Canonical Polyadic Decomposition of Third-Order Tensors of Which the Dimensions are Strictly Smaller than the Rank*

Lieven De Lathauwer

December 1, 2011

1 Case $I_1 \leqslant I_2 \leqslant I_3 = R - 1$

We consider the Canonical Polyadic Decomposition (CPD) of a rank-R tensor $\mathcal{T} \in \mathbb{K}^{I_1 \times I_2 \times I_3}$, with $I_1 \leqslant I_2 \leqslant I_3$ and $R = I_3 + 1$:

$$\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r. \tag{1}$$

Contributions:

- A new, relaxed, algebraic condition on $\mathbf{a}_r$, $\mathbf{b}_r$, $\mathbf{c}_r$, $1 \leqslant r \leqslant R$ that is sufficient for essential uniqueness of the CPD.
- Under this condition the CP factors can be obtained by means of standard linear algebra.

*Available in SISTA publication engine as: L. De Lathauwer, *CPD Study, Part IV*, Tech. Report 11-222, ESAT-SISTA, K.U.Leuven (Leuven, Belgium), 2011.

- Algorithm for the estimation of the CP factors in the case of an approximate CPD.
- Generic uniqueness for a given set (I_1, I_2, I_3, R) is established by checking the algebraic condition for one random sample.
- Similar approach in the partially symmetric case ($\mathbf{a}_r = \mathbf{b}_r$, $1 \leqslant r \leqslant R$) and in the symmetric case ($\mathbf{a}_r = \mathbf{b}_r = \mathbf{c}_r$, $1 \leqslant r \leqslant R$). (Generic bound will be different.)
- Verification for (4×4×4) tensors of rank 5 (symmetric and unsymmetric case).

In matrix format we have:

$$\mathbf{T}_{[1,2;3]} = (\mathbf{A} \odot \mathbf{B}) \cdot \mathbf{C}^T. \tag{2}$$

A necessary condition for uniqueness of the CPD is that $\mathbf{A} \odot \mathbf{B}$ has full column rank. For now we additionally assume that $k(\mathbf{C}) = I_3$. Denote by $\underline{\mathbf{C}}^T$ the $(I_3 \times I_3)$-matrix obtained from $\mathbf{C}^T$ by deleting the bottom row. Further define $\tilde{\mathbf{c}} = \underline{\mathbf{C}}^{-1}\mathbf{c}_R$ and

$$\mathbf{H} = \left[\mathbf{a}_1 \otimes \mathbf{b}_1 + \tilde{c}_{1,R}\mathbf{a}_R \otimes \mathbf{b}_R \quad \cdots \quad \mathbf{a}_{I_3} \otimes \mathbf{b}_{I_3} + \tilde{c}_{I_3,R}\mathbf{a}_R \otimes \mathbf{b}_R\right]. \tag{3}$$

We have:

$$\mathbf{T}_{[1,2;3]} = \mathbf{H} \cdot \underline{\mathbf{C}}^T. \tag{4}$$

Let $\mathbf{f}_1$, $\mathbf{f}_2$, …, $\mathbf{f}_{I_3}$ denote a basis for the column space of $\mathbf{T}_{[1,2;3]}$. We have:

$$\begin{aligned} \mathbf{T}_{[1,2;3]} &= \mathbf{F} \cdot \mathbf{G}^T, & (5)\\ \mathbf{F} &= [\mathbf{f}_1 \cdots \mathbf{f}_{I_3}] = \mathbf{H} \cdot \mathbf{M}^T, & (6)\\ \mathbf{G}^T &= \mathbf{M}^{-T} \cdot \underline{\mathbf{C}}^T, & (7) \end{aligned}$$

in which $\mathbf{M} \in \mathbb{K}^{I_3 \times I_3}$ is nonsingular.

For the computation of the CPD we will use the fact that the columns of $\mathbf{H}$ represent rank-2 matrices. Let Φ denote the rank-2 detecting mapping, introduced in [1]. Further, $\mathbf{F}_{i_3} = (\mathbf{f}_{i_3})_{[1;2]} \in \mathbb{K}^{I_1 \times I_2}$, $1 \leqslant i_3 \leqslant I_3$. We have:

$$\begin{aligned} \Phi(\mathbf{F}_{k_1}, \mathbf{F}_{k_2}, \mathbf{F}_{k_3}) &= \sum_{l_1,l_2,l_3=1}^{I_3} m_{k_1 l_1} m_{k_2 l_2} m_{k_3 l_3} \Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \\ &\quad \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_3}\mathbf{b}_{l_3}^T + \tilde{c}_{l_3,R}\mathbf{a}_R\mathbf{b}_R^T). \end{aligned} \tag{8}$$

We have the following properties:

- $\Phi(\cdot,\cdot,\cdot)$ is symmetric in its arguments.
- $$\Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T) = \mathcal{O}, \quad 1 \leqslant l_1 \leqslant I_3.$$
- $\Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T) = 2\tilde{c}_{l_1,R}\Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T, \mathbf{a}_R\mathbf{b}_R^T)$
  and
  $\Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T) = 2\tilde{c}_{l_2,R}\Phi(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T, \mathbf{a}_R\mathbf{b}_R^T)$, $1 \leqslant l_1 < l_2 \leqslant I_3$.

We look for symmetric tensors $\mathcal{X} \in \mathbb{K}^{I_3 \times I_3 \times I_3}$ that satisfy

$$\sum_{k_1,k_2,k_3=1}^{I_3} x_{k_1,k_2,k_3}\Phi(\mathbf{F}_{k_1}, \mathbf{F}_{k_2}, \mathbf{F}_{k_3}) = \mathcal{O}. \tag{9}$$

We stack $\Phi(\mathbf{F}_{k_1}, \mathbf{F}_{k_2}, \mathbf{F}_{k_3})$, $1 \leqslant k_1 \leqslant k_2 \leqslant k_3 \leqslant I_3$, in a matrix $\bar{\mathbf{F}} \in \mathbb{K}^{I_1^3 I_2^3 \times C_{I_3+2}^3}$. We determine the vectors $\mathbf{x}$ in the null space of $\bar{\mathbf{F}}$, which can then be mapped to tensors $\mathcal{X}$. We make the crucial assumption that the set of tensors $\{\Phi(\mathbf{a}_{r_1}\mathbf{b}_{r_1}^T, \mathbf{a}_{r_2}\mathbf{b}_{r_2}^T, \mathbf{a}_{r_3}\mathbf{b}_{r_3}^T) | 1 \leqslant r_1 < r_2 < r_3 \leqslant R\}$ is linearly independent. Under this assumption, the null space of $\bar{\mathbf{F}}$ has dimension C_R^2. Denote $\mathbf{W} = \mathbf{M}^{-T}$. Combination of (8) and (9) shows that the vectors in the null space represent tensors that are linear combinations of:

$$\mathbf{w}_k \otimes \mathbf{w}_k \otimes \mathbf{w}_k, \quad 1 \leqslant k \leqslant I_3, \tag{10}$$

$$\tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2}, \quad 1 \leqslant k_1 < k_2 \leqslant I_3, \tag{11}$$

in which $\tilde{\mathbf{w}}_{k_1,k_2} = -\tilde{c}_{k_2,R}\mathbf{w}_{k_1} + \tilde{c}_{k_1,R}\mathbf{w}_{k_2}$.

If we stack the tensors obtained from a basis of the null space of $\bar{\mathbf{F}}$ in a super-tensor $\mathcal{Y} \in \mathbb{K}^{I_3 \times I_3 \times I_3 \times C_R^2}$, then this super-tensor admits the CPD

$$\mathcal{Y} = \sum_{k=1}^{I_3} \mathbf{w}_k \otimes \mathbf{w}_k \otimes \mathbf{w}_k \otimes \mathbf{v}_k + \sum_{k_1=1}^{I_3} \sum_{k_2=k_1+1}^{I_3} \tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2} \otimes \mathbf{v}_{k_1,k_2}. \tag{12}$$

The vectors $\mathbf{v}_k$, $1 \leqslant k \leqslant I_3$, and $\mathbf{v}_{k_1,k_2}$, $1 \leqslant k_1 < k_2 \leqslant I_3$, represent the coefficients of the tensors (10) and (11), respectively, in the linear combinations that yield the basis vectors of the null space of $\bar{\mathbf{F}}$. They can be stacked in a matrix $\mathbf{V} \in \mathbb{K}^{C_R^2 \times C_R^2}$. This matrix has full rank, since the basis vectors are linearly independent. We make the assumption that CPD (12) is unique. A sufficient condition for this is that (i) the set of vectors $\{\mathbf{w}_k \otimes \mathbf{w}_k | 1 \leqslant k \leqslant I_3\} \cup \{\tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2} | 1 \leqslant k_1 < k_2 \leqslant I_3\}$ is linearly independent and that (ii) the set $\{\mathbf{w}_k | 1 \leqslant k \leqslant I_3\} \cup \{\tilde{\mathbf{w}}_{k_1,k_2} | 1 \leqslant k_1 < k_2 \leqslant I_3\}$ does not contain proportional vectors. Indeed, knowing that $\mathbf{V}$ has full rank, the condition implies that the CPD

$$\mathcal{Y}_{[1;2,3;4]} = \sum_{k=1}^{I_3} \mathbf{w}_k \otimes (\mathbf{w}_k \otimes \mathbf{w}_k) \otimes \mathbf{v}_k + \sum_{k_1=1}^{I_3} \sum_{k_2=k_1+1}^{I_3} \tilde{\mathbf{w}}_{k_1,k_2} \otimes (\tilde{\mathbf{w}}_{k_1,k_2} \otimes \tilde{\mathbf{w}}_{k_1,k_2}) \otimes \mathbf{v}_{k_1,k_2} \tag{13}$$

is unique, see for instance [2, 3]. In the case of an exact decomposition, (13) follows from a matrix EVD. (Exploiting the symmetry of $\mathbf{w}_k \cdot \mathbf{w}_k^T$ and $\tilde{\mathbf{w}}_{k_1,k_2} \cdot \tilde{\mathbf{w}}_{k_1,k_2}^T$, one may work with an $(I_3 \times \frac{I_3(I_3+1)}{2} \times C_R^2)$-dimensional version of $\mathcal{Y}_{[1;2,3;4]}$.) In the case of an approximate decomposition, the EVD may be used to initialize an algorithm for fitting (12).

In the case of an exact decomposition, one can, in the set $\mathsf{W} = \{\mathbf{w}_k | 1 \leqslant k \leqslant I_3\} \cup \{\tilde{\mathbf{w}}_{k_1,k_2} | 1 \leqslant k_1 < k_2 \leqslant I_3\}$, find C_R^3 subsets of three vectors that are linearly dependent. These subsets can be characterized as follows. Choose $R-3$ different columns of $\mathbf{C}$, say $\mathbf{c}_1$, $\mathbf{c}_2$, $\ldots$, $\mathbf{c}_{R-3}$. The vectors of the corresponding subset are all three orthogonal to $\mathbf{c}_1$, $\mathbf{c}_2$, $\ldots$, $\mathbf{c}_{R-3}$. The first vector is additionally orthogonal to $\mathbf{c}_{R-2}$, the second vector is instead orthogonal to $\mathbf{c}_{R-1}$, and the third vector is orthogonal to $\mathbf{c}_R$. The different subsets correspond to different choices for the columns of $\mathbf{C}$ to which the three vectors in the set are orthogonal.

Let $\{k_1, k_2, \ldots, k_{R-3}\} \cup \{l_{R-2}, l_{R-1}, l_R\} = \{1, 2, \ldots, R\}$. Let $\mathbf{W}_{l_{R-2},l_{R-1},l_R} \in \mathbb{K}^{I_3 \times 3}$ be the rank-2 matrix containing the vectors of W that are orthogonal to $\mathbf{c}_{k_r}$, $1 \leqslant r \leqslant R-3$. Define

$$\mathbf{E}_{l_{R-2},l_{R-1},l_R} = \mathbf{F} \cdot \mathbf{W}_{l_{R-2},l_{R-1},l_R}. \tag{14}$$

The entries of $\mathbf{E}_{l_{R-2},l_{R-1},l_R} \in \mathbb{K}^{I_1 I_2 \times 3}$ can be stacked in a tensor $\mathcal{E}_{l_{R-2},l_{R-1},l_R} \in \mathbb{K}^{I_1 \times I_2 \times 3}$ of which the CPD is given by:

$$\mathcal{E}_{l_{R-2},l_{R-1},l_R} = \mathbf{a}_{l_{R-2}} \otimes \mathbf{b}_{l_{R-2}} \otimes \bar{\mathbf{c}}_1 + \mathbf{a}_{l_{R-1}} \otimes \mathbf{b}_{l_{R-1}} \otimes \bar{\mathbf{c}}_2 + \mathbf{a}_{l_R} \otimes \mathbf{b}_{l_R} \otimes \bar{\mathbf{c}}_3, \tag{15}$$

where $\bar{\mathbf{C}} = [\bar{\mathbf{c}}_1\ \bar{\mathbf{c}}_2\ \bar{\mathbf{c}}_3]$ has rank 2 since $\mathbf{W}_{l_{R-2},l_{R-1},l_R}$ has rank 2. The vectors $\mathbf{a}_{l_{R-2}}$, $\mathbf{b}_{l_{R-2}}$, $\mathbf{a}_{l_{R-1}}$, $\mathbf{b}_{l_{R-1}}$, $\mathbf{a}_{l_R}$, $\mathbf{b}_{l_R}$ can be found from this decomposition if $[\mathbf{a}_{l_{R-2}}\ \mathbf{a}_{l_{R-1}}\ \mathbf{a}_{l_R}]$ and $[\mathbf{b}_{l_{R-2}}\ \mathbf{b}_{l_{R-1}}\ \mathbf{b}_{l_R}]$ have full column rank.

In the case of an exact decomposition, the full matrices $\mathbf{A}$ and $\mathbf{B}$ can be found by starting from $\lceil \frac{R}{3} \rceil$ subsets of W that correspond to matrices $\mathbf{W}_{l_{R-2},l_{R-1},l_R}$ of which the column spaces minimally intersect. The matrix $\mathbf{C}$ can then be found from (2).

In the case of an approximate decomposition, one can in principle start from the C_R^3 subsets of W that are maximally linearly dependent, to obtain $3C_R^3$ vectors that are estimates of the columns of $\mathbf{A}$ and $\mathbf{B}$. These can then be clustered in R groups of C_{R-1}^2 vectors, and the cluster centers can be used to initialize an algorithm for fitting (1).

Remark 1. Instead of working with $\bar{\mathbf{F}} \in \mathbb{K}^{I_1^3 I_2^3 \times C^3_{I_3+2}}$, one may work with a reduced version $\tilde{\mathbf{F}} \in \mathbb{K}^{C^3_{I_1} C^3_{I_2} \times C^3_{I_3+2}}$, as explained in [1]. Instead of estimating the null space of $\bar{\mathbf{F}}$ or $\tilde{\mathbf{F}}$, one may also estimate the null space of $\bar{\mathbf{F}}^H \cdot \bar{\mathbf{F}} \in \mathbb{K}^{C^3_{I_3+2} \times C^3_{I_3+2}}$. The latter matrix can be computed efficiently by computing $(\mathbf{f}_{i_1} \otimes \mathbf{f}_{i_2} \otimes \mathbf{f}_{i_3})^H (\mathbf{f}_{j_1} \otimes \mathbf{f}_{j_2} \otimes \mathbf{f}_{j_3})$ as $(\mathbf{f}_{i_1}^H \mathbf{f}_{j_1})(\mathbf{f}_{i_2}^H \mathbf{f}_{j_2})(\mathbf{f}_{i_3}^H \mathbf{f}_{j_3})$. The disavantage is that the condition number is squared.

2 Case $I_1 \leqslant I_2 \leqslant I_3 = R-K$, $1 < K < \min(I_1, I_2)-1$

We use the rank-$(K+1)$ detecting mapping Φ_{K+1}, introduced in [1].

Consider $K = 2$ by way of example. We have:

$$\begin{aligned} \Phi_3(&\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \\ &\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \\ &\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T, \\ &\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T) = \mathcal{O}, \qquad 1 \leqslant l_1 \leqslant I_3. \end{aligned}$$

The vectors

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T),\end{gathered}$$

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T)\end{gathered}$$

and

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_1}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T)\end{gathered}$$

are proportional to

$$\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T, \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T, \mathbf{a}_{R-1}\mathbf{b}_{R-1}^T, \mathbf{a}_R\mathbf{b}_R^T),$$

$1 \leqslant l_1 < l_2 \leqslant I_3$.

The vectors

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_3}\mathbf{b}_{l_3}^T + \tilde{c}_{l_3,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_3,R}\mathbf{a}_R\mathbf{b}_R^T),\end{gathered}$$

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_3}\mathbf{b}_{l_3}^T + \tilde{c}_{l_3,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_3,R}\mathbf{a}_R\mathbf{b}_R^T),\end{gathered}$$

and

$$\begin{gathered}\Phi_3(\mathbf{a}_{l_1}\mathbf{b}_{l_1}^T + \tilde{c}_{l_1,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_1,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_2}\mathbf{b}_{l_2}^T + \tilde{c}_{l_2,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_2,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_3}\mathbf{b}_{l_3}^T + \tilde{c}_{l_3,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_3,R}\mathbf{a}_R\mathbf{b}_R^T,\\ \mathbf{a}_{l_3}\mathbf{b}_{l_3}^T + \tilde{c}_{l_3,R-1}\mathbf{a}_{R-1}\mathbf{b}_{R-1}^T + \tilde{c}_{l_3,R}\mathbf{a}_R\mathbf{b}_R^T),\end{gathered}$$

are expected to be linearly independent, $1 \leqslant l_1 < l_2 < l_3 \leqslant I_3$.

References

[1] A link between the decomposition of a third-order tensor in rank-(L,L,1) terms and simultaneous matrix diagonalization, Tech. report.

[2] R.A. Harshman, *Foundations of the PARAFAC procedure*: *Model and conditions for an "explanatory" multi-mode factor analysis*, UCLA Working Papers in Phonetics, 16 (1970), pp. 1–84.

[3] S.E. Leurgans, R.T. Ross, and R.B. Abel, *A decomposition for three-way arrays*, SIAM J. Matrix Anal. Appl., 14 (1993), pp. 1064–1083.