Learning Partial Synchronization Regimes with Imposed Qualitative Behavior on an Array of Chua's Oscillators

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Abstract—We demonstrate that using a numerical optimization framework it is possible to explore and analyze the information processing capabilities of non-linear oscillator networks. It will be shown how a properly formulated cost function can be used to achieve synchronization and at the same time impose qualitative behavior on an array of chaotic oscillators. Our approach aims to add an application motivated aspect to existing results that so far focused on conditions for synchronization. To demonstrate the potential of the optimization framework we show on a simple case study that network configurations corresponding to partial synchronization regimes can be learned. This can also be done with imposing various qualitative behavior on individual oscillators.

1. Introduction

Research on synchronization in coupled chaotic oscillators was so far mainly focusing on deriving conditions for synchronization. In [6] global symmetries of a network of identical, diffusively coupled oscillators were used to classify linear invariant manifolds corresponding to partial synchronization regimes and to analyze their global asymptotic stability. In [2] the link between graph topology and stability of global synchronization was clarified. In [1] it was shown that stability of partial synchronization regimes depends on the individual oscillator dynamics and on the choice of state variables used in the coupling. However the existence of partially synchronized clusters depends only on the network topology.

These findings are of primary importance on the way to unravel the rules that govern complex dynamics witnessed in networks of coupled oscillators. The advantage is that these rules reflect the most fundamental mathematical properties of coupled oscillators independent of individual dynamics of oscillators and from initial conditions. However it is not clear how these results can be applied with affordable computational complexity to larger networks or to networks with a structure that is not very simple. Real-world coupled dynamical systems are much more complex than those used in the above mentioned studies.

This paper presents a case study showing how global optimization methods can be used to reveal rules of synchronization potentially playing a role in information processing.

2. Statement of the problem

We consider an array composed of nonlinear oscillators:

$$\dot{x}_{r,c} = F(x_{r,c}; A_{r,c}) + \sum_{k,l \in S_\rho(r,c)} G_{r,c} P x_{k,l},\ r, c = 1, ..., M \in \mathbb{N}_0 \tag{1}$$

where $x_{r,c} \in \mathbb{R}^d$ represents the state vector of the r, c^{th} oscillator; $F : \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^d$ is a nonlinear function depending on parameters $A_{r,c} \in \mathbb{R}^p$ that define the cells; $G \in \mathbb{R}^{M \times M}$ is a weight matrix and $P = (p_{i,j}) \in \mathbb{R}^{d \times d}$ with $p_{i,j} \in \{0, 1\}$ determines which state variables couple the oscillators and $S_\rho(r, c)$ defines a fixed neighborhood topology with neighborhood radius ρ. We investigated how a network with specific properties can be learned, e.g. by solving the optimization problem:

$$\min_{A,G} U(\{X_{st}(t_0), X_{st}(t_1), ..., X_{st}(t_n)\}) \tag{2}$$

where $U(.)$ denotes the cost function and $X_{st} = (x^1_{st}(t), x^2_{st}(t), ..., x^d_{st}(t)) : \mathbb{R} \to \mathbb{R}^{M^2 d}$ is a stationary solution of (1) for a given initial condition and $t_0 < t_1 < ... < t_n < \infty$. Additional inequality constraints may apply to A and G. $U(.)$ is constructed in such a way that dynamical properties of individual cells or cell populations fulfill the desired requirements. These requirements may include different types of synchronization and stability criteria, desired Lyapunov spectrum or embedding dimension of the attractor, etc. We assume that $F_{r,c}$ can provide a rich enough set of dynamical behaviors.

One fundamental issue to be solved is to find network configurations that make a given set of cells in the array synchronize with additional requirements on qualitative behavior. An important advantage of the proposed approach compared to other studies is that the only requirement on the vector field defining individual oscillators is that they permit the solution to exist and being unique. In this study we used oscillators with continuous vectors fields. On the other hand, in some cases this liberty may result in an optimization problem that is very hard to solve if it is possible at all.

3. Global Optimization Framework

Inspired by e.g. [7] where it was shown that difficult tasks can be learned for dynamical systems, we conceived a novel approach on the way to investigate the signal and image processing ability of a network defined above. The global optimization framework is generic, i.e. no assumption was made about network size or topology. The system-level diagram of the framework is shown on Figure 1.

The first task is to define a proper cost function. The most straightforward cost function uses only the time evolution of the network. Theoretical knowledge can also be incorporated, e.g. inspired by [2] we can prescribe a specific information traffic distribution for links between cells to be synchronized and for links to the rest of the array. This way theoretical results can be transformed into system design laws.

Definition of the training set means that we explicitly prescribe which cells are expected to be synchronized. Alternatively we can only prescribe a number of cells that have to be synchronized. The latter approach is easier to formulate and can be used as a first test to check whether the evolution of the network confirms that the cost function was properly formulated.

Prescribing specific cell configurations to be synchronized is a more tedious task the number of possibilities grows exponentially even if we drop repeating configurations due to the rotational symmetry of a rectangular array. This extra complexity however pays off abundantly since it can reveal the capabilities and limitations of the network in well defined signal and image processing tasks. The choice of optimization algorithm is an intricate task. Since integrating the network for a given parameter set can last long, a key factor here is to choose algorithms that require a low number of function evaluations to find the global optimum.

In [1] examples were shown indicating that partial (phase) synchronization regimes persist even with 100% of parameter mismatch. Although we lack generic theoretical knowledge about how parameter variation affects synchronization error, such examples - together with results shown later in this paper - indicate that cost surfaces minimizing synchronization error are smooth with several local minima. Therefore a candidate optimization algorithm must perform global search first.

In [8] a new global optimization algorithm called Coupled Simulated Annealing (CSA) is introduced. In CSA the annealing temperatures of several Simulated Annealing processes are interconnected in order to improve performance in convergence speed and to increase the probability of exploring all basins of attraction in a given number of cost function evaluations. In addition, the coupling also allows the control of the variance of the acceptance probabilities by the acceptance temperature. The sensitivity of the acceptance temperature on its initial value is therefore reduced. In addition, variance control guides optimization to quasi-optimal runs, i.e. it balances global search done at high temperatures with gradually switching to local search with the temperature cooling. The number of cost function evaluations per individual processes decreases exponentially when the number of optimizers is increased linearly. This also makes CSA a good candidate since interactions between solutions decrease the number of cost function evaluations to reach a given energy threshold. Once a solution is found in the framework, its sensitivity to variations in initial conditions and parameter variations have to be tested.

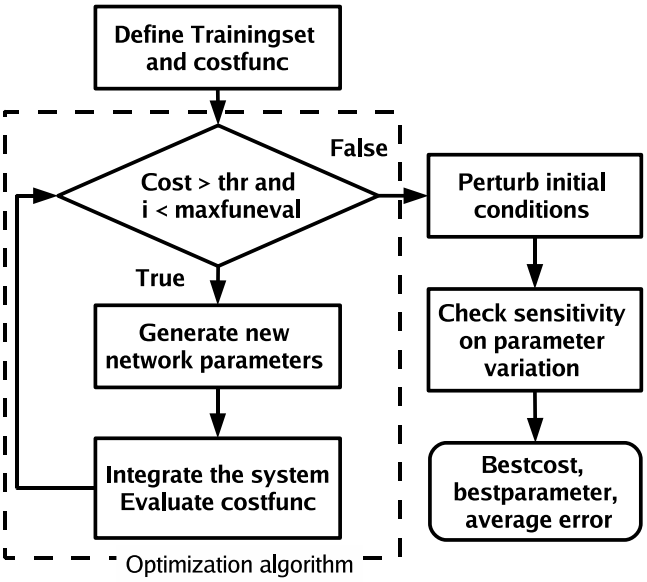

Figure 1: Overview of the investigated global optimization framework that teaches a network of chaotic oscillators to match qualitative behavior requirements formulated in the cost function.

4. Case study: Partial synchronization with imposed dynamics on a coupled array composed of Chua's oscillators

To demonstrate the potential of the optimization framework we applied it to investigate for rules related to the onset and stability of partial synchronization regimes.

4.1. Cost function for synchronization

Defining a cost function for synchronization is a very intricate task since there is no widely accepted measure of synchronization. During our study we investigated for a method that is

- able to capture different forms of synchronization (complete-, phase-, lag-, generalized)
- provides the measure in a way that does not need any human interpretation, i.e. can be embedded in the cost function

- computationally economical.

Currently we analyzed complete synchronization of a prescribed configuration of cells. This aim can be reached in a computationally cheap manner by minimizing

$$U_{std} = \sum_{t_j=1}^{n} \sum_{i=1}^{d} \frac{\exp\left(\frac{1}{N_S} \sum\limits_{k\in\Pi_S} \left(x_k^i(t_j) - \overline{x}_S^i(t_j)\right)^2\right)^{\frac{1}{2}}}{\exp\left(\frac{1}{N_R} \sum\limits_{k\in\Pi_R} \left(x_k^i(t_j) - \overline{x}_R^i(t_j)\right)^2\right)^{\frac{1}{2}}} \quad (3)$$

where $x_S^i(t_j) \in \Pi_S$ represents a number of snapshots taken from the time evolution of cells to be synchronized, $x_R^i(t_j) \in \Pi_R$ represents those that are not to be synchronized. $\Pi_S, \Pi_R \subset \{1, ..., M\} \times \{1, ..., M\}$ denote cell populations with N_S standing for the total number of cells to be synchronized and N_R for the number of cells to be desynchronized and $\Pi_S \bigcap \Pi_R = \emptyset$. $\overline{x}_S(t_j)$ is the spatial average for each time instance of the target population of cells to be synchronized and $\overline{x}_S(t_j)$ is the average for the rest of the cells. Briefly, for a set of time instances we require the spatial standard deviance of the target population to vanish whereas the ratio of spatial standard deviances between the target population and the rest is maximized.

4.1.1. Cost function imposing qualitative behavior

In this study we consider chaotic, limit cycle and equilibrium point qualitative behavior. Lyapunov exponents are widely used to measure qualitative properties of a dynamical system. The cost function used to impose any of these modes on either the population required to be in sync or on the rest is:

$$U_{lyap}(\Omega, G) = \sum_{j\in\Theta_c} e^{-\lambda_{max}(X_j;A,G)} + \sum_{j\in\Theta_e} e^{\lambda_{max}(X_j;A,G)} + \sum_{j\in\Theta_l} |\lambda_{max}(X_j; A, G)| \quad (4)$$

where λ_{max} is the maximum Lyapunov exponent estimated from the time series of an oscillator (for a fast implementation see [4]). Cells are also partitioned into mutually exclusive sets according to the qualitative behavior we want to impose on them after initial transients lapsed during the evolution of the array. Chaotic behavior is imposed on cells denoted by Θ_c convergence to one of the equilibria is imposed on cells denoted by Θ_e and limit cycle behavior on cells denoted by Θ_l.

5. Simulation results

We performed learning for 2x2, 3x3 and 7x7 on arrays of Chua's oscillators with hyperbolic tangent non-linearity, with eight-connected local topology, zero-flux boundary conditions and simulated in Matlab with evolution times large enough to analyze the long-term stationary behavior of the array ($T = 200$). According to [5], the dynamics of a cell is given by:

$$\frac{d}{dt}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a & -a \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -af(x^1) \end{pmatrix}. \quad (5)$$

Variable x^1 was coupled to x^3, i.e. $P =$ [0 0 0; 0 0 0; 1 0 0] in (1). Both the coupling matrix and the matrix of bifurcation parameters (a in (5)) of each cell were included in the learning process. On Figure 2 we show illustrative results for the 2x2 case where the target population to be synchronized was the bottom-left and upper-right cells. CSA with 16 individual optimizers and 200 population iterations ran for 20 minutes using ode45 with relative tolerance 10^{-3} on a Linux machine powered by an AMD Opteron 250. Similar qualitative effects can be observed for 3x3 and 7x7 arrays. An important result of this study is that we showed that it is possible to learn partial synchronization regimes on arrays having different qualitative behavior. Figure 2 shows two examples for partial synchronization: chaotic dynamics emerges when limit cycle and convergent oscillators are coupled, but the same cells can synchronize also when coupling makes chaotic oscillators become limit-cycle oscillators.

6. Discussion and conclusions

In [3] Liu et al. claim that when coupling is added to a network of chaotic oscillators with double- or multi-scroll attractor, Lyapunov exponents being zero in the uncoupled system become positive. They suggest that this rule is general, however, the example shown on Figure 2 indicates that this is not always true since we were able to learn configurations where the opposite effect was happening. This example demonstrates, that the proposed approach can be a useful tool to test hypotheses made on coupled networks of chaotic oscillators. Rules (either theoretical or empirical) believed as generic can be transformed into a cost function. Then a global optimization process can maximize the cost function, i.e. it looks for solutions that violate the rule formulated as a cost function. Finding no such solution can be a strong evidence supporting the validity of the original rule. Confirming generic (theoretical) rules via global optimization might seem improper. However, deriving sound theoretical results is extremely difficult when dealing with coupled networks of chaotic oscillators. Therefore the method showed in this paper can provide valuable feedback. This study showed that by using a global optimization framework it is possible to get insight into the complex interactions of coupled chaotic oscillators. Our method can be useful in guiding investigations and confirming results from theoretical analysis.

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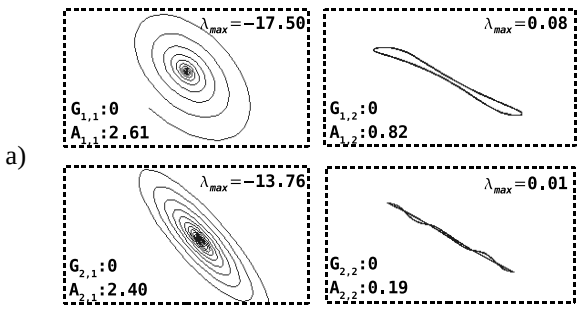

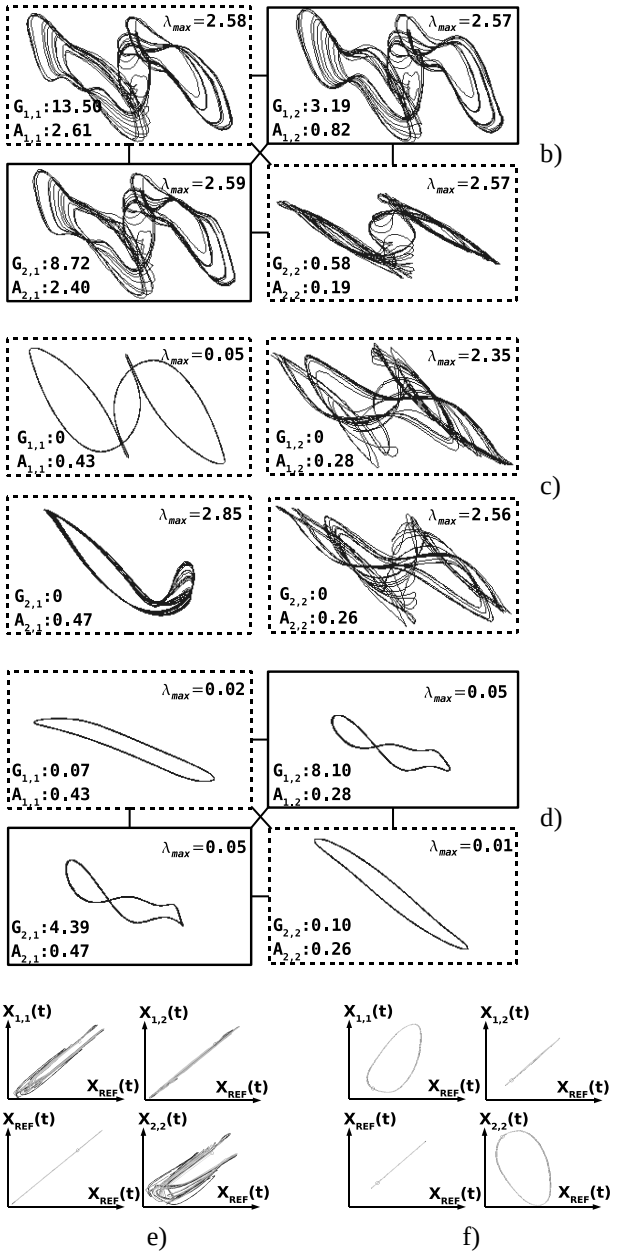

Figure 2: Trajectories showing the stationary behavior from time T to $2T$ of two qualitative behavior types of Chua's oscillator arrays. For both types, cells (2,1) and (1,2) were learned to synchronize. For a,b,e) positive maximum Lyapunov exponents (λ_{max}) were imposed on all cells. For c,d,f): zero λ_{max} were imposed on all cells. On a)-d) continuous line denotes synchronizing cells, dotted line denotes desynchronization. On a,c) trajectories of the array without coupling are shown. b) and d) illustrate the behavior when coupling is added to array a) and c) respectively. e,f) Synchronization errors shown as $norm(X_{REF})$ versus $norm(X_{r,c})$ between a chosen reference cell and all other cells. Errors of array b) and d) are shown on e) and f) respectively. The coupling and bifurcation parameters $G_{r,c}$ and $A_{r,cr}$ are shown below each oscillator.