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System inversion with application to filtering and smoothing in the presence of unknown inputs[1]

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[1]This report is available by anonymous ftp from *ftp.esat.kuleuven.ac.be* in the directory *pub/sista/gillijns/reports/TR-07-08.pdf*

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Abstract

A new procedure for left inversion of linear discrete-time dynamical systems is introduced. The procedure is based on estimation theory and allows to design full order and reduced order inverses that reconstruct the system input and the system state with arbitrary time delays. It is shown that if a certain matrix pair is observable, a matrix parameter in the design can be determined so that the poles of the inverse system are arbitrary assigned. The unobservable modes of the pair turn out to be zeros of the system, however, not necessarily all zeros are unobservable modes. Consequently, the procedure can also be used for stable inversion of non-minimum phase systems. It is illustrated in a numerical example that increasing the delay of reconstruction above the inherent delay of the system is crucial for stable inversion of non-minimum phase systems.

An extension of the inversion procedure to combined deterministic-stochastic systems is considered. It is shown that the two matrix parameters can now be determined so that the reconstructed input vector and state vector are minimal-variance unbiased. The resulting estimators generalize existing results in optimal filtering and smoothing in the presence of unknown inputs.

1 Introduction

System inversion deals with deriving a system whose input is the output of the original system and vice versa. The problem of inverting linear dynamical system has received a lot of attention in the past due to its strong connection to control and estimation theory. It has applications in such areas as fault detection and isolation [33], geophysical estimation [22], simultaneous stabilization of dynamical systems [6] and adaptive tracking control [30]. In this paper, the inversion problem is reconsidered from the viewpoint of estimation theory.

A brief history of the major develpoments in system inversion is schematically shown in Figure 1. The earliest systematic approach to system inversion is due to Brockett and Mesarovic [5, 4] who considered scalar systems and derived necessary and sufficient conditions for invertibility as well as an inversion algorithm. Important contributions in the inversion of multivariable system were obtained by Sain and Massey [28] and Silverman [29]. The idea behind the so-called *structure algorithm* of Silverman is to incorporate time derivatives of the measurements (or delayed measurements in the discrete-time case) in the output equation in such a way that a transformed system is obtained for which the inversion problem is trivial. The resulting inverse system then consists of a bank of differentiators (or delay elements in the discrete-time case) followed by the inverse of the transformed system. This inverse of the transformed system has been called the *dynamical portion* of the inverse system by Silverman. Silverman obtained the insight that the order of the dynamical portion can be reduced so that an inverse is obtained which can be realized with exactly the same number of differentiators (or delay elements) as the original system. Sain and Massey also considered inverses which consist of a bank of delay elements followed by a dynamical system. In their approach, however, the number of differentiators needed to realize the inverse can be higher than the order of the original system. An important concept introduced by Sain and Massey, is the *inherent delay* of a discrete-time system, which is the minimal delay with which the input of the system can be reconstructed from knowledge of the system outputs. The inversion procedure of Sain and Massey allows to derive inverses with arbitrary delay larger than or equal to the inherent delay. Silverman's structure algorithm, on the other hand, is limited to inverses which reconstruct the system input with delay equal to the inherent delay.

A disadvantage of the inversion procedures described above is that they can yield unstable inverses. In most applications, however, a stable inverse is desired since this introduces robustness against errors in the initial state. Moylan [24] was the first to develop an algorithm that returns a stable inverse if the original system has no unstable zeros. His algorithm is close to that of Silverman, however, for efficiency of calculation the order of the intermediate system is reduced at every step of the iterative algorithm. A very straightforward approach to stable inversion is given by Antsaklis [2]. His treatment is based on feedback control and allows to assign the poles of the inverse system, except those that equal the zeros of the given system. The method is, however, limited to *instantaneous* inverses, that is, inverses that reconstructs the input without delay.

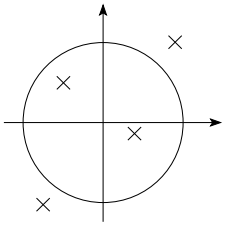
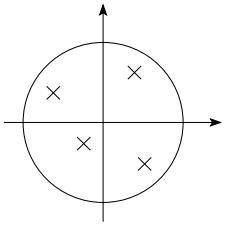
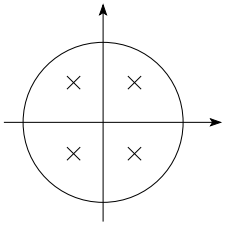
1965		
Brockett & Mesarovic[5]		inversion algorithms and criteria for scalar systems
1968-1969		
Silverman[29] Sain & Massey[28]		inversion algorithms and criteria for MIMO systems
1977		
Moylan[24]		stable inversion
1978		
Antsaklis[2]		pole placement, however limited to instantaneous inverses

Figure 1: *History of system inversion*

In a first contribution, we extend existing results by addressing the problem of pole placement for inverses with arbitrary delay. To this aim, a new and straightforward procedure for left inversion based on estimation theory is introduced. The procedure is most closely related to that of Sain and Massey in the sense that the same bank of differentiators is considered. During the design of the dynamical portion, two matrix parameters can be freely chosen. We prove that if a certain matrix pair is observable, one of these parameters can be determined so that the poles of the dynamical portion are arbitrary assigned. This pair turns out to be observable if the system has no zeros, however, having no zeros is not necessary for observability. Consequently, the method can also be used for stable inversion of certain types of non-minimum phase systems, as will also be illustrated. Based on the theory of reduced order observers, a new procedure to reduce the order of the dynamical portion is derived as well as methods and conditions under which the poles of the reduced order dynamical portion can be assigned. Finally, conditions are established under which a dynamical portion of order zero exists.

If the systems is also affected by noise, the input vector and the state vector can not be exactly reconstructed. A convenient way to invert such a combined deterministic-stochastic system is to estimate the state vector and the input vector so that a certain optimality condition obtains. Numerous methods to deal with input disturbances in linear estimation can be found in the literature, such as robust Kalman filtering techniques [26], methods based on state augmentation [1], moving horizon approaches [27], sliding mode observers [10] and unknown input observers [22, 8, 18]. In this paper, we are concerned with unknown input observers, where the input disturbance is assumed to be completely unknown and where no assumption is made about its properties.

Full order unknown input observers [36, 9, 22, 8, 18] and reduced order observers [34, 16, 25, 21] have been developed for discrete-time systems as well as for continuous-time systems. A rigorous and straightforward method is given by [17, 18]. The approach consists in transforming the system into a system that is decoupled from the unknown inputs. The standard filtering techniques can then be applied to estimate the system state. Another approach consists in parameterizing the filter equations and then calculating the optimal parameters by minimizing the trace of the state covariance matrix under an unbiasedness condition [22, 7, 8].

During the last years, research has also shifted towards joint input-state estimation [19, 13, 14, 35, 12] and time-delayed state estimation or fixed-lag smoothing [20, 31, 12, 32]. The latter method simplifies the existence conditions of the unknown input filters. However, as shown by Sundaram and Hadjicostis [32], a consequence of time-delayed estimation is that the noise affecting the system becomes correlated with the estimation error. They handled this problem by increasing the dimension of the estimator appropriately.

In a second contribution, we address the problem of optimal state and input estimation for combined deterministic-stochastic systems from the viewpoint of the results obtained in the first contribution. We develop state estimators and joint input-state estimators for the one step ahead prediction problem, the filtering problem and the fixed-lag smoothing problem.

Following the approach of [17, 18], it is first shown that the optimal state estimation problem is easily converted into a standard estimation problem for a system with correlated process and measurement noise. This result extends the work of [17, 18] from the problem of optimal filtering to state estimation with arbitrary delay. Secondly, a joint input-state estimator is considered that takes the form of the dynamical portion considered in the first contribution and the two matrix parameters are determined so that the estimates of the system state and the unknown input have minimal variance. The derivation of the optimal state estimator follows to some extent that of [32], however, it is shown that the order of the estimator does not need to be increased to deal with the correlation. The input estimator generalizes the results of [13, 14].

This paper consists of two parts. The first part is addressed in section 2 and deals with the inversion of deterministic systems, that is, with the first contribution. The second part is addressed in section 3 and deals with the inversion of combined deterministic-stochastic systems, that is, with the second contribution.

We will use the following notations: $\mathbb{E}[\cdot]$ denotes the expected value of a random variable, $\hat{a}$ denotes an estimate of the variable a, I denotes an identity matrix of appropriate dimensions, A^{T} denotes the transpose of a matrix A, $A^{(1)}$ denotes any $\{1\}$-inverse of A (i.e. any matrix satisfying $AA^{(1)}A = A$ [3]), $A^{\dagger}$ denotes the Moore-Penrose generalized inverse of A and finally, $\breve{A}$ denotes $[A\ 0]$.

2 Inversion of deterministic systems

Consider the LTI discrete-time system

$$\mathcal{S} : \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k, \end{array} \right. \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector, and $y_k \in \mathbb{R}^p$ is the measurement. The input vector u_k is deterministic, but unknown. Throughout the paper, we will assume that $\mathcal{S}$ is observable and controllable. For clarity of explanation, we will only consider time-invariant systems, however most results are easily generalized to time-varying systems.

In terms of the $z-$transform, $\mathcal{S}$ has the transfer function matrix

$$H(z) = C(zI - A)^{-1}B + D,$$

that is, $Y(z) = H(z)U(z)$. The following definitions are made in parallel with those preceding [28].

Definition 1. *A system is said to be an $L-$delay left inverse of $\mathcal{S}$ if its transfer matrix $H_L(z)$ satisfies $H_L(z)H(z) = z^{-L}I_m$.*

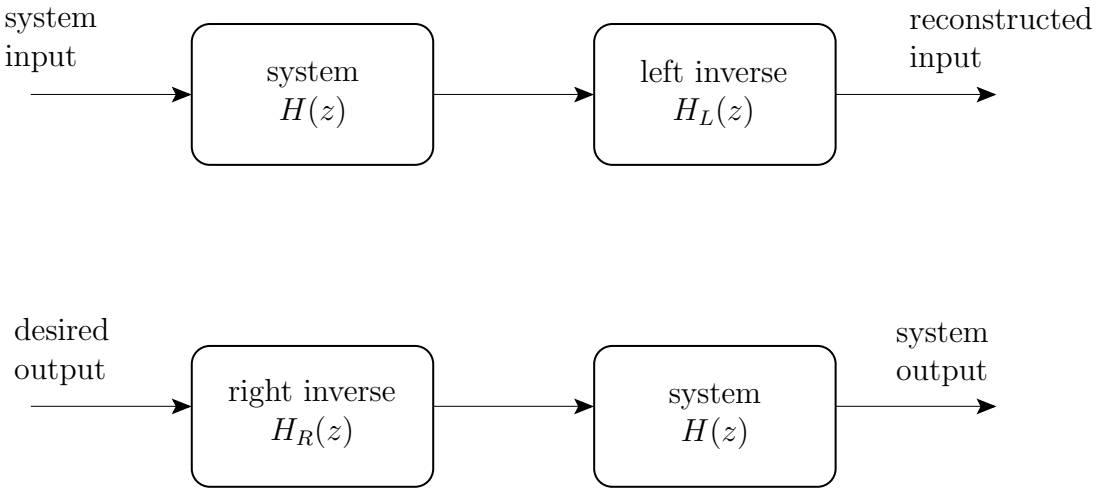


Figure 2: *Left inversion vs right inversion*

Definition 2. *The system $\mathcal{S}$ is said to be left invertible if it has an $L-$delay left inverse for some finite non-negative integer L. The least non-negative integer L for which an $L-$delay left inverse exists is called the inherent delay of $\mathcal{S}$.*

The output $Y_L(z)$ of the series connection of $H(z)$ and $H_L(z)$ equals

$$Y_L(z) = H_L(z)H(z)U(z) = z^{-L}U(z).$$

An $L-$delay left inverse of $\mathcal{S}$ thus reconstructs the input signal applied to $\mathcal{S}$ with L steps delay. A left inverse that computes the input without delay will also be called an *instantaneous* left inverse. Dual to a left inverse, is a right inverse, that is, a system that computes an input signal such that $\mathcal{S}$ has a desired output. A right inverse can thus be seen as an open loop controller. The duality between left and right inversion is schematically shown in Figure 2.

In this paper, we consider only left inversion. However, due to the duality between left and right inversion, all results derived in the paper can be easily translated to right inversion.

2.1 Problem formulation

The main inversion procedures for MIMO systems were introduced by Silverman [29] and Sain and Massey [28]. In both approaches, the inverse system consists of a bank of delay elements followed by a dynamical system. In accordance to [29], this dynamical system will be called the *dynamical portion* of the inverse system in the remainder of the paper. Silverman obtained with his structure algorithm the interesting insight that, given an

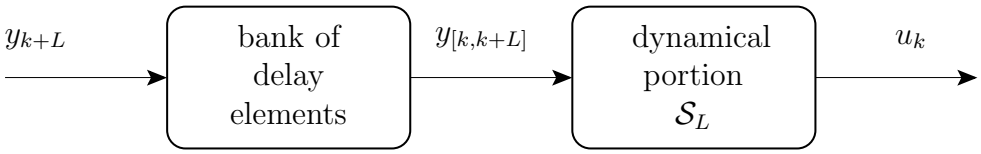


Figure 3: *Structure of the $L-$delay left inverse system*

invertible system, there always exists a bank of delay elements and a dynamical portion such that the corresponding inverse system can be realized with exactly the same number of delay elements as the original system. Sain and Massey, on the other hand, considered $L-$delay left inverses that consist of a bank of pL delay elements followed by a dynamical system. Since pL can be larger than n, this may yield inverses which consist of more delay elements than the original system, especially if L is large. An advantage of the approach of Sain and Massey is that it allows to derive inverses of arbitrary delay (of course larger than the inherent delay of the system), whereas the approach of Silverman is limited to inverses with delay equal to the inherent delay. Increasing the delay above the inherent delay will seem crucial for stable inversion of non-minimum phase systems.

Like Sain and Massey, we consider left inverses which consist of a bank of delay elements with input y_{k+L} and output

$$y_{[k,k+L]} := \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+L} \end{bmatrix},$$

followed by a dynamical portion. The dynamical portion will be denoted by $\mathcal{S}_L$ and takes the form

$$\mathcal{S}_L : \left\{ \begin{array}{rcl} z_{k+1} & = & A_L z_k + B_L y_{[k,k+L]} \\ u_k & = & C_L z_k + D_L y_{[k,k+L]}, \end{array} \right. \tag{2}$$

where $z_k \in \mathbb{R}^{n_z}$. The structure of the inverse system is schematically shown in Figure 3. In case $\mathcal{S}_L$ has the same order as $\mathcal{S}$ ($n_z = n$), we will talk about full order inversion. If $n_z < n$, the corresponding inverse system will be called a reduced order inverse of $\mathcal{S}$.

As discussed above, this approach can yield inverses that are realized with more than n delay elements. In most applications, however, the order of $\mathcal{S}_L$ is of more importance than the number of delay elements needed to realize the inverse system since the inverse system itself is mostly not used for input reconstruction, but rather the dynamical portion. Therefore, most attention in the remainder of the paper will be given to the properties of the dynamical portion.

The remainder of the first part of this first paper will deal with identifying the system matrices of $\mathcal{S}_L$. In section 2.2, the problem of full order inversion is considered. The main results of that section is the derivation of the general form of the system matrices A_L, B_L, C_L and D_L for a full order system $\mathcal{S}_L$ that reconstructs both the state vector and

the input vector of $\mathcal{S}$. The problem of stable inversion is addressed in section 2.3, where conditions are derived under which the poles of $\mathcal{S}_L$ can be assigned as well as methods to assign the poles. The main result of that section is that pole placement is possible if a certain matrix pair is observable. The unobservable modes of that pair turn out to be zeros of the system. In section 2.4, a method is introduced to derive reduced order inverses as well as conditions under which the poles of the reduced order dynamical portion can be assigned. Finally, in section 2.5, the theory is illustrated with two numerical examples.

2.2 Full order inversion

It is well known that system invertibility implies state reconstructibility, that is, if the input vector of $\mathcal{S}$ can be reconstructed, also its state vector can be reconstructed. This has led to the design of left inverses that reconstruct both the state vector and the input vector [29, 28].

In this section, we consider the most simple case, that is, the case of full order inversion with known initial state. In a first contribution, the general form of the system matrices A_L and B_L is determined so that the state vector of $\mathcal{S}_L$ is a reconstruction of the state vector of $\mathcal{S}$, that is, so that for $z_0 = x_0$, it holds that $z_k = x_k$, for $k = 1, 2, \ldots$ In a second contribution, the general form of the corresponding matrices C_L and D_L is derived.

2.2.1 State reconstruction

Before deriving the general form of A_L and B_L, some notations have to be introduced. Defining

$$u_{[k,k+L]} := \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+L} \end{bmatrix},$$

the response of $\mathcal{S}$ over $L+1$ consecutive time units can be written as

$$y_{[k,k+L]} = \mathcal{O}_L x_k + \mathcal{H}_L u_{[k,k+L]}, \tag{3}$$

where the matrices $\mathcal{O}_L$ and $\mathcal{H}_L$ obey the recursions

$$\mathcal{O}_0 = C,$$

$$\mathcal{O}_k = \begin{bmatrix} C \\ \mathcal{O}_{k-1}A \end{bmatrix}, \quad k \geq 1,$$

and

$$\mathcal{H}_0 = D,$$

$$\mathcal{H}_k = \begin{bmatrix} D & 0 \\ \mathcal{O}_{k-1}B & \mathcal{H}_{k-1} \end{bmatrix}, \quad k \geq 1,$$

respectively.

The general form of A_L and B_L is then given in the following Theorem.

Theorem 1. *If and only if*

$$\textit{rank}\, \mathcal{H}_L = \textit{rank}\, \mathcal{H}_{L-1} + \textit{rank} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{6}$$

with rank $\mathcal{H}_{-1} := 0$, x_{k+1} *can be reconstructed as a linear combination of* x_k *and* $y_{[k,k+L]}$. *The general form of the reconstruction can be written as*

$$x_{k+1} = (A - K\mathcal{O}_L)x_k + Ky_{[k,k+L]}, \tag{7}$$

where K *is given by*

$$K = \breve{B}\mathcal{H}_L^{(1)} + Z\Sigma_L, \tag{8}$$

with Z *an arbitrary matrix parameter and* $\Sigma_L := I - \mathcal{H}_L\mathcal{H}_L^{(1)}$.

In the proof of Theorem 1, we make use of the following Lemma.

Lemma 1. *A matrix* K *satisfying* $K\mathcal{H}_L = \breve{B}$ *exists if and only if condition* (6) *obtains.*

Proof. A necessary and sufficient condition for the existence of a solution to $K\mathcal{H}_L = \breve{B}$ is

$$\text{rank} \begin{bmatrix} \breve{B} \\ \mathcal{H}_L \end{bmatrix} = \text{rank}\, \mathcal{H}_L.$$

Noting that

$$\begin{aligned} \text{rank} \begin{bmatrix} \breve{B} \\ \mathcal{H}_L \end{bmatrix} &= \text{rank} \begin{bmatrix} B & 0 \\ D & 0 \\ \mathcal{O}_{L-1}B & \mathcal{H}_{L-1} \end{bmatrix}, \\ &= \text{rank}\, \mathcal{H}_{L-1} + \text{rank} \begin{bmatrix} B \\ D \end{bmatrix}, \end{aligned}$$

concludes the proof. □

The proof of Theorem 1 can now be given.

Proof. Consider a linear combination of x_k and $y_{[k,k+L]}$,

$$A_L x_k + B_L y_{[k,k+L]}, \tag{9}$$

where $A_L \in \mathbb{R}^{n\times n}$ and $B_L \in \mathbb{R}^{n\times p(L+1)}$ still have to be determined. Using (3), (9) is rewritten as

$$(A_L + B_L\mathcal{O}_L)x_k + B_L\mathcal{H}_L u_{[k,k+L]}. \tag{10}$$

Expression (10) equals x_{k+1} for all possible x_k and all possible $u_{[k,k+L]}$ if and only if

$$A_L + B_L\mathcal{O}_L = A \tag{11}$$

and

$$B_L\mathcal{H}_L = \breve{B}. \tag{12}$$

It follows from Lemma 1 that a solution to (12) exists if and only if condition (6) obtains, in which case the general form of the solution is given by $B_L = \breve{B}\mathcal{H}_L^{(1)} + Z\Sigma_L$, see e.g. [3]. Finally, substituting (11) and (12) in (10), yields (7). □

Since the $n \times p(L+1)$ matrix parameter Z can be freely chosen, the matrix K in the linear combination (7) is in general not unique. It is unique only if $\mathcal{H}_L$ has full rank, which occurs only for a square system ($p = m$) with invertible D. Indeed, if $\mathcal{H}_L$ is invertible, $\Sigma_L = 0$ so that the unique matrix K is given by $K = \breve{B}\mathcal{H}_L^{-1} = [BD^{-1}\ 0]$.

The state reconstructor (7) can be seen as an observer which is decoupled from unknown inputs. In case of filtering ($L = 1$), condition (6) becomes

$$\text{rank}\begin{bmatrix} D & 0 \\ CB & D \end{bmatrix} = \text{rank } D + \text{rank}\begin{bmatrix} B \\ D \end{bmatrix},$$

which is the well-known necessary condition for the existence of a disturbance decoupled filter in the presence of unknown inputs [17, 8]. The relation to unknown input decoupling will be discussed in more detail in section 3.2.1.

The matrix Σ_L is not full rank if $\mathcal{H}_L \neq 0$. This follows from the following Lemma.

Lemma 2. *rank* $\Sigma_L = p(L+1) - $ *rank* $\mathcal{H}_L$

Proof. First, note that $\mathcal{H}_L\mathcal{H}_L^{(1)}$ is idempotent. Consequently, the rank of Σ_L is given by

$$\begin{aligned} \text{rank } \Sigma_L &= \text{rank } (I - \mathcal{H}_L\mathcal{H}_L^{(1)}), \\ &= p(L+1) - \text{rank } (\mathcal{H}_L\mathcal{H}_L^{(1)}), \\ &= p(L+1) - \text{rank } \mathcal{H}_L, \end{aligned}$$

see [3]. □

Consequently, no generality is lost by replacing $Z\Sigma_L$ in (8) by $\bar{Z}\bar{\Sigma}_L$, where the $(p(L+1) - \text{rank } \mathcal{H}_L) \times (p(L+1))$ matrix $\bar{\Sigma}_L$ has full row rank and is row equivalent to Σ_L and where the $n \times (p(L+1) - \text{rank } \mathcal{H}_L)$ matrix $\bar{Z}$ is arbitrary. This fact may be used to reduce the number of computations and to reduce the number of delay elements needed to realize the inverse system. However, for clarity of explanation, this issue is not addressed any further, but is left as an exercise for the reader.

2.2.2 Input reconstruction

The following Theorem yields the general form of the matrices C_L and D_L of the input reconstructor.

Theorem 2. *If and only if*

$$\textit{rank } \mathcal{H}_L - \textit{rank } \mathcal{H}_{L-1} = m, \tag{13}$$

u_k can be reconstructed as a linear combination of $y_{[k,k+L]}$ and x_k. The general form of the reconstruction can be written as

$$u_k = M(y_{[k,k+L]} - \mathcal{O}_L x_k), \tag{14}$$

where M is given by

$$M = \breve{I}_m \mathcal{H}_L^{(1)} + U\Sigma_L \tag{15}$$

with U an arbitrary matrix parameter.

In the proof of Theorem 2, we make use of the following Lemma.

Lemma 3. *A matrix M satisfying $M\mathcal{H}_L = \breve{I}_m$ exists if and only if condition* (13) *obtains.*

Proof. A necessary and sufficient condition for the existence of a solution to $M\mathcal{H}_L = \breve{I}_m$ is

$$\text{rank} \begin{bmatrix} \breve{I}_m \\ \mathcal{H}_L \end{bmatrix} = \text{rank } \mathcal{H}_L.$$

Noting that

$$\text{rank} \begin{bmatrix} \breve{I}_m \\ \mathcal{H}_L \end{bmatrix} = \text{rank } \mathcal{H}_{L-1} + m,$$

concludes the proof. □

The proof of Theorem 2 can now be given.

Proof. Consider a linear combination of x_k and $y_{[k,k+L]}$,

$$C_L x_k + D_L y_{[k,k+L]}, \tag{16}$$

where $C_L \in \mathbb{R}^{m\times n}$ and $D_L \in \mathbb{R}^{m\times p(L+1)}$ still have to be determined. Using (3), (16) is rewritten as

$$(C_L + D_L\mathcal{O}_L)x_k + D_L\mathcal{H}_L u_{[k,k+L]}. \tag{17}$$

Expression (17) equals u_k for all possible x_k and all possible $u_{[k,k+L]}$ if and only if

$$C_L = -D_L\mathcal{O}_L$$

and

$$D_L\mathcal{H}_L = \breve{I}_m. \tag{18}$$

It follows from Lemma 3 that a solution to (18) exists if and only if condition (13) obtains, in which case the general form of the solution is given by $D_L = \breve{I}_m\mathcal{H}_L^{(1)} + U\Sigma_L$, see e.g. [3]. □

Condition (13) is exactly the necessary and sufficient condition for $L-$delay left invertibility derived by Sain and Massey [28]. Note that this conditions implies $p \geq m$, that is, a necessary condition for $L-$delay left invertibility is that the dimension of the output vector of $\mathcal{S}$ is at least the dimension of the input vector. Also, it follows from condition (13) that $\mathcal{S}$ has an instantaneous left inverse if D has full column rank. Finally, note that condition (13) is stronger than condition (6), which again shows that system invertibility implies state reconstructibility.

Since the $m \times p(L+1)$ matrix parameter U can be freely chosen, the matrix M in the linear combination (14) is not unique. It is unique only if $\mathcal{H}_L$ has full rank, in which case the unique matrix M is given by $M = \breve{I}_m\mathcal{H}_L^{-1} = [D^{-1}\ 0]$.

2.2.3 The inverse system

Summarizing the results of the previous two sections, yields the following Theorem.

Theorem 3. *Let condition* (13) *obtain, then the system*

$$\mathcal{S}_L : \left\{ \begin{array}{rcl} x_{k+1} & = & (A - K\mathcal{O}_L)x_k + Ky_{[k,k+L]} \\ u_k & = & -M\mathcal{O}_L x_k + My_{[k,k+L]}, \end{array} \right. \tag{19}$$

with K given by (8) *and M by* (15) *is a dynamical portion of an $L-$delay left inverse of $\mathcal{S}$ for all possible values of U and Z.*

As already discussed, if $\mathcal{H}_L$ has full rank, the matrix parameters K and M are unique, so that $\mathcal{S}_L$ is unique. In addition, no matter what delay L is chosen, $\mathcal{S}_L$ always reduces to

$$\mathcal{S}'_L : \left\{ \begin{array}{rcl} x_{k+1} & = & (A - BD^{-1}C)x_k + BD^{-1}y_k \\ u_k & = & -D^{-1}Cx_k + D^{-1}y_k. \end{array} \right.$$

This uniqueness will have negative consequences for the developments in the remainder of the paper. For example, tuning the stability of the inverse by placing its poles is not possible for such systems.

As a special case of Theorem 3, consider a system $\mathcal{S}$ with full rank D. Then it follows from condition (13) that $\mathcal{S}$ is instantaneously left invertible. Making the choice $U = 0, Z = 0$ and taking as $\{1\}$-inverse the Moore-Penrose generalized inverse, yields the well-known inverse system

$$\mathcal{S}''_L : \left\{ \begin{array}{rcl} x_{k+1} & = & (A - BD^{\dagger}C)x_k + BD^{\dagger}y_k \\ u_k & = & -D^{\dagger}Cx_k + D^{\dagger}y_k, \end{array} \right. \tag{20}$$

which can also be derived by pre-multiplying the output equation of $\mathcal{S}$ by $D^{\dagger}$. If D does not have full column rank, the output equation of (20) does not exactly reconstruct u_k. However, it has to be stressed that $-D^{\dagger}Cx_k + D^{\dagger}y_k$ then still is the minimum norm solution of $Du_k = y_k - C_k x_k$.

2.3 Stable inversion

So far, we have assumed that the initial state of the system is known. In that case, stability of the inverse system is not important for input reconstruction. If the initial state is unknown, however, an unstable inverse system will blow up the error in the initial state. Moylan [24] was the first to develop an algorithm which always yields a stable inverse if the original system has no unstable zeros. One year later, Antsaklis [2] derived conditions and methods to assign the poles of the inverse system at arbitrary locations. His approach to pole assignement is based on standard control theory. However, his treatment is limited to systems with instantaneous inverses.

In this section, we extend the work of Antsaklis to $L-$delay left invertible systems. We use the freedom in the choice of the matrix parameter Z to tune the poles of the inverse system. The main contributions of this section are (i) the derivation of a condition under which a stable $L-$delay left inverse exists, (ii) the derivation of conditions under which the poles of the inverse system can be assigned, and (iii) a method to assign the poles at arbitrary locations.

In case the initial state of $\mathcal{S}$ is unknown, it is more convenient to consider $\mathcal{S}_L$ as a joint input-state estimator and to denote the state vector and input vector of $\mathcal{S}_L$ by $\hat{x}_k$ and $\hat{u}_k$, that is, by estimates of x_k and u_k. The interpretation of the inverse system as a joint input-state estimator is schematically show in Figure 4.

Defining the error in $\hat{x}_k$ by $\tilde{x}_k := x_k - \hat{x}_k$, it follows from (19) that if condition (6) obtains,

$$\tilde{x}_{k+1} = (A - K\mathcal{O}_L)\tilde{x}_k. \tag{21}$$

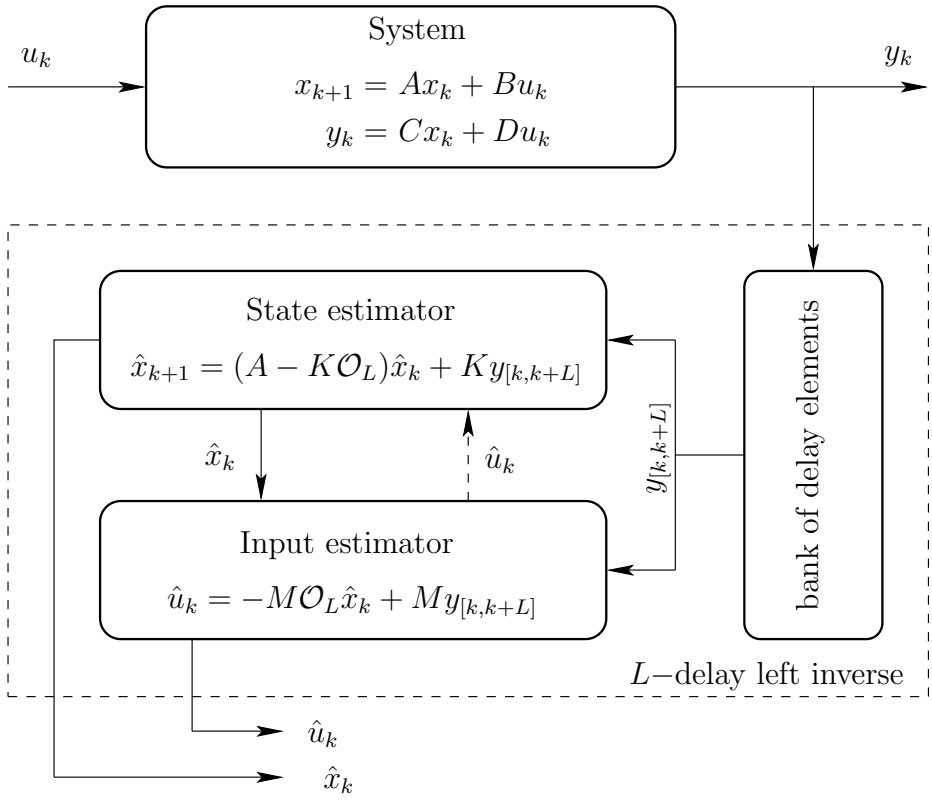


Figure 4: *Interpretation of the inverse system as an estimator*

Similarly, defining $\tilde{u}_k := u_k - \hat{u}_k$, it follows from (19) that for an $L-$delay left invertible system

$$\tilde{u}_k = -M\mathcal{O}_L\tilde{x}_k. \tag{22}$$

We conclude from (21) and (22) that, if and only if all eigenvalues of $A - K\mathcal{O}_L$ lie inside the unit circle, the state vector and the output vector of $\mathcal{S}_L$ converge to the state vector and the input vector of $\mathcal{S}$ respectively, starting from any initial state $\hat{x}_0$. Since

$$A - K\mathcal{O}_L = (A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L) - Z(\Sigma_L\mathcal{O}_L),$$

the position of the eigenvalues of $A - K\mathcal{O}_L$ can be influenced by the choice of the matrix parameter Z. The following Theorem immediately follows from control theory.

Theorem 4. *1. If and only if $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is observable, Z can be chosen so that all eigenvalues of $A - K\mathcal{O}_L$ are arbitrary assigned.*

2. If and only if $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is detectable, Z can be chosen so that all eigenvalues of $A - K\mathcal{O}_L$ lie inside the unit circle.

A relation between the unobservable modes of $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ and the rank of a matrix pencil is given in the following Lemma.

Lemma 4. *Let condition* (6) *obtain, then the unobservable modes $\lambda \in \mathbb{C}$ of $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ satisfy*

$$\textit{rank} \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} < n + \textit{rank} \begin{bmatrix} B \\ D \end{bmatrix}. \tag{23}$$

Proof. The unobservable modes of $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ are those $\lambda \in \mathbb{C}$ for which

$$\text{rank} \begin{bmatrix} \lambda I - (A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L) \\ -\Sigma_L\mathcal{O}_L \end{bmatrix} < n.$$

This implies

$$\text{rank} \begin{bmatrix} \lambda I - (A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L) & \breve{B} \\ -\Sigma_L\mathcal{O}_L & \mathcal{H}_L \end{bmatrix} < n + \text{rank} \begin{bmatrix} \breve{B} \\ \mathcal{H}_L \end{bmatrix}. \tag{24}$$

Using the fact that

$$\begin{aligned} \text{rank} \begin{bmatrix} \lambda I - (A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L) & \breve{B} \\ -\Sigma_L\mathcal{O}_L & \mathcal{H}_L \end{bmatrix} &= \text{rank} \begin{bmatrix} \lambda I - A & \breve{B} \\ -\mathcal{O}_L & \mathcal{H}_L \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{H}_L^{(1)}\mathcal{O}_L & I \end{bmatrix}, \\ &= \text{rank} \begin{bmatrix} \lambda I - A & \breve{B} \\ -\mathcal{O}_L & \mathcal{H}_L \end{bmatrix}, \end{aligned}$$

(24) is rewritten as

$$\text{rank} \begin{bmatrix} \lambda I - A & \breve{B} \\ -\mathcal{O}_L & \mathcal{H}_L \end{bmatrix} < n + \text{rank } \mathcal{H}_L. \tag{25}$$

Finally, if condition (6) obtains, substituting the inequality

$$\begin{aligned} \text{rank} \begin{bmatrix} \lambda I - A & \breve{B} \\ -\mathcal{O}_L & \mathcal{H}_L \end{bmatrix} &= \text{rank} \begin{bmatrix} \lambda I - A & B & 0 \\ -C & D & 0 \\ -\mathcal{O}_{L-1}A & \mathcal{O}_{L-1}B & \mathcal{H}_{L-1} \end{bmatrix}, \\ &\geq \text{rank} \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} + \text{rank } \mathcal{H}_{L-1}, \end{aligned}$$

in (25), yields (23). □

For an $L-$delay left invertible system, Lemma 4 yields a relation between the unobservable modes of $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ and the transmission zeros of $\mathcal{S}$.

Lemma 5. *If $\mathcal{S}$ is $L-$delay left invertible, the unobservable modes of $(A-\breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ are transmission zeros of $\mathcal{S}$.*

The following theorem is a direct consequence of Theorem 4 and Lemma 5.

Theorem 5. *Consider an $L-$delay left invertible system $\mathcal{S}$. Then,*

1. *if $\mathcal{S}$ has no zeros, Z can be chosen so that all the poles of $\mathcal{S}_L$ are arbitrary assigned;*
2. *if $\mathcal{S}$ has no unstable zeros, Z can be chosen so that $\mathcal{S}_L$ is stable.*
3. *More specifically, let $\mathcal{S}$ have k zeros, then at most k poles of $\mathcal{S}_L$ will equal zeros of $\mathcal{S}$ while the other poles can be arbitrary assigned.*

It is important to note that Lemma 4 and Lemma 5 only hold in one direction, that is, not necessarily all transmission zeros of $\mathcal{S}$ are unobservable modes of $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$. This means that also for a system with (possibly unstable) zeros there can exist L for which the poles of $\mathcal{S}_L$ are assignable. The pair $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is more likely to be observable for larger L. For example, it may happen that the pair is not observable for L equal to the inherent delay of the system, but is observable for a certain L larger than the inherent delay. Indeed, in section 2.5.2 we will consider such an example.

Considering only state reconstruction, we have from Theorem 4 and Lemma 4 the following Theorem.

Theorem 6. *Consider a system $\mathcal{S}$ for which condition* (6) *obtains. If*

$$\textit{rank} \begin{bmatrix} \lambda I - A & B \\ -C & D \end{bmatrix} = n + \textit{rank} \begin{bmatrix} B \\ D \end{bmatrix} \tag{26}$$

$\forall \lambda \in \mathbb{C}$ *with* $|\lambda| \geq 1$, *the matrix parameter* Z *can be chosen so that the state reconstructor* (7) *is stable. In addition, if* (26) *holds* $\forall \lambda \in \mathbb{C}$, *the poles of the state reconstructor can be arbitrary assigned.*

Note that Theorem 6 extends the sufficient conditions for the existence of stable state reconstructors in the presence of unknown inputs from $L = 1$ (filtering) [17] to arbitrary L.

2.4 Stable reduced order inversion

Silverman [29] noticed that the order of the dynamical portion obtained with his structure algorithm can be reduced in such a way that the resulting inverse system can be realized with exactly the same number of delay elements as the original system. His approach to reduced order inversion is based on calculating part of the state vector directly from the measurements. A rigorous approach to the design of minimal order inverses is given by Yuan [37] and Emre and Silverman [11]. Their methods are based on the concepts of elementary null sequences and minimal dynamical covers, respectively.

A point which has not yet received a lot of attention is that even if the observability and controllability matrix of an inverse system are full rank, there may exist an inverse system of lower order. This may seem contrary to one of the main theorems of systems theory at first, however, note that the input of an inverse system is not free, but is constrained to be the output of another dynamical system.

The aim of the present section is not to derive inverse system which can be realized with a minimal number of delay elements, but rather to reduce the order of the dynamical portion $\mathcal{S}_L$. Our approach to reduced order inversion is most closely related to that of Silverman in the sense that we also calculate part of the state vector (eventually up to a similarity transformation) directly from the measurements. Reducing the order of $\mathcal{S}_L$ is then straightforward and has some similarities to the design of reduced order state observers [23]. The main contributions of this section are (i) the derivation of conditions and methods to reduce the order of the dynamical portion and (ii) the derivation of conditions and methods to arbitrary assign the poles of the reduced order dynamical portion.

2.4.1 Reduced order inversion

First, note that (3) can be decoupled from $u_{[k,k+L]}$ by pre-multiplying left and right hand side by Σ_L,

$$\Sigma_L y_{[k,k+L]} = \Sigma_L \mathcal{O}_L x_k. \tag{27}$$

Equation (27) yields a relation between x_k and $y_{[k,k+L]}$ which allows to calculate part of x_k directly from the measurements. Let the singular value decomposition of $\Sigma_L \mathcal{O}_L$ be given

by

$$\Sigma_L \mathcal{O}_L = X \Xi V^{\mathsf{T}}, \tag{28}$$

where $X \in \mathbb{R}^{p(L+1)\times p(L+1)}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal and where $\Xi \in \mathbb{R}^{p(L+1)\times n}$ contains the $r = \text{rank } \Sigma_L \mathcal{O}_L$ singular values on its main diagonal. Note that (28) can be rewritten as

$$\Sigma_L \mathcal{O}_L = \tilde{X} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^{\mathsf{T}},$$

where the i-th column of $\tilde{X}$ equals the i-th column of X multiplied by the i-th singular value, $i = 1, 2, \ldots, r$. Consequently, pre-multiplying (27) by $\tilde{X}^{-1}$, yields

$$\tilde{X}^{-1} \Sigma_L y_{[k,k+L]} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^{\mathsf{T}} x_k.$$

Define $\bar{x}_k := V^{\mathsf{T}} x_k$, and partition $\bar{x}_k$ as $\bar{x}_k^{\mathsf{T}} = [\bar{z}_k^{\mathsf{T}} \; z_k^{\mathsf{T}}]^{\mathsf{T}}$ with $\bar{z}_k \in \mathbb{R}^r$. Also, define $\bar{X} := \tilde{X}^{-1}\Sigma_L$ and partition $\bar{X}$ as $\bar{X}^{\mathsf{T}} = [\bar{X}_1^{\mathsf{T}} \; \bar{X}_2^{\mathsf{T}}]^{\mathsf{T}}$ with $\bar{X}_1 \in \mathbb{R}^{r\times p(L+1)}$. Then

$$\bar{z}_k = \bar{X}_1 y_{[k,k+L]}, \tag{29}$$

that is, $\bar{z}_k$ can be computed directly from $y_{[k,k+L]}$.

The remaining part of the derivation is closely related to the design of reduced order observers [23]. Since $\bar{z}_k$ is known, we consider as state equation of the reduced order dynamical portion a dynamic equation for z_k. First, note that under a similarity transformation with transformation matrix V^{T}, the pair $(A - K\mathcal{O}_L, K)$ becomes $(\bar{A}, \bar{B})$ with $\bar{A} := V^{\mathsf{T}}(A - K\mathcal{O}_L)V$ and $\bar{B} := V^{\mathsf{T}}K$. Consequently, under the similarity transformation, the state equation of $\mathcal{S}_L$ can be written as

$$\begin{bmatrix} \bar{z}_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{z}_k \\ z_k \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} y_{[k,k+L]}. \tag{30}$$

The dynamic equation for z_k is then easily extracted from (30) and (29),

$$z_{k+1} = \bar{A}_{22} z_k + (\bar{B}_2 + \bar{A}_{21}\bar{X}_1) y_{[k,k+L]}. \tag{31}$$

For the output equation of the reduced dynamical portion, we first rewrite the output equation of $\mathcal{S}_L$ as

$$u_k = -M\mathcal{O}_L V \bar{x}_k + M y_{[k,k+L]}.$$

Defining $\bar{C} := -M\mathcal{O}_L V$ and partitioning $\bar{C}$ as $\bar{C} = [\bar{C}_1 \; \bar{C}_2]$, with $\bar{C}_1 \in \mathbb{R}^{m\times r}$, yields

$$u_k = \bar{C}_1 \bar{z}_k + \bar{C}_2 z_k + M y_{[k,k+L]}. \tag{32}$$

Finally, substituting (29) in (32), yields the output equation of the reduced order dynamical portion,

$$u_k = \bar{C}_2 z_k + (M + \bar{C}_1 \bar{X}_1) y_{[k,k+L]}. \tag{33}$$

Combining (33) and (31) yields the dynamical portion

$$\mathcal{S}_L''' : \left\{ \begin{array}{rcl} z_{k+1} & = & \bar{A}_{22} z_k + (\bar{B}_2 + \bar{A}_{21}\bar{X}_1) y_{[k,k+L]} \\ u_k & = & \bar{C}_2 z_k + (M + \bar{C}_1 \bar{X}_1) y_{[k,k+L]}, \end{array} \right.$$

of order $n-r$. This dynamical portion can, however, be unstable even though all eigenvalues of $A - K\mathcal{O}_L$ lie inside the unit circle. The problem of assigning the poles of the reduced order inverse system is addressed in the next section.

2.4.2 Stable reduced order system inversion

Note that (31) can be rewritten as

$$z_{k+1} = (\bar{A}_{22} - N\bar{A}_{12}) z_k + \bar{B}_2 y_{[k,k+L]} + \bar{A}_{21}\bar{z}_k + N(\bar{z}_{k+1} - \bar{A}_{11}\bar{z}_k - \bar{B}_1 y_{[k,k+L]}), \tag{34}$$

where N is an arbitrary matrix. The freedom in the choice of N will be used to assign the poles of the reduced order dynamical portion. The term $\bar{z}_{k+1}$ in (34) can be eliminated by defining $s_k := z_k - N\bar{z}_k$. Together with (29), this yields the following state equation,

$$s_{k+1} = (\bar{A}_{22} - N\bar{A}_{12}) s_k + \tilde{B} y_{[k,k+L]}, \tag{35}$$

where $\tilde{B} := \bar{B}_2 - N\bar{B}_1 + \bar{A}_{21}\bar{X}_1 - N\bar{A}_{11}\bar{X}_1 + \bar{A}_{22}N\bar{X}_1 - N\bar{A}_{12}N\bar{X}_1$.

For the output equation, we first rewrite (32) as

$$u_k = \bar{C}_1 \bar{z}_k + \bar{C}_2 (s_k + N\bar{z}_k) + M y_{[k,k+L]}. \tag{36}$$

Substituting (29) in (36), yields the output equation of the reduced order dynamical portion,

$$u_k = \bar{C}_2 s_k + \tilde{C} y_{[k,k+L]}, \tag{37}$$

where $\tilde{C} := M + \bar{C}_1\bar{X}_1 + \bar{C}_2 N\bar{X}_1$.

Finally, combining (35) and (37) yields the dynamical portion

$$\mathcal{S}_L'''' : \left\{ \begin{array}{rcl} s_{k+1} & = & (\bar{A}_{22} - N\bar{A}_{12}) s_k + \tilde{B} y_{[k,k+L]} \\ u_k & = & \bar{C}_2 s_k + \tilde{C} y_{[k,k+L]}, \end{array} \right. \tag{38}$$

of order $n - r$ whose poles can be assigned by the choice of the matrix parameter N. Note that x_k can be reconstructed from s_k and $y_{[k,k+L]}$ as

$$x_k = V \left[\begin{array}{c} \bar{X}_1 y_{[k,k+L]} \\ s_k + N\bar{X}_1 y_{[k,k+L]} \end{array} \right].$$

The derivation of the conditions under which N can be chosen so that the poles of $\mathcal{S}_L''''$ are assigned, is based on the following Lemma.

Lemma 6. *If* $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ *is observable, then so is* $(\bar{A}_{22}, \bar{A}_{12})$.

Proof. First, note that $(A-\breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is observable if and only if $(A-K\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is observable. Consider the system

$$\mathcal{S}_d : \left\{ \begin{array}{rcl} x_{k+1} & = & (A - K\mathcal{O}_L)x_k + Ky_{[k,k+L]} \\ \Sigma_L y_{[k,k+L]} & = & \Sigma_L\mathcal{O}_L x_k \end{array} \right.$$

which is decoupled from u_k. If $(A - K\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is observable, it follows that knowledge of y_k over a finite time interval is sufficient to determine x_0 and thus also z_0. Now, consider the system

$$\begin{aligned} z_{k+1} &= \bar{A}_{22}z_k + (\bar{B}_2 y_{[k,k+L]} + \bar{A}_{21}\bar{z}_k), \\ \bar{z}_{k+1} + \bar{A}_{11}\bar{z}_k + \bar{B}_1 y_{[k,k+L]} &= \bar{A}_{12}z_k, \end{aligned}$$

of which the input $\bar{B}_2 y_{[k,k+L]} + \bar{A}_{21}\bar{z}_k$ and the output $\bar{z}_{k+1} + \bar{A}_{11}\bar{z}_k + \bar{B}_1 y_{[k,k+L]}$ can be calculated from knowledge of y_k over a finite time interval. Since also z_0 can be calculated from knowledge of y_k over a finite time interval, it follows that $(\bar{A}_{22}, \bar{A}_{12})$ must be observable. □

The following theorem is a direct consequence of Lemma 6 and Lemma 5.

Theorem 7. *Let* $\mathcal{S}$ *be an* n*-th order* $L-$*delay left invertible system without zeros, then there exists an* $(n - \text{rank } \Sigma_L\mathcal{O}_L)$*-th order dynamical portion of the form* (38) *whose poles can be arbitrary assigned.*

It follows from Theorem 7 that if rank $\Sigma_L\mathcal{O}_L = n$, there exists a dynamical portion of order 0. Indeed, if rank $\Sigma_L\mathcal{O}_L = n$, it follows from (27) that x_k can be reconstructed from $y_{[k,k+L]}$ as

$$x_k = (\Sigma_L\mathcal{O}_L)^\dagger \Sigma_L y_{[k,k+L]}. \tag{39}$$

Substituting (39) in the output equation of (19), yields

$$u_k = M(I - \mathcal{O}_L(\Sigma_L\mathcal{O}_L)^\dagger\Sigma_L)y_{[k,k+L]},$$

which is the desired 0-th order dynamical portion.

2.5 Numerical examples

We consider two numerical examples. The first example deals with stable reduced order inversion, the second example with stable full order inversion of a non-minimum phase system.

2.5.1 Stable reduced order inversion

Consider the minimal LTI system

$$x_{k+1} = \begin{bmatrix} 0.6 & -0.45 \\ 0 & 0.3 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_k, \tag{40a}$$

$$y_k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k. \tag{40b}$$

The system has poles at 0.6 and 0.3 and is thus stable. Furthermore, since the $D-$matrix has full column rank, it has an instantaneous left inverse. The instantaneous left inverse (20), however, has a pole at 1.3 and is thus unstable.

Nevertheless, since the system has no zeros, it follows from Theorem 5 that there exists a stable instantaneous left inverse of which the poles can be arbitrary placed. Suppose that in a certain application we want two complex conjugate poles: $0.7 \pm 0.6i$. These poles can be assigned by calculating the matrix Z so that the eigenvalues of $(A - BD^\dagger C) - Z(I - DD^\dagger)C$ are at the desired locations, which can be achieved using pole placement. Together with the choice $U = 0$, this yields the inverse system

$$x_{k+1} = \begin{bmatrix} -0.86 & -2.91 \\ 0.96 & 2.26 \end{bmatrix} x_k + \begin{bmatrix} 1.46 & 1 \\ -0.96 & -1 \end{bmatrix} y_k, \tag{41a}$$

$$u_k = \begin{bmatrix} 0 & -1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 1 \end{bmatrix} y_k. \tag{41b}$$

In Figure 5, the inverse system (41) is simulated starting from an arbitrary initial state, but with inputs equal to the outputs of (40). Since the inverse system is stable, the output of (41) converges to the input of (40). Convergence is rather slow because the poles of (41) lie close to the unit circle.

Note that the observability and controllability matrix of (41) have full rank. However, there exist left inverses of lower order than (41). Indeed, since rank $(I - DD^\dagger)C = 1$, it follows from Theorem 7 that there exists an instantaneous left inverse of order 1 of which the poles can be arbitrary assigned. Suppose that we want a first order left inverse with the *deadbeat* property, that is, starting from any initial state, it should exactly reconstructs the state vector and the input vector of (40) from $k = 1$. This is achieved by placing the pole of the reduced order inverse at 0, which yields

$$\begin{aligned} \bar{x}_{k+1} &= 0\bar{x}_k + \begin{bmatrix} 1.4708 & -1.4142 \end{bmatrix} y_k, \\ u_k &= -0.7071\bar{x}_k + \begin{bmatrix} 1.7333 & 1.0000 \end{bmatrix} y_k. \end{aligned}$$

Note that in order to realize this inverse system only 1 delay element is needed. Using Silverman's structure algorithm, at least 2 delay elements will be needed.

2.5.2 Stable full order inversion of a non-minimum phase system

Consider the minimal LTI SISO system

$$x_{k+1} = \begin{bmatrix} 0.5 & 0.25 \\ 0.25 & -0.6 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \tag{42a}$$

$$y_k = \begin{bmatrix} -1 & 0.1 \end{bmatrix} x_k. \tag{42b}$$

The system has a transmission zero at 3 and is thus non-minimum phase. Its inherent delay is 1, however the pair $(A - \breve{B}\mathcal{H}_1^{(1)}\mathcal{O}_1, \Sigma_1\mathcal{O}_1)$ is not detectable so that there does not exist a stable full order $1-$delay left inverse. Since the unobservable modes of the pair are zeros of the system, one of the poles of the inverse system will equal the zero of the system, so that the $1-$delay left inverse system is unstable.

Quite remarkably, the observability matrix of the pair $(A-\breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ has full column rank for $L \geq 6$, which means that for $L \geq 6$, the poles of $\mathcal{S}_L$ can be arbitrary assigned. However, for values of L just above 6, one has to take special care about numerical problems when assigning the poles, since the condition number of the observability matrix of the pair $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ is very large, as can be seen in Figure 6. With increasing L, the condition number of the observability matrix decreases and becomes almost constant for $L \geq 30$.

This result indicates that (i) increasing the delay of reconstruction above the inherent delay seems crucial for stable inversion of non-minimum phase systems, at least for this example, and (ii) the input of a non-minimum phase system can be reconstructed without running into numerical problems only if the delay of reconstruction is much larger than the inherent delay of the system. Further research should investigate the inversion of systems with (unstable) zeros in more detail.

3 Inversion of combined deterministic-stochastic system

As discussed in the previous section, if $\mathcal{S}$ has no zeros, the poles of $\mathcal{S}_L$ can be moved arbitrary towards zero, resulting in a dynamical portion with the deadbeat property. It is, however, well-known that deadbeat observers are highly sensitive to noise. In case of a combined deterministic-stochastic system, it is therefore more convenient to design the observer parameters so that a certain optimality condition obtains. This is the subject of the second part of this paper.

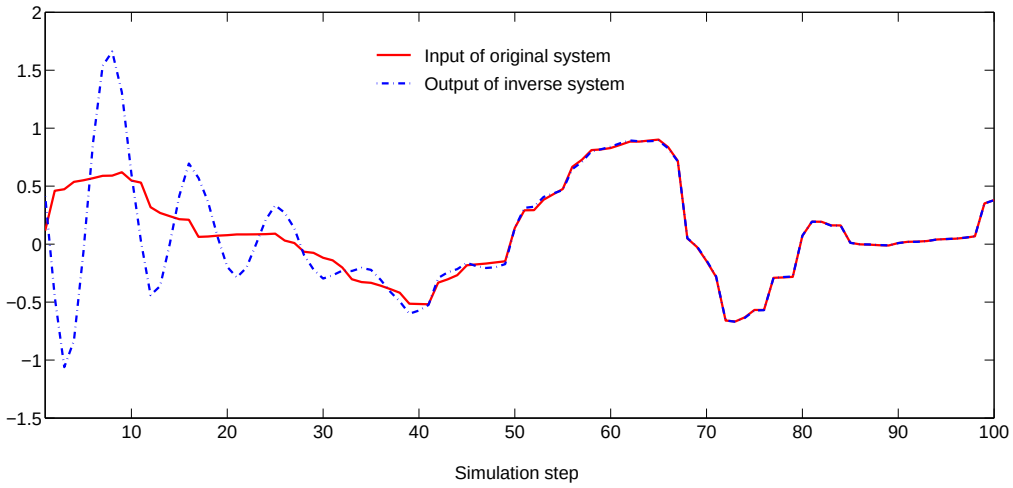


Figure 5: Starting from an arbitrary initial state, the output of the left inverse (41) converges to the input of the original system (40).

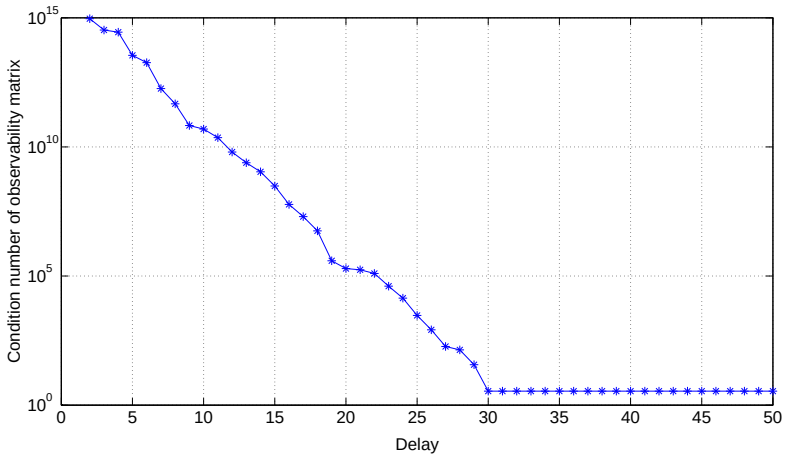


Figure 6: Condition number of the observability matrix of the pair $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L\mathcal{O}_L)$ as function of the delay L for the non-minimum phase system (42).

3.1 Problem formulation

Consider the linear discrete-time system

$$\mathcal{S}' : \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k + w_k \\ y_k & = & Cx_k + Du_k + v_k, \end{array} \right. \tag{43}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector, and $y_k \in \mathbb{R}^p$ is the measurement. The input vector u_k is deterministic, but unknown. The process noise $w_k \in \mathbb{R}^n$ and the measurement noise $v_k \in \mathbb{R}^p$ are assumed to be zero-mean random signals with

$$\mathbb{E}\left\{ \left[\begin{array}{c} w_k \\ v_k \end{array} \right] \left[\begin{array}{cc} w_j^{\mathsf{T}} & v_j^{\mathsf{T}} \end{array} \right] \right\} = \left[\begin{array}{cc} Q & 0 \\ 0 & R \end{array} \right] \delta_{k-j}, \tag{44}$$

where $Q \geq 0$ and $R > 0$. The results derived in this section are easily extended to systems with both known and unknown inputs, to systems with correlated process and measurement noise and to time-varying systems. However, for conciseness of equations, these extensions will not be considered.

The objective of this second part of the paper is threefold. The first objective, the problem of optimal state estimation for the system $\mathcal{S}'$, is addressed in section 3.2. Two state estimation procedures are considered. The procedure developed in section 3.2.1 follows the approach of [17] by transforming the optimal state estimation problem into a standard estimation problem for a system with correlated process and measurement noise. It extends existing results for unknown input decoupled state estimation from the case of optimal filtering to the case of state estimation with arbitrary time delay. In section 3.2.2, a state estimator of the form (7) is considered where the gain matrix is determined so that the estimate of the state vector has minimal variance. The resulting estimator extends the work of [32] by showing that the order of the estimator needs not to be increased. The second objective, the problem of optimal input estimation, is addressed in section 3.3. The third objective, the derivation of a joint input-state estimator where the state estimator and input estimator exchange information in both directions is addressed in section 3.4. The resulting estimator is unbiased if and only if condition (13) obtains, and thereby extends the results of [13, 14]. Finally, in section 3.5, the theory is illustrated with a numerical example.

3.2 Optimal state estimation

3.2.1 Unknown input decoupling

The concept of *unknown input decoupling* yields a rigorous and straightforward approach to the design of optimal filters for systems with unknown inputs [17, 18]. The idea behind unknown input decoupling is to transform the state equation of a system with unknown

inputs into an equivalent state equation which is decoupled from the unknown input. By also deriving an output equation which is decoupled from the unknown input, standard filtering techniques, like e.g. the Kalman filter, can be employed to estimate the system state.

The main contribution of this section is a straightforward derivation of an unknown input decoupled system which generalizes the results of [18] from the case of filtering to the case of optimal state estimation with arbitrary delay.

Before deriving the unknown input decoupled system, some notations have to be introduced. By recursively defining

$$\mathcal{N}_0 := 0, \qquad \mathcal{N}_1 := \begin{bmatrix} 0 \\ C \end{bmatrix},$$
$$\mathcal{N}_k := \begin{bmatrix} 0 & 0 \\ \mathcal{O}_{k-1} & \mathcal{N}_{k-1} \end{bmatrix}, \quad k \geq 2,$$

the response of $\mathcal{S}'$ over $L+1$ consecutive time units can be written as

$$y_{[k,k+L]} = \mathcal{O}_L x_k + \mathcal{H}_L u_{[k,k+L]} + \mathcal{N}_L w_{[k,k+L-1]} + v_{[k,k+L]}. \tag{45}$$

The derivation of the unknown input decoupled system is then based on the following Lemma.

Lemma 7. *If and only if condition* (6) *obtains,* Bu_k *can be expressed as*

$$Bu_k = K(y_{[k,k+L]} - \mathcal{O}_L x_k) - K\mathcal{N}_L w_{[k,k+L-1]} - K v_{[k,k+L]}, \tag{46}$$

where the general form of K *is given by* (8).

Proof. Using (45), (46) is rewritten as $(\breve{B} - K\mathcal{H}_L)u_{[k,k+L]} = 0$. It follows from Lemma 1 that the latter equation has a unique solution for all $u_{[k,k+L]}$ if and only if condition (6) obtains. □

Substituting (46) in (43), yields the following state equation,

$$x_{k+1} = (A - K\mathcal{O}_L)x_k + K y_{[k,k+L]} + (\breve{I}_n - K\mathcal{N}_L)w_{[k,k+L-1]} - K v_{[k,k+L]}.$$

This state equation is decoupled from the unknown input, but yet is equivalent to the state equation of $\mathcal{S}'$ if condition (6) obtains. An output equation which is decoupled from u_k is obtained by pre-multiplying (45) by Σ_L, which yields

$$\Sigma_L y_{[k,k+L]} = \Sigma_L \mathcal{O}_L x_k + \Sigma_L \mathcal{N}_L w_{[k,k+L-1]} + \Sigma_L v_{[k,k+L]}. \tag{47}$$

Since Σ_L does not have full rank, the $p(L+1)$ linear equations of (47) are not linearly independent. Therefore, Σ_L in (47) may also be replaced by $\bar{\Sigma}_L$.

Summarizing, the unknown input decoupled system is given by

$$\mathcal{S}'_d : \left\{ \begin{array}{rcl} x_{k+1} & = & (A - K\mathcal{O}_L)x_k + Ky_{[k,k+L]} + \bar{w}_k \\ \bar{\Sigma}_L y_{[k,k+L]} & = & \bar{\Sigma}_L \mathcal{O}_L x_k + \bar{v}_k, \end{array} \right.$$

where the process noise $\bar{w}_k := (\breve{I}_n - K\mathcal{N}_L)w_{[k,k+L-1]} - Kv_{[k,k+L]}$ is correlated to the measurement noise $\bar{v}_k := \bar{\Sigma}_L \mathcal{N}_L w_{[k,k+L-1]} + \bar{\Sigma}_L v_{[k,k+L]}$. The optimal state estimation problem for $\mathcal{S}'$ has thus been transformed into a standard Kalman filtering problem for a system with correlated process and measurement noise. Indeed, by treating $y_{[k,k+L]}$ as input and $\bar{\Sigma}_L y_{[k,k+L]}$ as output, standard estimation techniques like the Kalman filter can be employed to obtain optimal estimates of x_k. In order to be able to place the poles of the state estimator, the pair $(A - K\mathcal{O}_L, \bar{\Sigma}_L \mathcal{O}_L)$ must be observable. It is straightforward to prove that $(A - K\mathcal{O}_L, \bar{\Sigma}_L \mathcal{O}_L)$ is observable if and only if $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L \mathcal{O}_L)$ is observable. A sufficient condition for the latter pair to be observable was given in Theorem 6.

Summarizing, the unknown input decoupled system $\mathcal{S}'_d$ is appropriate for state estimation if condition (6) obtains and if the pair $(A - \breve{B}\mathcal{H}_L^{(1)}\mathcal{O}_L, \Sigma_L \mathcal{O}_L)$ is observable. These conditions extends the results of [15, 17] from the case of filtering ($L = 1$) to the case of state estimation with arbitrary delay.

3.2.2 Unbiased minimum-variance state estimation

An alternative state estimator will now be derived by calculating the parameters of the state reconstructor (7) so that the estimates of the system state are unbiased minimum-variance. The derivation runs to some extent parallel to that in [32], however, it is shown that the order of the estimator needs not to be increased in order to deal with the correlation between the estimation error and the system noise.

We consider a state estimator of the form

$$\hat{x}_{k+1} = A\hat{x}_k + K_k \tilde{y}_{[k,k+L]}, \tag{48}$$

where

$$\tilde{y}_{[k,k+L]} := y_{[k,k+L]} - \bar{y}_{[k,k+L]}, \tag{49}$$

with $\bar{y}_{[k,k+L]} := \mathcal{O}_L \hat{x}_k$. We assume that an unbiased estimate $\hat{x}_0$ of the initial state is available with covariance matrix P_0^x.

We now derive the value of K_k which yields the unbiased estimate $\hat{x}_{k+1}$ of minimal variance. Using (43) and (48), we obtain the following expression for the error in $\hat{x}_{k+1}$,

$$\begin{aligned} \tilde{x}_{k+1} &:= x_{k+1} - \hat{x}_{k+1}, \\ &= A\tilde{x}_k + Bu_k + w_k - K_k \tilde{y}_{[k,k+L]}. \end{aligned} \tag{50}$$

Furthermore, it follows from (43) that

$$\tilde{y}_{[k,k+L]} = \mathcal{O}_L \tilde{x}_k + \mathcal{H}_L u_{[k,k+L]} + \mathcal{N}_L w_{[k,k+L-1]} + v_{[k,k+L]}, \tag{51}$$

Substituting (51) in (50), yields

$$\tilde{x}_{k+1} = (A - K_k\mathcal{O}_L)\tilde{x}_k + (\breve{B} - K_k\mathcal{H}_L)u_{[k,k+L]} + (\breve{I}_n - K_k\mathcal{N}_L)w_{[k,k+L-1]} - K_k v_{[k,k+L]}. \tag{52}$$

We conclude from (52) that the estimator (48) is unbiased for all possible $u_{[k,k+L]}$ if and only if K_k satisfies $K_k\mathcal{H}_L = \breve{B}$. It follows from Lemma 1 and Theorem 1 that a solution to the latter equation exists if and only if condition (6) obtains, in which case the general solution is given by

$$K_k = \breve{B}\mathcal{H}_L^{(1)} + Z_k\Sigma_L, \tag{53}$$

where Z_k is an arbitrary matrix.

Lemma 8. *Let K_k be given by* (53)*, then the estimator* (48) *is unbiased if and only if condition* (6) *obtains.*

We now calculate the value of Z_k that minimizes the trace of the error covariance matrix $P_{k+1}^x := \mathbb{E}[\tilde{x}_{k+1}\tilde{x}_{k+1}^\mathsf{T}]$. It follows from (52) that P_{k+1}^x is given by

$$P_{k+1}^x = K_k\bar{R}_k K_k^\mathsf{T} - K_k\bar{S}_k^\mathsf{T} - \bar{S}_k K_k^\mathsf{T} + \bar{T}_k, \tag{54}$$

where

$$\begin{aligned}
\bar{R}_k &= \mathbb{E}[(\tilde{y}_{[k,k+L]} - \mathcal{H}_L u_{[k,k+L]})(\tilde{y}_{[k,k+L]} - \mathcal{H}_L u_{[k,k+L]})^\mathsf{T}], \\
&= \mathcal{O}_L P_k^x \mathcal{O}_L^\mathsf{T} + R_{[k,k+L]} + \mathcal{N}_L Q_{[k,k+L-1]}\mathcal{N}_L^\mathsf{T} + \mathcal{O}_L P_k^{xw}\mathcal{N}_L^\mathsf{T} + \mathcal{N}_L P_k^{wx}\mathcal{O}_L^\mathsf{T} + \mathcal{O}_L P_k^{xv} + P_k^{vx}\mathcal{O}_L^\mathsf{T}, \\
\bar{S}_k &= AP_k^x\mathcal{O}_L^\mathsf{T} + \breve{I}_n Q_{[k,k+L-1]}\mathcal{N}_L^\mathsf{T} + \breve{I}_n P_k^{wx}\mathcal{O}_L^\mathsf{T} + AP_k^{xw}\mathcal{N}_L^\mathsf{T} + AP_k^{xv}, \\
\bar{T}_k &= AP_k^x A^\mathsf{T} + Q + AP_k^{xw}\breve{I}_n^\mathsf{T} + \breve{I}_n P_k^{wx} A^\mathsf{T},
\end{aligned}$$

with

$$\begin{aligned}
R_{[k,k+L]} &:= \mathbb{E}[v_{[k,k+L]}v_{[k,k+L]}^\mathsf{T}], \\
Q_{[k,k+L-1]} &:= \mathbb{E}[w_{[k,k+L-1]}w_{[k,k+L-1]}^\mathsf{T}], \\
P_k^{xw} = (P_k^{wx})^\mathsf{T} &:= \mathbb{E}[\tilde{x}_k w_{[k,k+L-1]}^\mathsf{T}], \\
P_k^{xv} = (P_k^{vx})^\mathsf{T} &:= \mathbb{E}[\tilde{x}_k v_{[k,k+L]}^\mathsf{T}].
\end{aligned}$$

Closed form expressions for $R_{[k,k+L]}$ and $Q_{[k,k+L-1]}$ are obtained from (44), which yields $R_{[k,k+L]} = \mathrm{diag}_0^{L+1}(R, R, \ldots, R)$ and $Q_{[k,k+L-1]} = \mathrm{diag}_0^{L}(Q, Q, \ldots, Q)$, where $\mathrm{diag}_j^i(\cdot)$ denotes a matrix with the i entries between the brackets on the j-th diagonal above the main

diagonal and zeros elsewhere. Using (52), we obtain the following closed form expressions for P_k^{xw} and P_k^{xv},

$$\begin{aligned} P_k^{xw} &= \mathbb{E}[\tilde{x}_k w_{[k,k+L-1]}^{\mathsf{T}}], \\ &= (\breve{I}_n - K_{k-1}\mathcal{N}_L)\mathbb{E}[w_{[k-1,k+L-2]} w_{[k,k+L-1]}^{\mathsf{T}}] + (A - K_{k-1}\mathcal{O}_L)\mathbb{E}[\tilde{x}_{k-1} w_{[k,k+L-1]}^{\mathsf{T}}], \\ &= \sum_{i=1}^{\min\{k,L-1\}} \left(\prod_{j=1}^{i-1}(A - K_{k-j}\mathcal{O}_L)\right)(\breve{I}_n - K_{k-i}\mathcal{N}_L)\mathrm{diag}_{-i}^{L}(Q, Q, \ldots, Q), \end{aligned}$$

and

$$\begin{aligned} P_k^{xv} &= \mathbb{E}[\tilde{x}_k v_{[k,k+L]}^{\mathsf{T}}], \\ &= -K_{k-1}\mathbb{E}[v_{[k-1,k+L-1]} v_{[k,k+L]}^{\mathsf{T}}] - (A - K_{k-1}\mathcal{O}_L)\mathbb{E}[\tilde{x}_{k-1} v_{[k,k+L]}^{\mathsf{T}}], \\ &= -\sum_{i=1}^{\min\{k,L\}} \left(\prod_{j=1}^{i-1}(A - K_{k-j}\mathcal{O}_L)\right) K_{k-i}\mathrm{diag}_{-i}^{L+1}(R, R, \ldots, R), \end{aligned}$$

where $\sum_i^j(\cdot) := 0$ for $i < j$ and $\prod_i^j(\cdot) := I$ for $i < j$.
Substituting (53) in (54), yields

$$\begin{aligned} P_{k+1}^{x} = Z_k\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}}Z_k^{\mathsf{T}} - Z_k\Sigma_L(\bar{S}_k - \breve{B}\mathcal{H}_L^{(1)}\bar{R}_k)^{\mathsf{T}} - (\bar{S}_k - \breve{B}\mathcal{H}_L^{(1)}\bar{R}_k)\Sigma_L^{\mathsf{T}}Z_k^{\mathsf{T}} \\ + \bar{T}_k + \breve{B}\mathcal{H}_L^{(1)}\bar{R}_k(\breve{B}\mathcal{H}_L^{(1)})^{\mathsf{T}} - \breve{B}\mathcal{H}_L^{(1)}\bar{S}_k^{\mathsf{T}} - \bar{S}_k(\breve{B}\mathcal{H}_L^{(1)})^{\mathsf{T}}. \end{aligned} \tag{55}$$

Uniqueness of the gain matrix Z_k minimizing the trace of (55) requires invertibility of $\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}}$. However, we known from Lemma 2 that Σ_L does not have full rank if $\mathcal{H}_L \neq 0$. Based on the rank of $\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}}$, we consider three cases.

Case 1: rank $\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}} = 0$
Note that this occurs e.g. when $\mathcal{H}_L$ has full row rank. The matrix Z_k minimizing the trace of (55) is then given by $Z_k = 0$, so that K_k becomes time-invariant,

$$\begin{aligned} K_k &= \breve{B}\mathcal{H}_L^{-1}, \\ P_{k+1}^{x} &= \bar{T}_k - \bar{S}_k\bar{R}_k^{-1}\bar{S}_k^{\mathsf{T}} + (\bar{S}_k\bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{-1})\bar{R}_k(\bar{S}_k\bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{-1})^{\mathsf{T}}. \end{aligned} \tag{56}$$

Case 2: $0 <$ rank $\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}} < p(L+1)$
The optimal gain matrix is not unique. One of the matrices Z_k minimizing the trace of (55) is obtained by the use of the Moore Penrose generalized inverse,

$$Z_k = (\bar{S}_k - \breve{B}\mathcal{H}_L^{(1)}\bar{R}_k)\Sigma_L^{\mathsf{T}}(\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}})^{\dagger}. \tag{57}$$

Substituting (57) in (53) and (55) yields

$$K_k = \breve{B}\mathcal{H}_L^{(1)} + (\bar{S}_k - \breve{B}\mathcal{H}_L^{(1)}\bar{R}_k)\Sigma_L^{\mathsf{T}}(\Sigma_L\bar{R}_k\Sigma_L^{\mathsf{T}})^{\dagger}\Sigma_L, \tag{58}$$

and

$$P_{k+1}^x = \bar{T}_k - \bar{S}_k \bar{R}_k^{-1} \bar{S}_k^{\mathsf{T}} + (\bar{S}_k \bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{(1)}) \tilde{R}_k (\bar{S}_k \bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{(1)})^{\mathsf{T}}, \tag{59}$$

respectively, with

$$\tilde{R}_k = \bar{R}_k - \bar{R}_k \Sigma_L^{\mathsf{T}} (\Sigma_L \bar{R}_k \Sigma_L^{\mathsf{T}})^{\dagger} \Sigma_L \bar{R}_k. \tag{60}$$

Case 3: rank $\Sigma_L \bar{R}_k \Sigma_L^{\mathsf{T}} = p(L+1)$
This can occur only if $\mathcal{S}'$ is unaffected by unknown inputs, that is, if $B = 0$ and $D = 0$. Since $\Sigma_L \bar{R}_k \Sigma_L^{\mathsf{T}}$ is invertible, the optimal gain matrix is unique and the generalized inverses of $\Sigma_L \bar{R}_k \Sigma_L^{\mathsf{T}}$ in (57)-(60) can be replaced by inverses. Furthermore, it is straightforward to verify that we obtain the Kalman filter equations for $L = 0$ and $L = 1$. For $L > 1$, we obtain fixed-lag smoothing formulas.

3.3 Unbiased minimum-variance input estimation

The problem of optimal input estimation in the presence of noise has been considered in [13, 14]. In this section, we extend the latter results from filtering to state estimation with arbitrary delay. Inspired by the output equation of $\mathcal{S}_L$, we consider an estimator of the form

$$\hat{u}_k = M_k \tilde{y}_{[k,k+L]}, \tag{61}$$

where $\tilde{y}_{[k,k+L]}$ is given by (49), that is, the input estimator uses information from the state estimator derived in the previous section.

The matrix M_k which yields the unbiased estimate $\hat{u}_k$ of minimal variance will now be determined. Using (61) and (51), we obtain the following expression for the error in $\hat{u}_k$,

$$\begin{aligned} \tilde{u}_k &:= u_k - \hat{u}_k, \\ &= (\breve{I}_m - M_k \mathcal{H}_L) u_{[k,k+L]} - M_k (\mathcal{O}_L \tilde{x}_k + \mathcal{N}_L w_{[k,k+L-1]} + v_{[k,k+L]}). \end{aligned} \tag{62}$$

It follows from (62), that the estimator (61) is unbiased for all possible $u_{[k,k+L]}$ if and only if M_k satisfies $M_k \mathcal{H}_L = \breve{I}_m$. We know from Lemma 3 that a solution to the latter equation exists if and only if $\mathcal{S}$ is $L-$delay left invertible, in which case the general solution is given by

$$M_k = \breve{I}_m \mathcal{H}_L^{(1)} + U_k \Sigma_L. \tag{63}$$

where U_k is an arbitrary matrix.

Lemma 9. *Let M_k be given by* (63)*, then the estimator* (61) *is unbiased if and only if condition* (13) *obtains.*

We now calculate the value of U_k which minimizes the trace of the covariance matrix $P_k^u := \mathbb{E}[\tilde{u}_k \tilde{u}_k^{\mathsf{T}}]$. It follows from (62) that P_k^u is given by

$$P_k^u = M_k \bar{R}_k M_k^{\mathsf{T}}. \tag{64}$$

Substituting (63) in (64), yields

$$P_k^u = U_k \Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} U_k^\mathsf{T} + U_k \Sigma_L \bar{R}_k (\breve{I}_m \mathcal{H}_L^{(1)})^\mathsf{T} + \breve{I}_m \mathcal{H}_L^{(1)} \bar{R}_k \Sigma_L^\mathsf{T} U_k^\mathsf{T} + \breve{I}_m \mathcal{H}_L^{(1)} \bar{R}_k (\breve{I}_m \mathcal{H}_L^{(1)})^\mathsf{T}. \quad (65)$$

Based on the rank of $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T}$, we distinguish two cases. Note that rank $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} = p(L+1)$ is not possible for an $L-$delay left invertible system. Therefore, this case is not considered.

Case 1: rank $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} = 0$
The matrix U_k minimizing the trace of (65) is given by $U_k = 0$, so that M_k becomes time-invariant,

$$\begin{aligned} M_k &= \breve{I}_m \mathcal{H}_L^{-1}, \\ P_k^u &= \breve{I}_m \mathcal{H}_L^{-1} \bar{R}_k (\breve{I}_m \mathcal{H}_L^{-1})^\mathsf{T}. \end{aligned} \quad (66)$$

Case 2: $0 <$ rank $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} < p(L+1)$
The matrix U_k minimizing the trace of (65) is not unique. One of the optimal gain matrices is given by

$$U_k = -\breve{I}_m \mathcal{H}_L^{(1)} \bar{R}_k \Sigma_L^\mathsf{T} (\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T})^\dagger. \quad (67)$$

Substituting (67) in (63) and (65) yields

$$M_k = \breve{I}_m \mathcal{H}_L^{(1)} \bar{R}_k^{-1} \tilde{R}_k, \quad (68)$$

and

$$P_k^u = \breve{I}_m \mathcal{H}_L^{(1)} \tilde{R}_k (\breve{I}_m \mathcal{H}_L^{(1)})^\mathsf{T}, \quad (69)$$

respectively.

3.4 Joint input-state estimation

Combining (48) and (61) yields a joint input-state estimator which is unbiased if condition (13) obtains. Furthermore, note that for K_k given by (53) and M_k by (63), the resulting estimator is time-varying and that at each time instant k it equals a dynamical portion of the form (2). Moreover, if the covariance matrices P_k^x and P_k^u converge for $k \to \infty$, also Z_k, K_k, U_k and M_k will converge. The joint input-state estimator then converges to the dynamical portion $\mathcal{S}_L$ with $Z = Z_\infty$ and $U = U_\infty$.

The state estimator and input estimator of the joint input-state estimator exchange information in one direction: the input estimator uses information from the state estimator. In this section, we show that if condition (13) obtains the state estimator implicitly estimates the unknown input. This yields a joint input-state estimator where information between the state estimator and the input estimator is exchanged in two directions.

Based on the rank of $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T}$, we distinguish two cases.

Case 1: rank $\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} = 0$
Using the fact that $\breve{B} = B\breve{I}_m$, it follows from (66) that (56) can be written as $K_k = BM_k$. The state estimator (48) can thus be written as

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + BM_k\tilde{y}_{[k,k+L]}, \\ &= A\hat{x}_k + B\hat{u}_k, \end{aligned} \tag{70}$$

where the last equality follows from the assumption that condition (13) obtains. We conclude that the state estimator implicitly estimates the unknown input. Combining (70) with (61) yields a joint input-state estimator where the estimators exchange information in both directions.

Case 2: $0 < \text{rank } \Sigma_L \bar{R}_k \Sigma_L^\mathsf{T} < p(L+1)$
It follows from (68) that (58) can be rewritten as

$$\begin{aligned} K_k &= \breve{B}\mathcal{H}_L^{(1)} \bar{R}_k^{-1} \tilde{R}_k + \bar{S}_k \Sigma_L^\mathsf{T} (\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T})^\dagger \Sigma_L, \\ &= BM_k + \bar{S}_k \Sigma_L^\mathsf{T} (\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T})^\dagger \Sigma_L. \end{aligned}$$

The state estimator (48) can thus be written as

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + \left[BM_k + \bar{S}_k \Sigma_L^\mathsf{T} (\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T})^\dagger \Sigma_L\right] \tilde{y}_{[k,k+L]}, \\ &= A\hat{x}_k + B\hat{u}_k + \bar{S}_k \Sigma_L^\mathsf{T} (\Sigma_L \bar{R}_k \Sigma_L^\mathsf{T})^\dagger \Sigma_L \tilde{y}_{[k,k+L]}, \end{aligned} \tag{71}$$

where the last equality follows from the assumption that condition (13) obtains. We conclude that the state estimator implicitly estimates the unknown input. It follows from (59) and (69) that the state covariance matrix can be written in function of the input covariance matrix as

$$\begin{aligned} P_{k+1}^x = \bar{T}_k + BP_k^u B^\mathsf{T} - \bar{S}_k \bar{R}_k^{-1} \bar{S}_k^\mathsf{T} - \breve{B}\mathcal{H}_L^{(1)} \tilde{R}_k (\breve{B}\mathcal{H}_L^{(1)})^\mathsf{T} \\ + \ (\bar{S}_k \bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{(1)}) \tilde{R}_k (\bar{S}_k \bar{R}_k^{-1} - \breve{B}\mathcal{H}_L^{(1)})^\mathsf{T}. \end{aligned}$$

Combining (71) with (61) yields a joint input-state estimator where the estimators exchange information in both directions.

3.5 Numerical example

In this example, we demonstrate the effect of smoothing on the estimation accuracy. Consider again the system (40), but now subject to process and measurement noise with covariance matrix of the form (44) where $Q = 10^{-3}I$ and $R = 10^{-2}I$. The initial state of the system equals $x_0 = 0$. An unbiased estimate $\hat{x}_0$ is available with $P_0^x = 10^{-2}I$.

Considering the joint input-state estimator, it is to be expected that larger values of L yield more accurate estimates because more measurements are taken into account. The estimates for $L = 0$ and $L = 5$ (smoothing) are compared in Figure 7. Smoothing clearly

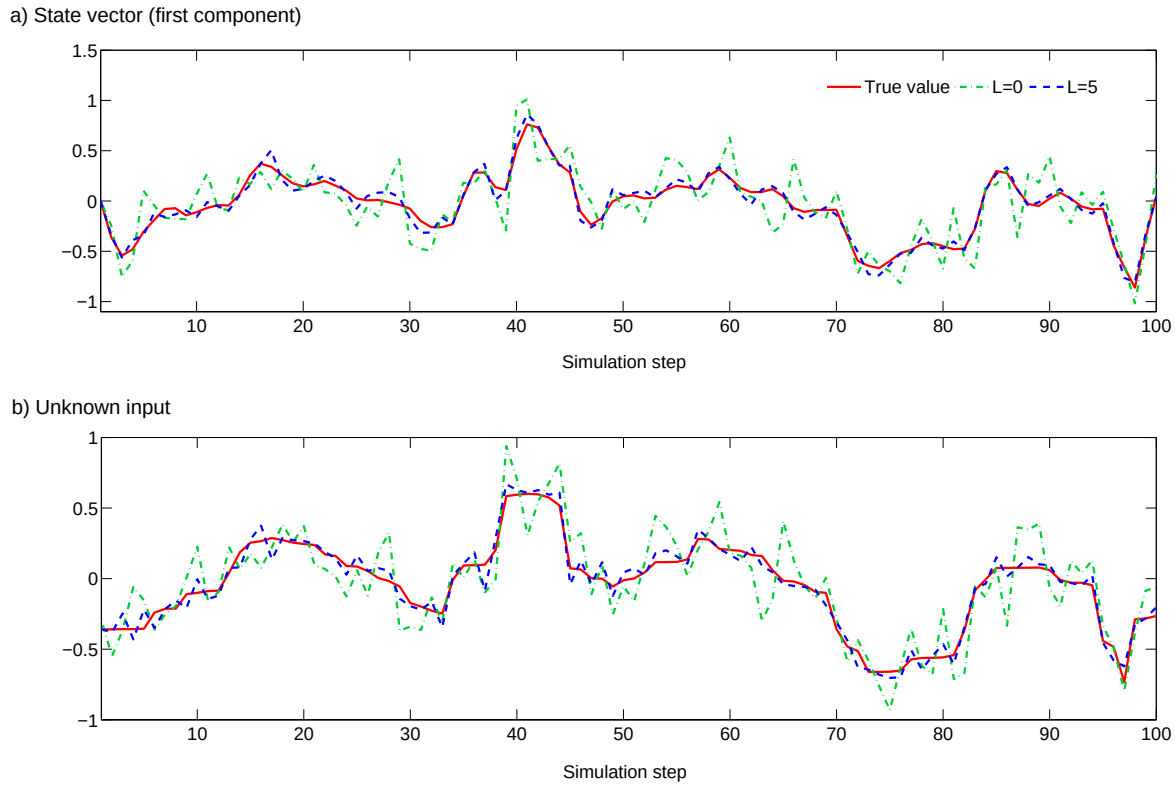


Figure 7: Effect of smoothing on the estimation accuracy.

increases estimation accuracy. This increase in accuracy can also be seen in the variances of the estimates. In both cases, the covariance matrices converge as $k \to \infty$. For $L = 0$, the covariance matrices converge to

$$P_\infty^x \approx \begin{bmatrix} 0.1644 & -0.0986 \\ -0.0986 & 0.0627 \end{bmatrix}, \quad P_\infty^u \approx 0.0406.$$

For $L = 5$, they converge to

$$P_\infty^x \approx \begin{bmatrix} 0.0078 & -0.0038 \\ -0.0038 & 0.0038 \end{bmatrix}, \quad P_\infty^u \approx 0.0035,$$

which indicates that the variance of the estimates is smallest for $L = 5$.

4 Conclusion and discussion

A new procedure for the inversion of linear dynamical systems based on estimation theory has been introduced. The procedure allows to design full order and reduced order inverses that reconstruct the system input and system state with arbitrary time delays. It was

proved that if the system has no zeros, a matrix parameter in the design of the inverse system can be determined so that the poles of the inverse system are arbitrary assigned. However, theoretical results and simulations also show that the method can be used for stable inversion of certain types of non-minimum phase systems. Increasing the delay of reconstruction above the inherent delay of the system seems crucial for inverting non-minimum phase systems. Further research should investigate the inversion of systems with (unstable) zeros in more detail.

Extensions of the inversion procedure to filtering and smoothing in the presence of both noise disturbances and systematic input disturbances have been considered. It was shown that the two matrix parameters can now be determined so that the estimates of the system state and system input are minimal-variance unbiased. Simulation results show that allowing a larger delay in reconstruction strongly increases estimation accuracy. Simulation results also indicate that the covariance matrices of the estimator converge. Further research should investigate conditions under which convergence occurs.

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