$

Examining (20) we see that the SNR can now be bounded

$$SNR_{k,\text{diag}}^n \geq \frac{\left(1-4(N-1)\alpha^2\right)^2}{1+(N-1)\tan^2\alpha} S_k \sigma_k^{-2} |h_k^{n,n}|^2$$

The capacity of user n is

$$c_{k,\text{diag}}^n = \log_2\left(1+SNR_{k,\text{diag}}^n \Gamma^{-1}\right)$$

which leads to (10).

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TABLE I

SIMULATION PARAMETERS

Number of DMT Tones	4096
Tone Width	4.3125 kHz
Symbol Rate	4 kHz
Coding Gain	3 dB
Noise Margin	6 dB
Symbol Error Probability	$< 10^{-7}$
Transmit PSD	Flat -60 dBm/Hz
FDD Band Plan	998
Cable Type	0.5 mm (24-Gauge)
Source/Load Resistance	135 Ohm
Noise Model	ETSI Type A

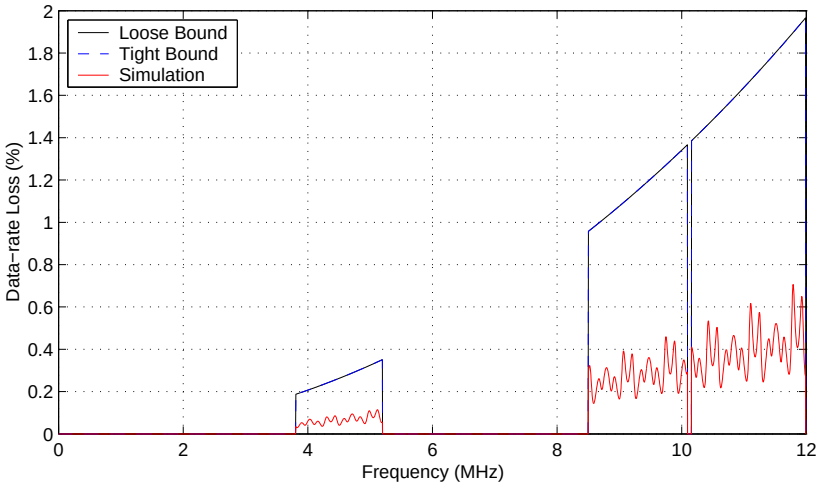


Fig. 1. US Data Rate Loss vs. Frequency (300 m)

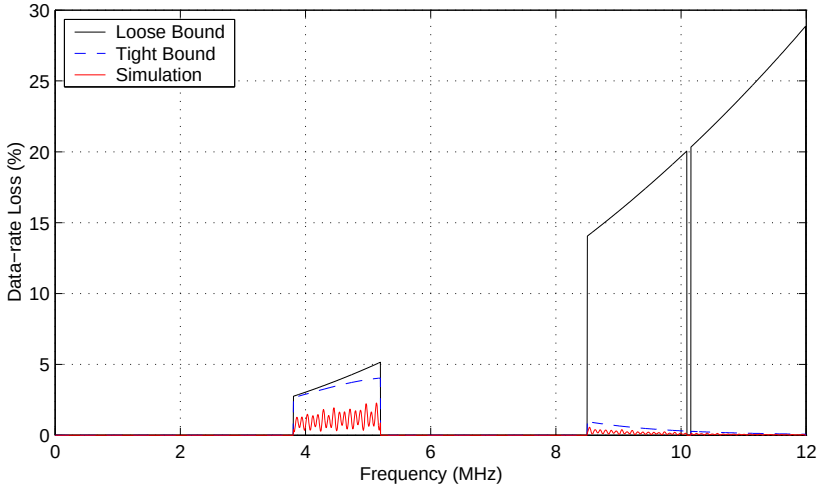


Fig. 2. US Data Rate Loss vs. Frequency (1200 m)

Fig. 3. US Data Rate Loss vs. Line Length

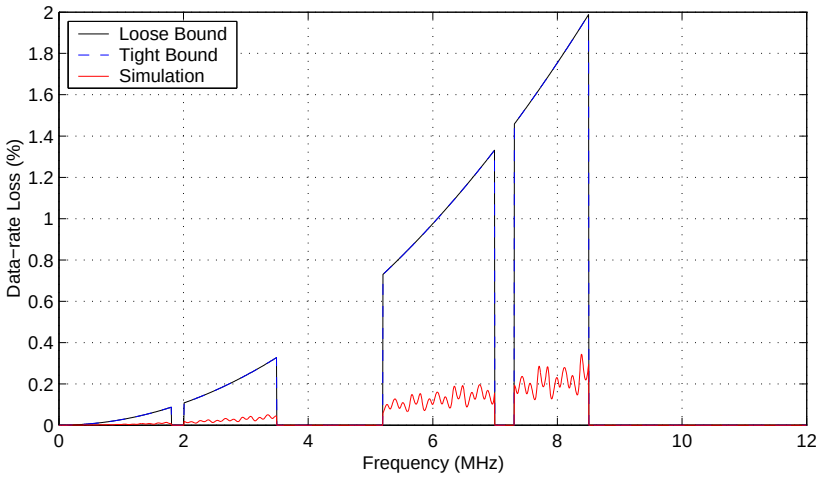


Fig. 4. DS Data Rate Loss vs. Frequency (300 m)

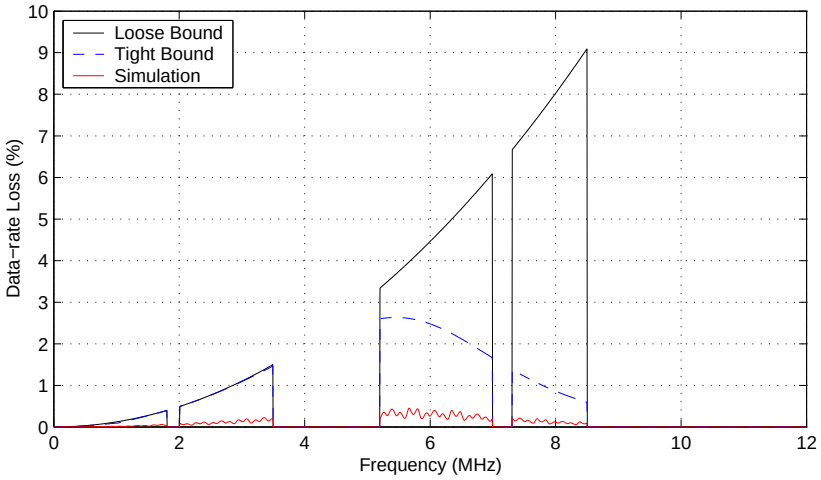


Fig. 5. DS Data Rate Loss vs. Frequency (1200 m)

Fig. 6. DS Data Rate Loss vs. Line Length