International Journal of Bifurcation and Chaos, Vol. 12, No. 4 (2002) 709–737
© World Scientific Publishing Company

GENERALIZED DECIMATION INVARIANT SEQUENCES IN DIMENSION N

A. BARBÉ
Katholieke Universiteit Leuven, Department of Electrical Engineering, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium
Andre.Barbe@esat.kuleuven.ac.be

F. VON HAESELER
CeVis Universität Bremen, Universitätsallee 29, 29359 Bremen, Germany

Received February 1, 2001; Revised March 26, 2001

We generalize the concept of one-dimensional decimation invariant sequences, i.e. sequences which are invariant under a specific rescaling, to dimension N. After discussing the elementary properties of decimation-invariant sequences, we focus our interest on their periodicity. Necessary and sufficient conditions for the existence of periodic decimation invariant sequences are presented.

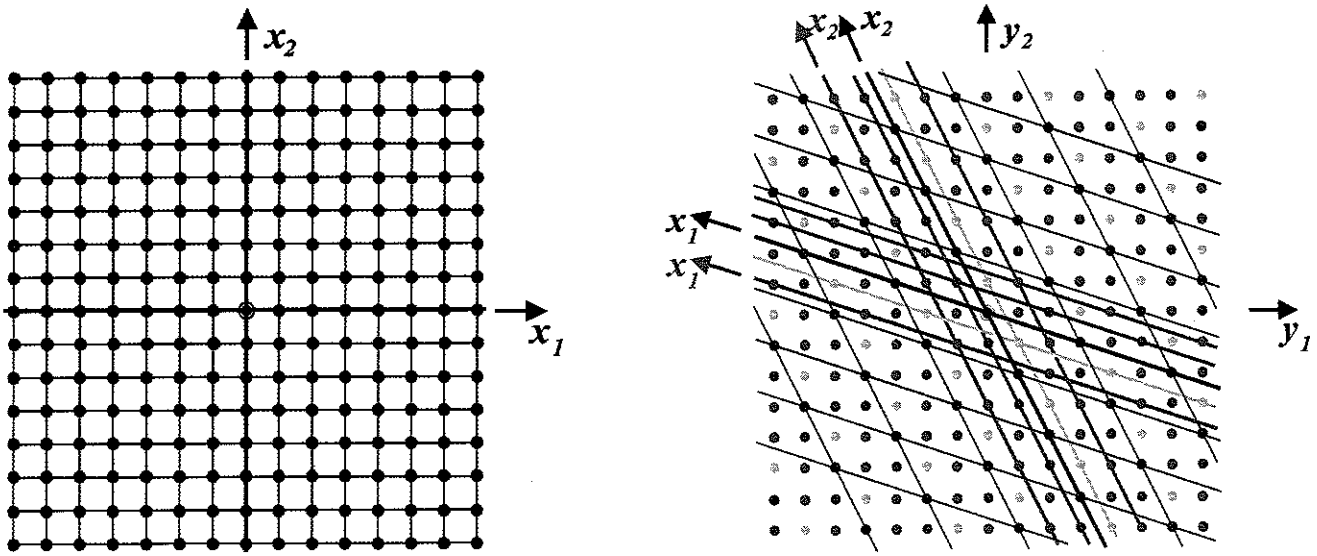
Keywords: Scaling invariance; renormalization; coarse-graining invariance; periodic structures; expansive maps; residue sets; pattern formation; fractals.

1. Introduction

Coarse-graining invariant two-dimensional sequences, generated as orbits of one- and two-dimensional cellular automata, were introduced in [Barbé *et al.*, 1995] and further investigated in [Barbé, 1996, 1997]. Such sequences are invariant under a so-called coarse-graining operation which provides a particular way of rescaling a sequence. In its most simple form, coarse-graining becomes a decimation, whereby the sequence under consideration is replaced by a regularly constructed subsequence. A sequence is called decimation invariant, when a particular set of decimations (i.e. subsequences) equals the original sequence (modulo a shift). Without any further reference to cellular automata, one-dimensional decimation invariant sequences were discussed in [Barbé & Skordev, 2001; Barbé & von Haeseler, 2000].

This paper deals with a twofold generalization of the decimation invariant sequences investigated earlier: apart from dealing with higher-dimensional sequences, it also considers decimations which are defined by using an *arbitrary* sublattice of the N-dimensional integer point-lattice (in contrast to the "square" sublattices used in [Barbé, 1997]. The following two-dimensional example will clarify this.

Consider the integer point-lattice $\mathbb{Z}^2$, called Λ (see Fig. 1(a), which also displays the usual orthogonal coordinate-system: a lattice point has vector coordinate $x = (x_1, x_2) \in \mathbb{Z}^2$). A two-dimensional sequence is defined by mapping this lattice into a finite set S. Graphically, such a sequence corresponds to a picture consisting of colored (square) pixels, the pixels representing the points in $\mathbb{Z}^2$, the colors the elements in S. Consider the sublattice Λ_0 represented by the black dots in Fig. 1(b). With the integer matrix $H = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$, this sublattice Λ_0 can be written as the set of points $\{y = Hx | x \in \Lambda\}$ (it is clear from the context that x and y should be interpreted as column vectors here). y is the coordinate vector of a point in the (y_1, y_2) coordinate system as indicated in Fig. 1(b).



(a) (b)

Fig. 1. (a) The $\mathbb{Z}^2$-lattice Λ. Each point $x = (x_1, x_2) \in \Lambda$ has its coordinates measured along the indicated axes. (b) Decomposition of Λ into five shifted sublattices $\Lambda_i = \{y = Hx + w_i | x \in \Lambda\}$, $i = 0, 1, \ldots, 4$; with $H = \begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$ and $w_0 = (0, 0)$, $w_1 = (1, 1)$, $w_2 = (1, -1)$, $w_3 = (-1, 1)$, $w_4 = (-1, -1)$. The coordinates $y = (y_1, y_2)$ are measured along the indicated (y_1, y_2)-axes. Λ_0 is displayed in black, Λ_1 in green, Λ_2 in yellow, Λ_3 in red, Λ_4 in blue. Λ_i have their own coordinate axes (x_1, x_2), as indicated. The coordinates of a Λ_i-point in the (x_1, x_2)-coordinate system are related to the corresponding coordinates in the (y_1, y_2)-system through $x = H^{-1}(y - w_i)$. A sequence defined on Λ is decimation invariant if the subsequences on the Λ_i-lattices (observed in their own coordinate system) are all identical to the original sequence.

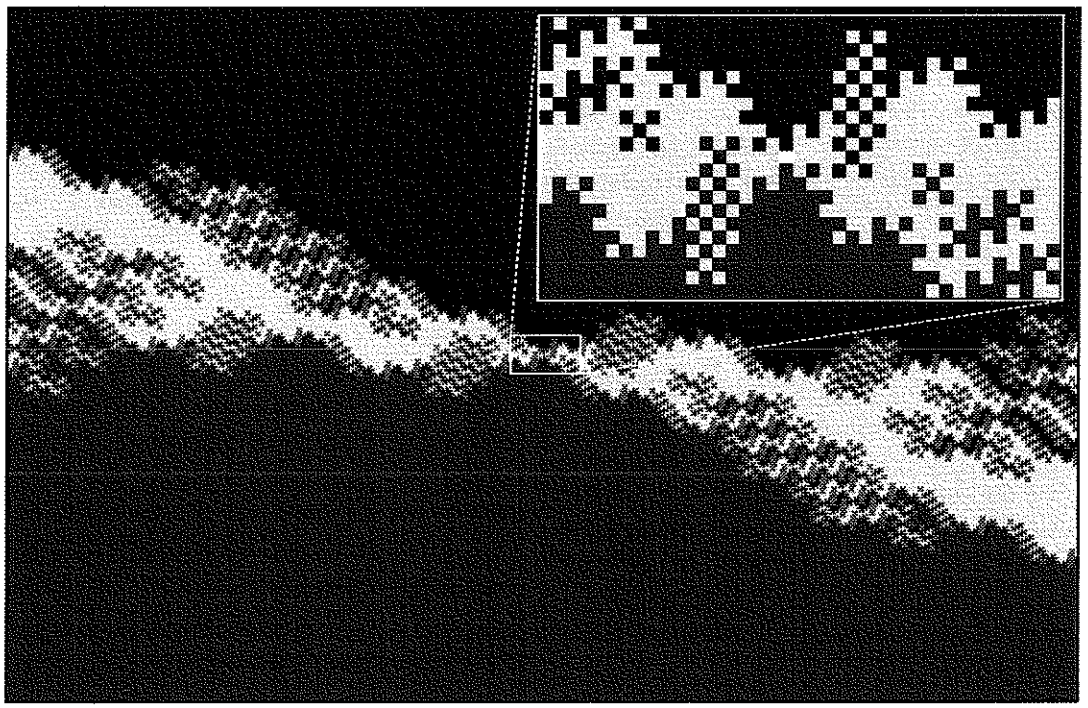


Fig. 2. Partial display of a two-dimensional decimation invariant sequence with respect to the decimations provided by the shifted sublattices Λ_i from Fig. 1(b) (display size is 620×400 pixels). The white pixel in the insert denotes the point $(0, 0)$.

All points in Λ can be covered by the union of the *shifted* sublattices $\Lambda_0, \Lambda_1, \ldots, \Lambda_4$, where Λ_1 to Λ_4 are properly shifted versions of Λ_0:

$$\Lambda_i = \Lambda_0 + w_i = \{y = Hx + w_i | x \in \Lambda\},$$

for some properly chosen $w_i \in \mathbb{Z}^2$, $i = 1, \ldots, 4$. Here we have taken $w_0 = (0, 0)$, $w_1 = (1, 1)$, $w_2 = (1, -1)$, $w_3 = (-1, 1)$, $w_4 = (-1, -1)$ (again to be considered as column vectors in the formulas). This set $W = \{w_i | i = 0, \ldots, 4\}$ is a so-called *residue-set* of H (see also the next section). Each shifted sublattice is displayed in a different color in Fig. 1(b). Let $(s(x))_{x\in\Lambda}$ be a sequence defined on Λ, with $s(x)$ an element in a finite set S. Then restricting this sequence to one of Λ_i defines a "*decimation*" $((s_i(x))_{x\in\Lambda}$ of the original sequence, i.e. the sequence with elements $s_i(x) = s(Hx + w_i)$, $x \in \Lambda$. This corresponds to attributing a new coordinate x to the element of the original sequence with coordinate $y = Hx + w_i$, inducing a certain *rescaling* related to H. A *decimation-invariant* sequence (DI-sequence) with respect to the given H and W, is then a sequence for which all (H, W)-related decimations equal the original sequence, i.e. $s(Hx + w_i) = s(x)$ for all $x \in \Lambda$, $w_i \in W$. For the present example, it means that the subsequences defined on all the Λ_i, $i = 0, \ldots, 4$, equal the original sequence defined on Λ. A solution is shown in Fig. 2.

The next section provides the formal setting and some basic facts (related to the (H, W)-defined sublattices) which are necessary for a general treatment of the DI-problem. Section 3 introduces the formal concepts and basic properties of decimation invariant sequences. Some DI-problems lead to periodic solutions; these form the topic of Sec. 4. In Sec. 5, the morphology of possible solutions is discussed through a set of well-chosen examples.

2. Basic Facts

An N-dimensional sequence will be treated as a map $f : \mathbb{Z}^N \to \mathbb{N}$. If $t \in \mathbb{Z}^N$, then $\sigma_t(f)$ denotes the shifted sequence which is defined as

$$\sigma_t(f)(s) = f(t + s).$$

An $N \times N$-matrix H with entries in $\mathbb{Z}$ induces, by multiplication from the right, a map from $\mathbb{Z}^N \to \mathbb{Z}^N$, which we also denote by H. Thus $H(s) = Hs$, whereby it is clear that the s in Hs should be considered as an N-dimensional column vector of integers.

H is defined to be expanding under a norm $\|\bullet\|$, if $\|H(s)\| \geq C\|s\|$, with $C > 1$. This occurs when all eigenvalues of H have modulus greater than 1, as, following [Lind, 1982], then there exists a norm $\|\bullet\|$ on $\mathbb{R}^n$ such that H is expanding with respect to this norm. This will be called the metric induced by H. In many cases however, the Euclidian norm will do, inducing $C = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of HH^T. From now on, we assume that H is always expanding and we agree that, if necessary, the metric induced by H will be used.

A finite set $W = \{w_1, \ldots, w_\Delta\} \subset \mathbb{Z}^N$, where $\Delta = |\det(H)|$, is called a **residue** set of H if for any $s \in \mathbb{Z}^N$ there exist unique $s' = \Xi_{H,W}(s) \in \mathbb{Z}^N$ and $w = \zeta_{H,W}(s) \in W$ such that

$$s = Hs' + w \tag{1}$$

holds. The maps $\Xi_{H,W} : \mathbb{Z}^n \to \mathbb{Z}^n$ and $\zeta_{H,W} : \mathbb{Z}^n \to W$ are called **image-part map** and **remainder-part map** (w.r.t. (H, W)), respectively. If there is no risk of confusion we simply write Ξ_W and ζ_W, or even Ξ and ζ.

For a two-dimensional matrix $H \in \mathbb{Z}^2$, a residue set W can easily be found by plotting the basis-vectors $h_1 = H\begin{bmatrix}1\\0\end{bmatrix}$, $h_2 = H\begin{bmatrix}0\\1\end{bmatrix}$ for the lattice $\Lambda_H = \{Hs | s \in \mathbb{Z}^2\}$ induced by H (see Fig. 3). This basis determines a parallelogram, the image of the unit square under the mapping H, as a basic lattice-cell. The integer points in the half-open basic parallelogram $\{\lambda_1 h_1 + \lambda_2 h_2 | \lambda_1, \lambda_2 \in \mathbb{R}, 0 \leq \lambda_1, \lambda_2 < 1\}$ form a residue set W of H, the *basic residue set*. Their number equals $|\det(H)|$. Then also any set of the form

$$W' = \{w_i + \alpha_i h_1 + \beta_i h_2 | w_i \in W;\ \alpha_i, \beta_i \in \mathbb{Z}\}$$

is a residue set for H. The extension to higher dimensions is clear: a basic residue set of H is formed by the integer points in the half-open parallelopiped generated by the column vectors in H.

As usual, Ξ^n denotes the nth iterate of Ξ. The set of fixed points of Ξ is denoted by $\mathrm{Fix}(\Xi)$, the set of periodic points is denoted by $\mathrm{Per}(\Xi)$. A point $s \in \mathbb{Z}^N$ is called **preperiodic** (w.r.t. Ξ), if there exists a $k \in \mathbb{N}$ such that $\Xi^k(s) \in \mathrm{Per}(\Xi)$.

Lemma 2.1. *If H is expanding and W a residue set of H, then every $s \in \mathbb{Z}^N$ is preperiodic w.r.t. Ξ.*

Proof. We have $s = H(\Xi(s)) + \zeta(s)$ for all $s \in \mathbb{Z}^N$. With respect to the norm $\|\bullet\|$ induced by H we

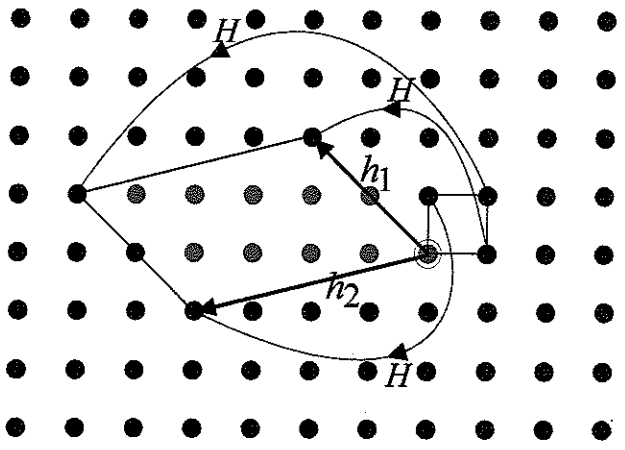


Fig. 3. Finding the basic residue set of $H = \begin{bmatrix} -2 & 4 \\ 2 & -1 \end{bmatrix}$. $h_1 = H\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$; $h_2 = H\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. The parallelogram is the image of the unit square under H. Points of the basic residue set are the ones displayed in red.

conclude that

$$\|\Xi(s)\| \leq \frac{\|s\| + R}{C}, \tag{2}$$

where $C > 1$ satisfies $\|H(s)\| \geq C\|s\|$ for all $s \in \mathbb{Z}^N$ and $R = \max\{\|w\| \,|\, w \in W\}$. Let $\rho = R/(C-1)$, then it follows that $\|\Xi(s)\| \leq \rho$ for all $s \in \mathbb{Z}^N$ with $\|s\| \leq \rho$ and $\|\Xi(s)\| < \|s\|$ when $\|s\| > \rho$. From this, and also from the discrete nature of the map Ξ, it follows that for any $s \in \mathbb{Z}^N$ there exists an $M \in \mathbb{N}$ such that $\|\Xi^m(s)\| \leq \rho$. Due to the finiteness of the elements in $\mathbb{Z}^n$ with $\|s\| \leq \rho$, the assertion follows. ∎

The proof in the above lemma shows that any orbit $(\Xi^k(s))_{k\in\mathbb{N}}$ eventually lands in the ball $\mathbf{B} = \{s \,|\, \|s\| \leq \rho\}$ and stays there. Also, as long as an orbit is outside $\mathbf{B}$, then its distance from the origin always decreases during its evolution.

Two points s_1 and s_2 are said to be Ξ-equivalent, $s_1 \sim_\Xi s_2$, if there exist n_1 and n_2 in $\mathbb{N}$ such that $\Xi^{n_1}(s_1) = \Xi^{n_2}(s_2)$. In other words, two points are **Ξ-equivalent** if they belong to orbits which eventually land in the same periodic orbit of Ξ. The quotient $\mathbb{Z}^n/\sim_\Xi$ is denoted by $P(\Xi)$.

If $s \in \mathbb{Z}^N$, then $\Xi^{-1}(s) = \{Hs + w | w \in W\}$ denotes the set of points which Ξ maps to s. Similarly, $\Xi^{-n}(s)$ $(n \in \mathbb{N})$ denotes the set of points which Ξ^n maps to s. Since every point $s \in \mathbb{Z}^N$ is preperiodic, we have that

$$\mathbb{Z}^N = \bigcup_{n\in\mathbb{N}} \Xi^{-n}(\mathrm{Per}(\Xi)).$$

for all $s \in \mathbb{Z}^N$. This observation leads to the definition of a generating set, which will be central in the construction of decimation-invariant sequences.

Definition 2.2. Let $H : \mathbb{Z}^N \to \mathbb{Z}^N$ be expanding and let W be a residue set for H. A set $S \subset \mathbb{Z}^N$ is called *generating* (w.r.t. $\Xi_{H,W}$) if

(i) $$\bigcup_{n\in\mathbb{N}} \Xi^{-n}(S) = \mathbb{Z}^N,$$

(ii) If $S' \subseteq S$ such that S' satisfies (i), then $S = S'$.

The first condition guarantees that S generates $\mathbb{Z}^N$ and the second condition establishes that S is minimal with this property. It is clear that a necessary and sufficient condition for S to be a generating set is that it contains one and only one element from each periodic orbit of Ξ.

Example 1. Let $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W = \{(2, 2), (0, -1)\}$. H is expanding under the Euclidian norm: $\|Hs\| \geq C\|s\|$, with $C = \sqrt{\lambda_{\max}(HH^T)} = \sqrt{2}$, and so $\rho = \sqrt{8}/(\sqrt{2}-1) \approx 6.8284$. The ball $\mathbf{B} = \{s \,|\, \|s\| \leq \rho\}$ contains the following six periodic orbits into which all Ξ-orbits arrive:

$$\begin{aligned}
&(-2, 2) \to (-2, 2)\\
&(1, 0) \to (1, 0)\\
&(0, 1) \to (1, 1) \to (-1, 0) \to (0, 1)\\
&(-2, 1) \to (0, 2) \to (-1, 1) \to (-2, 1)\\
&(-1, 2) \to (1, 2) \to (2, 1) \to (2, 0) \to (-1, -1)\\
&\qquad \to (-3, 0) \to (-1, 2)\\
&(-2, 0) \to (-3, 1) \to (-3, 2) \to (0, 3) \to (2, 2)\\
&\qquad \to (0, 0) \to (-2, 0).
\end{aligned}$$

$\mathrm{Fix}(\Xi) = \{(-2, 2), (1, 0)\}$, $\mathrm{Per}(\Xi)$ contains all the points appearing in the cycles above. $P(\Xi)$ contains six equivalence classes, each one containing all points that land in the same cycle.

For computational purposes, it is useful to represent a residue set W as an $N \times \Delta$ matrix $W_{[]}$ whose columns are the elements of W, i.e. $W_{[]} = [w_1, w_2, \ldots, w_\Delta]$. However, any interchange of columns in this matrix produces a new matrix which actually represents the same residue

set. Therefore, it will be convenient to consider distinct H-residue sets $W = \{w_1, w_2, \ldots, w_\Delta\}$ and $V = \{v_1, v_2, \ldots, v_\Delta\}$ as having an **H-consistent numbering**. By this it is meant that $v_i \equiv w_i \pmod{H}$, or $v_i = w_i + Hq$, where $q \in \mathbb{Z}^N$ (implying that w_i and v_i have similar positions in the point lattice generated by H). It follows that $\zeta_W(v_i) = w_i$ for all $i = 1, 2, \ldots, \Delta$. The condition for H-consistent numbering for two residue sets W and V is then implied by the matrix relationship $V_{[]} = W_{[]} + HQ$, where Q is an integer matrix, i.e. $Q \in Z^{N\times\Delta}$. Actually, when W is an H-residue set, then the set of all H-residue sets is given by the set of matrices

$$\mathcal{W}_H = \{W_{[]} + HQ | Q \in \mathbb{Z}^{N\times\Delta}\}. \tag{3}$$

3. Decimation Invariance and Basic Properties

We start with a straightforward adaption of the definition of one-dimensional p-decimation invariant sequences (p being an integer of absolute value ≥ 2), as given in [Barbé & Skordev, 2001], to the case involving an N-dimensional expansive map H with associated residue set W. Letting $\kappa_W = \{\kappa_w \in \mathbb{Z}^N | w \in W\}$, this leads to the concept of a **($\mathbf{H}$, $\mathbf{W}$, κ_W)-decimation invariant sequence** $f : \mathbb{Z}^N \to \mathbb{N}$ if for each $w \in W$

$$f(Hs + w) = f(s + \kappa_w) \tag{4}$$

holds for all $s \in \mathbb{Z}^N$. In other words, f is (H, W, κ_W)-decimation invariant when all (H, W)-induced decimations of f, i.e. all subsequences $(f(Hs + w))_{s\in\mathbb{Z}^N}$, equal the original sequence modulo a shift by κ_w.

Lemma 3.1. *Let Ξ and ζ be the image-part map and remainder-part map w.r.t. (H, W), respectively. If f is a (H, W, κ_W)-DI sequence and if V is a second residue set of H, then f is also an (H, V, κ_V)-DI sequence, with $\kappa_v = \Xi(v) + \kappa_{\zeta(v)}$ for $v \in V$.*

Proof. Since f is an (H, W, κ_W)-DI sequence, we derive for $v \in V$ that

$$f(Hs + v) \overset{(1)}{=} f(H(s + \Xi(v)) + \zeta(v)) \overset{(4)}{=} f(s + \Xi(v) + \kappa_{\zeta(v)}).$$

The assertion follows from the observation that (4) holds when setting $\kappa_v = \Xi(v) + \kappa_{\zeta(v)}$ for $v \in V$. ■

This lemma actually shows that the specification of a residue set W and of the associated κ_W is not uniquely determined. If there exists a pair (W, κ_W) such that (4) is satisfied, such a pair does indeed exist for any residue set V. It legitimizes the following

Definition 3.2. A sequence f is called **H-decimation invariant**, or H-DI sequence, if there is a (W, κ_W)-pair such that (4) is satisfied.

The next theorem however shows that there always exists a *normal form* of Eq. (4), by allowing the condition $\kappa_w = 0$ for all $w \in W$.

Theorem 3.3. *The sequence f is an H-DI sequence if and only if there exists a residue set W of H such that*

$$f(Hs + w) = f(s) \tag{5}$$

holds for all $s \in \mathbb{Z}^N$ and all $w \in W$.

Proof. If such a W exists, then f is an H-DI sequence, by definition.

Now suppose that f is an H-DI sequence, i.e.

$$f(Hs + v) = f(s + \kappa_v) \tag{6}$$

for a residue set V. The set

$$W = \{v - H\kappa_v | v \in V\} \tag{7}$$

is also an H-residue set and we have that

$$f(Hs + (v - H\kappa_v)) = f(H(s - \kappa_v) + v) \overset{(6)}{=} f(s - \kappa_v + \kappa_v) = f(s)$$

holds for all $s \in \mathbb{Z}^n$ and $v \in V$. Therefore the residue set W is of the desired type. ■

Equation (5) will be considered as the **normal form** for dealing with H-DI sequences. From here on, we will exclusively consider the normal form for all purposes where the context requires the specification of a corresponding residue set W. This will be done by referring to an H-DI sequence satisfying (5) as a **($\mathbf{H}$, $\mathbf{W}$)-DI sequence**.

The following result shows that the property of a sequence being H-DI is invariant under a shift.

Lemma 3.4. *Let f be an (H, W)-DI sequence. The shifted sequence $\sigma_t(f)$ is an (H, V)-DI sequence if and only if*

$$V = W + (H - I)t\,, \tag{8}$$

I being the identity matrix of the same dimension as H. Moreover, $V \neq W$ for $t \neq 0$.

Proof. Let f be an (H, W)-DI sequence and define $\kappa_w = \Xi_W(w + t) - t$ for all $w \in W$, with Ξ_W the image part map w.r.t. (H, W). We compute that

$$\begin{aligned}\sigma_t(f)(Hs+w) &= f(Hs+w+t)\\ &\overset{(1)}{=} f(H(s+1\Xi_W(w+t))+\zeta_W(w+t))\\ &\overset{(5)}{=} f(s+\Xi_W(w+t))\\ &= f(s+t+(\Xi_W(w+t)-t))\\ &= \sigma_t(f)(s+\kappa_w)\,.\end{aligned}$$

This already shows that $\sigma_t(f)$ is an (H, W, κ_W)-DI sequence. By Theorem 3.3, $\sigma_t(f)$ is a (H, V)-DI sequence with $V = \{w - H(\Xi_W(w + t) - t)|w \in W\}$. Using $w + t = H\Xi_W(w + t) + \zeta_W(w + t)$ yields $V = \{\zeta_W(w+t)+Ht-t|w \in W\} = \{\zeta_W(w+t)|w \in W\} + (H - I)t$. Observing that $\{\zeta_W(w + t)|w \in W\} = W$ gives (8).

The arguments for the *only if*-part take the reverse direction.

Since H is expanding, i.e. all eigenvalues have modulus > 1, it follows that $Ht \neq t$ for all $t \neq 0$. Therefore $V \neq W$ for all $t \neq 0$. ■

Consider a change of coordinates induced by a matrix $B \in Gl(N, \mathbb{Z}) = \{A \in \mathbb{Z}^{N\times N} | |\det A| = 1\}$ (i.e. $s \mapsto B^{-1}s$ defines the transition from the old coordinates to the new coordinates). Then the expanding map H in the old coordinates transforms into the expanding map $\hat{H} = B^{-1}HB$ in the new coordinates. H and $\hat{H}$ are called equivalent. Residue sets for equivalent matrices have clearly the same cardinality. The following lemma concerns the decimation invariance of a sequence considered in the new coordinates.

Lemma 3.5. *Let f be an (H, W)-DI sequence, and let $\hat{H} = B^{-1}HB$, with $B \in Gl(N, \mathbb{Z})$. Then the sequence g, defined by $g(s) = f(Bs)$ is $(\hat{H}, B^{-1}(W))$-DI.*

Proof. Observe that $g(\hat{H}s + B^{-1}(w)) = f(B(\hat{H}s + B^{-1}(w))) = f(HBs+w) \overset{f \text{ is } (H,W)-\text{DI}}{=} f(Bs) = g(s)$. ■

The construction of (H, W)-DI sequences relies on the next lemma.

Lemma 3.6. *If f is an (H, W)-DI sequence, and Ξ the image-part map w.r.t. (H, W), then*

$$f(\Xi(s)) = f(s)$$

for all $s \in \mathbb{Z}^N$.

Proof. The assertion follows from

$$f(s) \overset{(1)}{=} f(H\Xi(s) + \zeta(s)) \overset{(4)}{=} f(\Xi(s))\,. \quad ■$$

As a consequence, we note that the range of an (H, W)-DI sequence f is finite. To be more precise, the cardinality of $f(\mathbb{Z}^N)$ is at most equal to the number of different periodic orbits of Ξ, i.e. $|f(\mathbb{Z}^N)| = |P(\Xi)|$. An (H, W)-DI sequence with this property is said to be of **maximal diversity**.

The problem of finding all sequences which are (H, W)-DI, is referred to as the **(H, W)-DI problem**. The following theorem leads to its solution.

Theorem 3.7. *Let f_1, f_2 both be (H, W)-DI sequences, and let S be a generating set w.r.t. $\Xi_{H,W}$, then $f_1 = f_2$ if and only if $f_1(s) = f_2(s)$ for all $s \in S$.*

Proof. This follows from the fact that $f(t_1) = f(t_2)$ if $t_1 \sim_\Xi t_2$ for $t_1, t_2 \in \mathbb{Z}^N$. ■

Theorem 3.7 shows that an (H, W)-DI solution is completely determined by its values on a generating set S, which depends, through Ξ, on both H and W. It provides a way to find all solutions to a given (H, W)-DI problem. First a solution of maximal diversity is generated. To this end, determine a generating set S by taking one and only one element from each periodic orbit of Ξ (see Definition 2.2). Then attribute a *different* value $f(s) \in \mathbb{N}$ to each element s of this generating set. A solution of maximal diversity is constructed by letting $f(\Xi^{-n}(s)) = f(s)$ for $n = 1, 2, 3, \ldots$ and for all $s \in S$.

In the context of this paper, there is no need to distinguish between sequences which are **symbolically equivalent**, i.e. sequences which can be mapped into each other under a one to one symbolic substitution of sequence-values (which were considered to be elements in $\mathbb{N}$). Formally: two sequences $f_1 : \mathbb{Z}^N \to f_1(\mathbb{Z}^N)$ and $f_2 : \mathbb{Z}^N \to f_2(\mathbb{Z}^N)$ with finite range, respectively, are symbolically equivalent if there exists a bijective map $\alpha : f_1(\mathbb{Z}^N) \to f_2(\mathbb{Z}^N)$ such that $\alpha(f_1(s)) = f_2(s)$ holds for all $s \in \mathbb{Z}^N$. We write $f_1 \sim_S f_2$, and $[f]_S$ denotes the class of all sequences that are symbolically equivalent to f. Under symbolic equivalence, we can actually speak about *the* solution of maximal diversity.

From the solution of maximal diversity for a given (H, W)-DI problem, all other solutions can be obtained by **range-reduction** of the solution of maximal diversity, i.e. by identifying distinct sequence-values. This is equivalent to letting distinct elements of S map into a same value (and also the corresponding expansions of these elements under Ξ^{-n}, $n = 1, 2, \ldots$). As a consequence, the number of DI-sequences equals the number of partitions of the generating set S, the so-called Bell-number $B(|S|)$ of $|S| = |P(\Xi_{H,W})|$, [Lüneburg, 1989]. We summarize this in

Lemma 3.8. *Let $\mathcal{F}_{H,W}$ denote the set of all (H, W)-DI sequences modulo symbolic equivalence. Then this set is completely determined by the solution of maximal diversity f_{md} whose range has cardinality $|S| = |P(\Xi_{H,W})|$: it contains f_{md} and all its range-reduced versions. Moreover, $\mathcal{F}_{H,W}$ has cardinality $B(|S|)$.*

Example 2 (Example 1 continued). This example concerns the (H, W)-DI problem with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W = \{(2, 2), (0, -1)\}$. Selecting one and only one element from the six periodic orbits of the corresponding Ξ as obtained before produces for example the generating set $S = \{(-2, -2), (1, 0), (0, 1), (-2, 1)(-1, 2)(-2, 0)\}$. Figure 4 displays (a part) of the solution of maximal diversity as a pixel-based picture, where all pixels at locations $\Xi^{-n}(s)$ (for $n = 1$ to n sufficiently large) are given the color ("value") yellow for $s = (-2, 2)$, black for $s = (1, 0)$, green for $s = (0, 1)$, purple for $s = (-2, 1)$, blue for $s = (-1, 2)$, and red for $s = (-2, 0)$. A spiraling fractal-like pattern seems to emerge. Observe that the green pixel-set forms an overall periodic pattern. The same is true for the red, blue and purple pixel-sets. The yellow and black pixels produce the spiraling pattern. Solutions which are not of maximal diversity are obtained by equating two or more colors in this picture. For example, making all black pixels also yellow, will produce an overall periodic solution with five values (colors). In total, there are $B(6) = 203$ different solutions (patterns), i.e. solutions which are not symbolically equivalent. The number of nonperiodic solutions (showing the yellow and/or black spiral feature) equals $2 * B(5) = 104$.

We already know from Lemma 3.4 that, if $V = W + (H - I)t$, any (H, V)-DI sequence is the t-shifted version of a corresponding (H, W)-DI sequence. Now, the graphical representations of all shifted versions of a sequence essentially display the same pattern (whereby shifts are considered to be irrelevant). Therefore, we introduce the concept of **pattern-equivalence** between two sequences: Two sequences f_1 and f_2 are pattern-equivalent, and we write $f_1 \sim_P f_2$, if there exists a $t \in \mathbb{Z}^N$ such that $f_1 \sim_S \sigma_t(f_2)$. Then $[f]_P = \cup_{t \in \mathbb{Z}^N} [\sigma_t(f)]_S$ forms the class of all sequences which are pattern equivalent to f. The graphical representation of any sequence in $[f]_P$ is a representative for the associated pattern. Also, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sets of sequences, we say that $\mathcal{F}_1 \sim_P \mathcal{F}_2$ if, for all $f \in \mathcal{F}_1$, there is a $g \in \mathcal{F}_2$ such that $f \sim_P g$ and if, for all $g \in \mathcal{F}_2$, there is an $f \in \mathcal{F}_1$ such that $g \sim_P f$. The pattern-set associated with $\mathcal{F}$ is then the set of patterns associated with the sequences in $\mathcal{F}$. Lemma 3.4 then tells us that the solution-set $\mathcal{F}_{H,V}$, if $V = W + (H - I)t$, is pattern-equivalent to the solution-set $\mathcal{F}_{H,W}$, for all $t \in \mathbb{Z}^N$. We will now consider the question on pattern-equivalence between (H, V_γ)-DI solution-sets when V_γ ranges over *all* shifted versions of some fixed H-residue set W, i.e. when $V_\gamma = W + \gamma$, $\gamma \in \mathbb{Z}^N$.

First observe that when $\Gamma_0 = \{(H-I)t | t \in \mathbb{Z}^N\}$ represents the point lattice generated by the matrix $(H - I)$ and R is a residue set for $(H - I)$, then $\mathbb{Z}^N/\Gamma_0 = \{\Gamma_s = \Gamma_0 + s | s \in R\}$ forms a partition of $\mathbb{Z}^N$, consisting of the distinct shifted versions of Γ_0. The number of classes in this partition equals $|\det(H - I)|$. Lemma 3.4 can now be generalized as

Lemma 3.9. *Let W and $V_\gamma = W + \gamma$, $\gamma \in \mathbb{Z}^N$ be H-residue sets. Then*

$$\mathcal{F}_{H,V\gamma_1} \sim_P \mathcal{F}_{H,V\gamma_2} \quad \text{if } \gamma_1 - \gamma_2 \in \Gamma_0\,, \qquad (9)$$

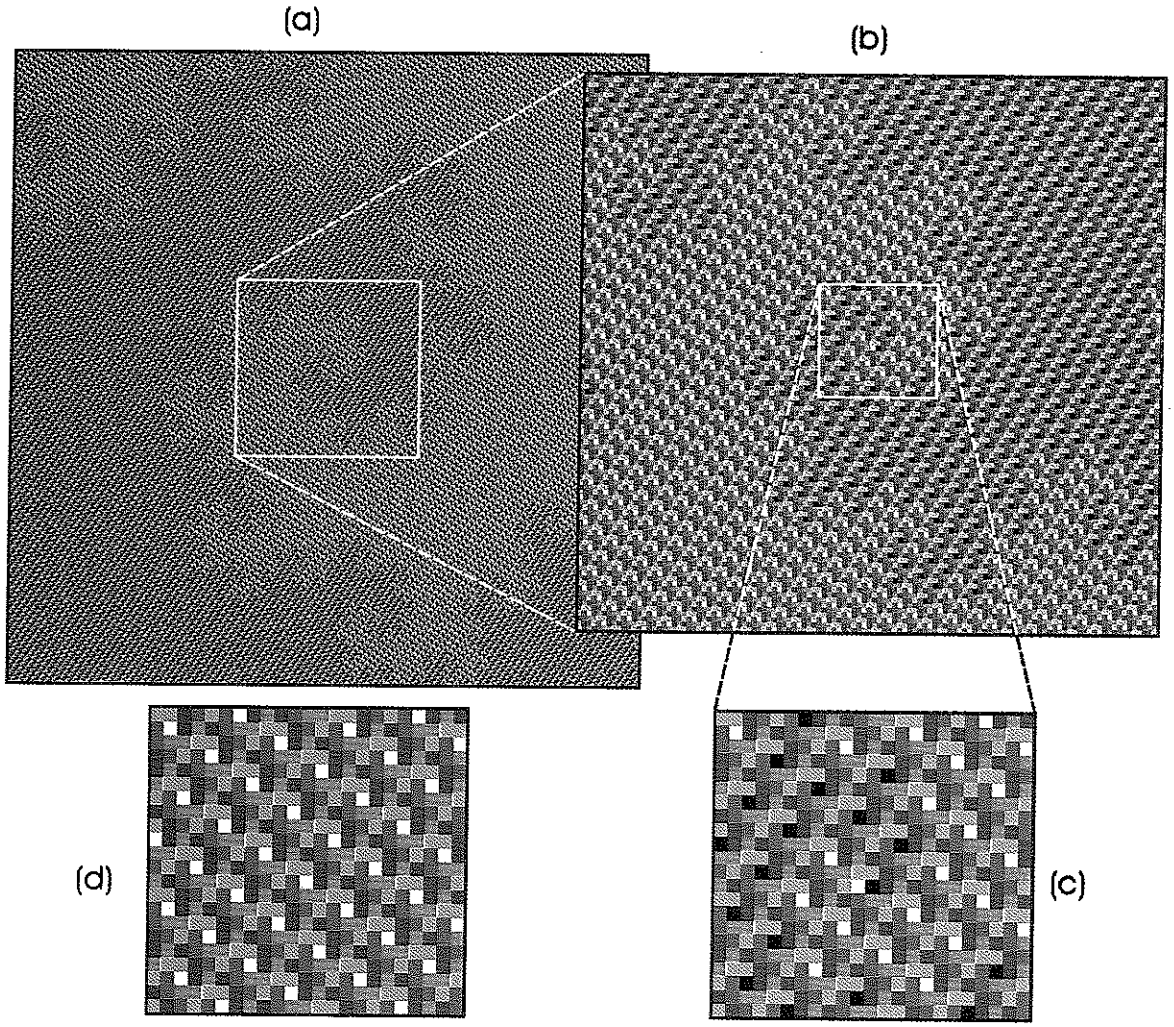


Fig. 4. (a) Representation as a pixel-based image of the 6-valued maximal diversity solution of the (H, W)-DI problem with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W = \{(2, 2), (0, -1)\}$, (b and c) consecutive enlargements. The red, green, blue and purple pixel-sets form a periodic pattern. The white pixel represents the location of the origin (it should actually be red). (d) A periodic solution obtained by making the black pixels in (c) also yellow.

There are at most $|\det(H-I)|$ *distinct pattern-sets associated with the solution-sets of* $\mathcal{F}_{V_\gamma}$, $\gamma \in \mathbb{Z}^N$.

Proof. As $\gamma_1 - \gamma_2 \in \Gamma_0$, it holds that $\gamma_1 = \gamma_2 + (H-I)t$ for some $t \in \mathbb{Z}^N$. Then

$$\begin{aligned} \mathcal{F}_{H,V\gamma_1} &= \mathcal{F}_{H,W+\gamma_2+(H-I)t} \\ &= \{f | f \text{ is } (H, W+\gamma_2+(H-I)t)\text{-DI}\} \\ &\overset{(8)}{=} \{\sigma_{-t}(f) | f \text{ is } (H, W+\gamma_2)\text{-DI}\} \\ &= \sigma_{-t}\mathcal{F}_{H,V\gamma_2}\,. \end{aligned}$$

The assertion concerning the number of pattern-sets results from the observation that (9) is also equivalent to $\mathcal{F}_{H,V_\gamma} \sim_P \mathcal{F}_{H,V_S}$ for all $\gamma \in \{s + \Gamma_0 = \Gamma_s\}$, $s \in R$ (R being an $(H-I)$-residue set). ■

Example 3. Consider the (H, V_γ)-DI problems with $H = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $V_\gamma = W + \gamma$, $\gamma \in \mathbb{Z}^2$, where $W = \{(0, 0), (-2, 0), (2, 0), (0, -2)(0, 2)\}$. As $|\det(H-I)| = 4$, only four different pattern-sets may result. These are given by the pattern-sets corresponding to the solutions of the $(H, V_\gamma = W+\gamma)$-DI problems with γ belonging to the $(H-I)$-residue set $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Figure 5 shows the corresponding solutions of maximal diversity (the complete solution-sets can be obtained by range-reduction of the solutions of maximal diversity).

Note that the pattern-sets $\mathcal{F}_{H,V_{\gamma_1}}$ and $\mathcal{F}_{H,V_{\gamma_2}}$ for γ_1 and γ_2 such that $\gamma_1 - \gamma_2 \notin \Gamma_0$ need not be different. For example, for the (H, V_γ)-DI problem

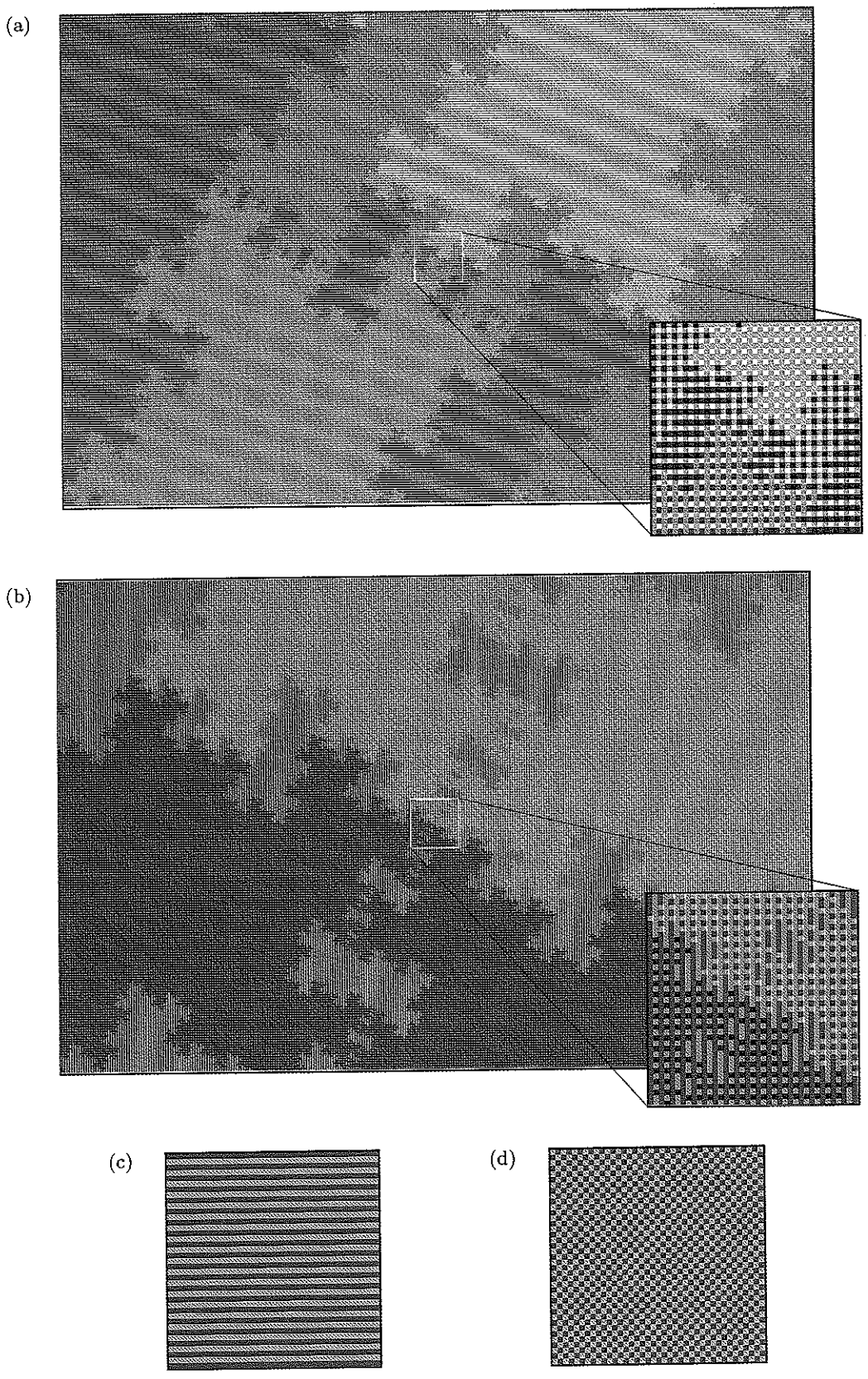


Fig. 5. Representative patterns of maximal diversity for the (H, V_γ)-DI problems with $H = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $V_\gamma = \{\{(0, 0), (2, 0)(-2, 0), (0, 2), (0, -2)\} + \gamma, \gamma \in \mathbb{Z}^2$. The representative patterns are obtained for (a) $\gamma = (0, 0)$; (b) $\gamma = (1, 0)$; (c) $\gamma = (0, 1)$; (d) $\gamma = (1, 1)$.

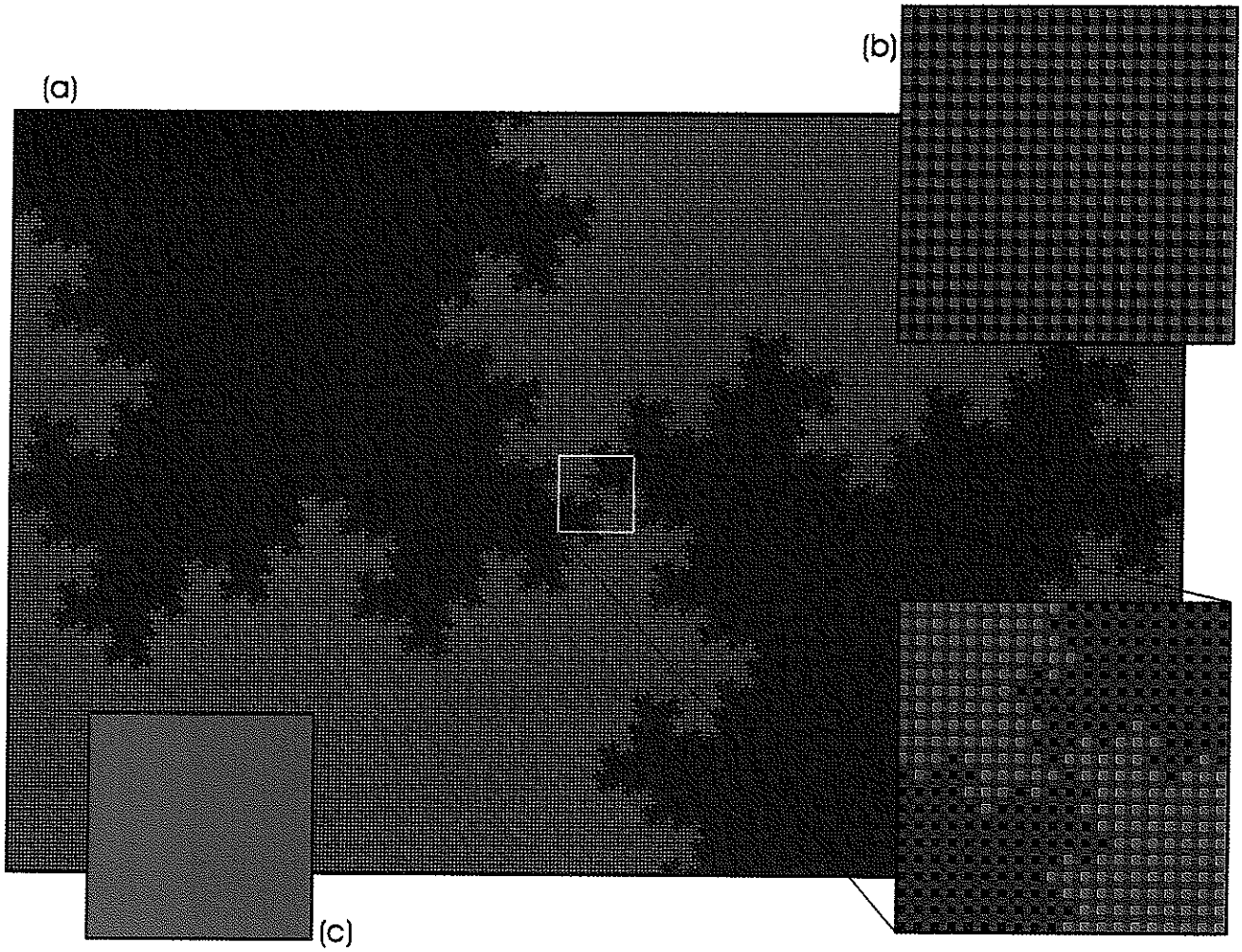


Fig. 6. Representative patterns of maximal diversity for the (H, V_γ)-DI problems with $H = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ and $V_\gamma = \{(0, 0), (2, 0)(-2, 0), (0, 2), (0, -2)\} + \gamma$, $\gamma \in \mathbb{Z}^2\}$. The representative patterns are obtained for (a) $\gamma = (0, 0)$; (b) $\gamma = (-2, 0)$, $(-1, -1)$, $(-3, -1)$, $(-2, -2)$ (periodic pattern); (c) $\gamma = (-1, 0)$, $(-3, 0)$, $(-2, -1)$, $(-1, -2)$, $(-3, -2)$ (trivial pattern).

with $H = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ and $V_\gamma = W + \gamma$, $\gamma \in \mathbb{Z}^2$ with $W = \{(0, 0), (2, 0), (-2, 0), (0, 2), (0, -2)\}$, there are $|\det(H - I)| = 10$ potentially different pattern-sets $\mathcal{F}_{H,V_\gamma}$ (corresponding to potentially different patterns of maximal diversity). However, there are only three different patterns of maximal diversity (see Fig. 6): W produces a fractal-like pattern of maximal diversity; $W + (-1, 0)$, $W + (-3, 0)$, $W+(-2, -1)$, $W+(-1, -2)$, $W+(-3, -2)$ all produce the trivial pattern; $W+(-2, 0)$, $W+(-1, -1)$, $W+(-3, -1)$, and $W+(-2, -2)$ produce the same periodic pattern.

All examples above allow nontrivial periodic solutions (at least by range-reduction of the solution of maximal diversity). That this is not always the case was already shown in the example of Fig. 2.

Let us now consider the question whether it is possible for an (H, W)-DI sequence to be also (H, V)-DI, where $W \neq V$. The following theorem shows that this is only the case when f has some **periodic direction** θ, i.e. when there is some $\theta \in \mathbb{Z}^N \backslash \{0\}$ such that $f(s) = f(s+\theta)$ for all $s \in \mathbb{Z}^N$. A sequence with no periodic direction will be called **nonperiodic**.

Theorem 3.10. *An H-DI sequence f has a unique normal representation as a (H, W)-DI sequence if and only if f is nonperiodic.*

Proof

1. If the normal representation is not unique, then f has a periodic direction.

Indeed, let f be (H, W)-DI as well as (H, V)-DI, with $W \neq V$. Then there is at least one $v \in V$ such that $v \notin W$. Because f is (H, V)-DI, it holds that for all $s \in \mathbb{Z}^N$

$$f(Hs + v) = f(s)\,. \tag{10}$$

With $v \overset{(1)}{=} H\Xi_W(v) + \zeta_W(v)$, this becomes

$$f(Hs + H\Xi_W(v) + \zeta_W(v)) = f(s)\,, \tag{11}$$

for all $s \in \mathbb{Z}^N$. Because f is (H, W)-DI, the left part of (11) also equals $f(s + \Xi_W(v))$. Thus we conclude that, for all $s \in \mathbb{Z}^N$:

$$f(s + \Xi_W(v)) = f(s)\,. \tag{12}$$

As $\Xi_W(v) \neq 0$ (otherwise $v = \zeta_W(v) \in W$), the sequence f has the periodic direction $\Xi_W(v)$.

2. If f is an H-DI sequence with periodic direction θ, i.e. $f(s + \theta) = f(s)$ for all $s \in \mathbb{Z}^N$, then the normal form is not unique.

Indeed, assume that f is (H, W)-DI. The periodicity implies that $f = \sigma_\theta(f)$. By Lemma 3.4, $\sigma_\theta(f)$ and thus also f are (H, V)-DI, with $V = W + (H - I)\theta$ and $V \neq W$ since $\theta \neq 0$. ■

The example presented thus far, and also Theorem 3.10 indicate that H-DI sequences with periodic features form a special class. The next section will investigate conditions for periodicity, and will also provide a characterization of the class of all residue sets which can appear in the normal representation of a *periodic H-DI sequence.*

4. Periodic H-DI Sequences

In this section we consider N-dimensional H-DI sequences which are periodic. A sequence $f : \mathbb{Z}^N \to \mathbb{Z}$ is called **periodic** of rank K if there exists a subgroup $\mathcal{G} \subset \mathbb{Z}^N$ (called **period**) of rank K such that $f(s + g) = f(s)$ holds for all $s \in \mathbb{Z}^N$ and all $g \in \mathcal{G}$. A subgroup $\mathcal{G}$ of rank K is given by

$$\mathcal{G} = \left\{ \sum_{j=1}^{K} \alpha_j g_j | \alpha_j \in \mathbb{Z} \right\},$$

where the $N \times K$-matrix $G = [g_1 g_2 \cdots g_K]$ formed by the N-dimensional column vectors $g_i \in \mathbb{Z}^N$, $i = 1 \ldots, K$, has rank K. The subgroup $\mathcal{G}$ has **full rank** when its rank equals N. A matrix $G \in \mathbb{Z}^{N \times K}$ is said to generate $\mathcal{G}$ if $G(\mathbb{Z}^K) = \mathcal{G}$, and we write $\langle G \rangle = \mathcal{G}$. In geometrical terms, $\mathcal{G}$ defines a sublattice in the integer point-lattice $\mathbb{Z}^N$ [Senechal, 1995]. Note that G is by no means uniquely determined: In fact if $P \in Gl(K, \mathbb{Z}) = \{A \in \mathbb{Z}^{K \times K} | \det(A) = \pm 1\}$, then GP generates the same group $\mathcal{G} = \langle G \rangle$.

If f has the period $\mathcal{G}$, then every subgroup of $\mathcal{G}$ is also a period of f. A period $\mathcal{G}$ is a **minimal period** of f if there exists no period $\mathcal{G}'$ of f such that $\mathcal{G}$ is a proper subgroup of $\mathcal{G}'$, i.e. a period of f is a maximal subgroup w.r.t. inclusion among all subgroups of $\mathbb{Z}^N$ which are possible periods of f. For full-rank periods, it means that $|\det G|$ is minimal. The constant sequence is an H-DI sequence for all H: it is the **trivial solution** for any (H, W)-DI problem. It has the **trivial period** $\mathbb{Z}^N$ as minimal period.

Observe that, if $\mathcal{G}$ is a period of f, then any $g \in \mathcal{G}$ is a periodic direction for f; if f has a periodic direction $g \notin \mathcal{G}$, then $\mathcal{G}$ is not minimal.

In analogy to DI-sequences of maximal diversity, we introduce here also **periodic DI-sequences of maximal diversity**: An (H, W)-DI sequence f with period $\mathcal{G}$ is of maximal diversity if the cardinality of $f(\mathbb{Z}^N)$ is greater than or equal to the cardinality of any other (H, W)-DI sequence g with the same period.

The first theorem substantiates the observation in the previous section that an H-DI sequence is periodic if and only if it has more than one normal representation as an (H, W)-DI sequence.

Theorem 4.1. *An (H, W)-DI sequence has period $\mathcal{G}$ if and only if it is also (H, V)-DI, for all V of the form*

$$V = \{w + Hg_w | w \in W\} \tag{13}$$

with $g_w \in \mathcal{G}$. Moreover, $\mathcal{G}$ is a minimal period, if f is (H, V)-DI with V exclusively of the form (13).

Proof

1. If f is a periodic (H, W)-DI sequence with period $\mathcal{G}$, then it is also (H, V)-DI with V of the form (13). Indeed, let V be of the given form, i.e. $v_w = w + Hg_w \in V$ for all $w \in W$. Then, for all $s \in \mathbb{Z}^N$: $f(Hs + v_w) = f(Hs + w + Hg_w) = f(H(s + g_w) + w) \overset{(\alpha)}{=} f(s + g_w) \overset{(\beta)}{=} f(s)$ (equality (α) follows from f being (H, W)-DI, and (β) from its periodicity). The equality between the first and last terms in this chain of equalities then implies that f is (H, V)-DI.

2. If f is (H, V)-DI for all V satisfying (13), then it has period $\mathcal{G}$.

Indeed, (H, V)-DI of f implies that

$$f(Hs + w + Hg_w) = f(s). \tag{14}$$

As this holds for all $g_w \in \mathcal{G}$, it also follows that f is (H, W)-DI (take $g_w = 0$), implying that

$$f(Hs + w + Hg_w) \overset{(5)}{=} f(s + g_w). \tag{15}$$

Combining (14) and (15) gives $f(s + g_w) = f(s)$ for all $s \in Z^N$ and for all $g_w \in \mathcal{G}$. It follows that $\mathcal{G}$ is a period of f.

Suppose that $\mathcal{G}$ is not minimal, then there is an additional $g' \notin \mathcal{G}$ such that $\langle g' \rangle$ is also a period, and by (13), this would imply that the sequence is also (H, V')-DI for $V' = \{w + Hg' | g' \in \langle g' \rangle, w \in W\}$, which does not belong to the set of V-residue sets of the form (13). ■

Remark. Writing the residue sets as $N \times \Delta$ matrices, and G being an $N \times K$ generator matrix of $\mathcal{G}$ of rank K, expression (13) becomes

$$V_{[]} \in \mathcal{W} = \{W_{[]} + HGQ | Q \in \mathbb{Z}^{K \times \Delta}\}. \tag{16}$$

Compare to (3), and observe also that this set is invariant under any transformation $G \mapsto GP$, $P \in Gl(K, \mathbb{Z})$, as it should, because $\langle G \rangle = \langle GP \rangle = \mathcal{G}$.

Example 4 (Example 1 continued). The periodic pattern in Fig. 4(d) is a solution for an (H, W)-DI problem with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W_{[]} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}$. The period is $\mathcal{G} = \langle G \rangle = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}$. The previous theorem says that this periodic sequence is also an (H, V)-DI sequence for all V of the form $V_{[]} = W_{[]} + HGQ$, $Q \in \mathbb{Z}^{2\times 2}$. For example, $V_{[]} = W_{[]} + HG\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 10 \end{bmatrix}$. Figure 7(a) shows the solution of maximal diversity of the corresponding (H, V)-DI problem: it is a sequence with 16 different values (colors). Again a spiraling fractal-like structure is hidden in this pattern, as shown in the enlargements (b) and (c). Identifying some of the colors produces the periodic pattern (d), which is identical to the periodic solution in Fig. 4(d).

The previous theorem required the specification of a W-residue set for which there exists a $\mathcal{G}$-periodic (H, W)-DI sequence. The rest of this section will deal with conditions between H, W, $\mathcal{G}$ which allow or even ensure the existence of an (H, W)-DI sequence having $\mathcal{G}$ as a minimal period. It starts with some necessary conditions.

Lemma 4.2. *Let f be an H-DI sequence and let $\mathcal{G}$ be a minimal period of f. Then the following statements hold*

(i)
$$H(\mathcal{G}) \subset \mathcal{G} \tag{17}$$

(ii)
$$\text{If } t \in \mathbb{Z}^N \text{ such that } Ht \in \mathcal{G}, \quad \text{then } t \in \mathcal{G}. \tag{18}$$

(iii)
$$\Xi(W + \mathcal{G}) \subseteq \mathcal{G} \tag{19}$$

Proof. Recall that Ξ and ζ are the image-part and remainder-part map with respect to (H, W), where W is an H-residue set appearing in the normal form for f.

(i) For $s \in \mathbb{Z}^N$ and $g \in \mathcal{G}$ we have

$$\begin{aligned} f(s + Hg) &\overset{(1)}{=} f(\zeta(s) + H(\Xi(s) + g)) \\ &= f(\Xi(s) + g) \text{ (f is DI-sequence)} \\ &= f(\Xi(s)) \text{ (periodicity of f)} \\ &= f(s) \text{ (Lemma 3.6)}. \end{aligned}$$

As this holds for all $s \in \mathbb{Z}^N$ and since $\mathcal{G}$ is a minimal period it follows that $Hg \in \mathcal{G}$.

(ii) For $t \in \mathbb{Z}^N$ such that $Ht \in \mathcal{G}$ we have that

$$\begin{aligned} f(s) &= f(s + Ht) \text{ (periodicity)} \\ &\overset{(1)}{=} f(\zeta(s) + H(\Xi(s) + t)) \\ &= f(\Xi(s) + t) \text{ (f is DI)}. \end{aligned}$$

On the other hand, f satisfies $f(s) = f(\Xi(s))$ (Lemma 3.6), which yields $f(\Xi(s)) = f(\Xi(s) + t)$ for all $s \in \mathbb{Z}^N$. Since $\Xi : \mathbb{Z}^N \to \mathbb{Z}^N$ is surjective it follows $f(s) = f(s + t)$ for all $s \in \mathbb{Z}^N$. The minimality of $\mathcal{G}$ implies that $t \in \mathcal{G}$.

(iii) For $s \in \mathbb{Z}^N$, $w \in W$ and $g \in \mathcal{G}$ we have

$$\begin{aligned} f(s + g) &= f(s) \\ &= f(w + Hs) \text{ (f is DI)} \\ &= f(w + g + Hs) \text{ (periodicity)} \\ &\overset{(1)}{=} f(\zeta(w + g) + H(\Xi(w + g) + s)) \\ &= f(s + \Xi(w + g)) \text{ (f is DI)}. \end{aligned}$$

Again, the minimality of $\mathcal{G}$ implies that $\Xi(w + g) \in \mathcal{G}$. ■

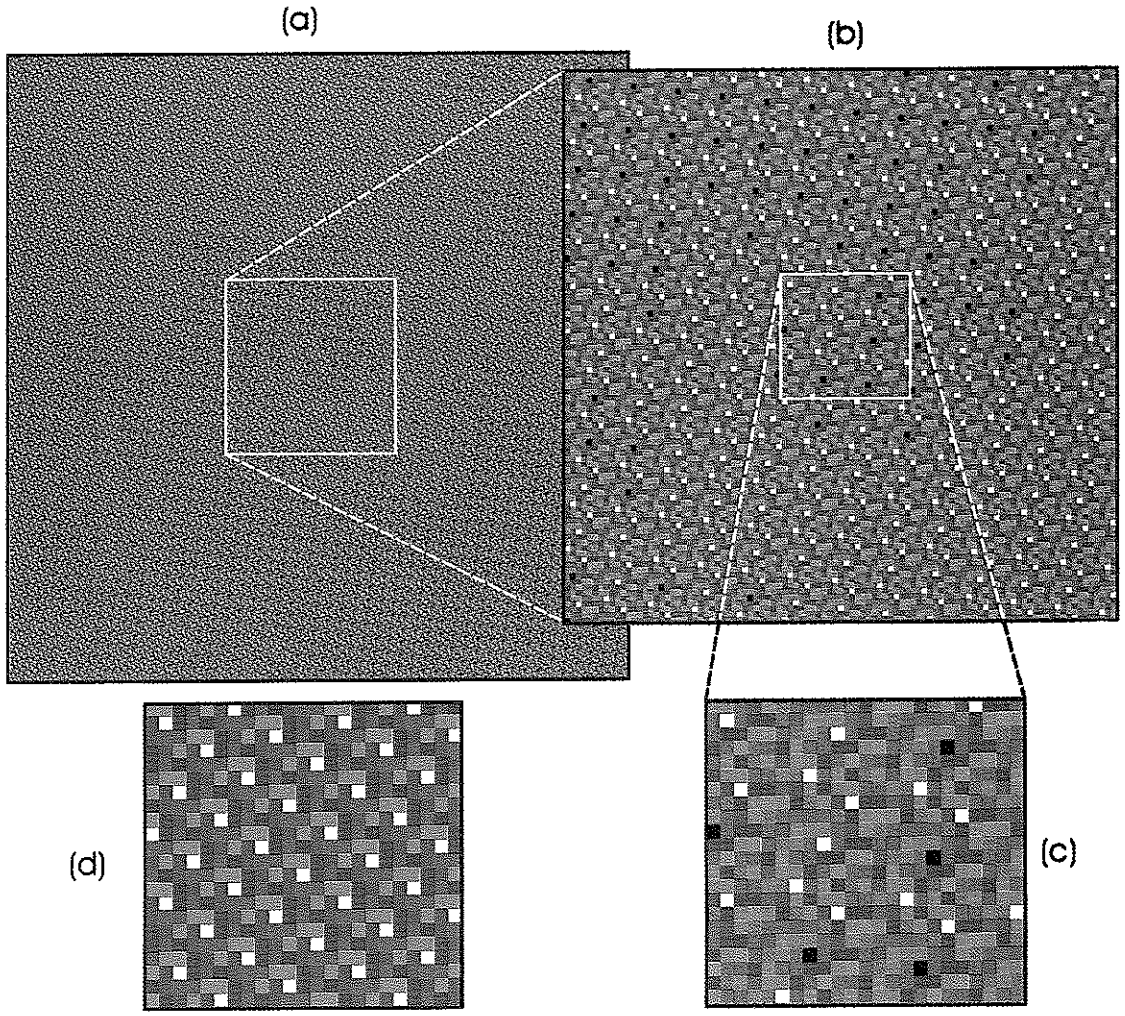


Fig. 7. (a) The 16-valued maximal diversity solution of the (H, V)-DI problem with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $V = \{(2, 2), (3, 10)\}$, (b and c) consecutive enlargements. The white and black pixels form a fractal-like spiral embedded in a periodic background. This solution of maximal diversity can be turned into an overall periodic solution of maximal diversity (make all black pixels white). As can be seen from (c), this solution has period $\left\langle \begin{bmatrix} 8 & 1 \\ -1 & 8 \end{bmatrix} \right\rangle$. (d) As shown in Example 4, following Theorem 4.1, identifying more colors in the solution of maximal diversity must also generate this periodic solution which is the same as the one displayed in Fig. 4(d) $\left(\text{period } \left\langle G = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix} \right\rangle\right)$.

In what follows, we will exclusively consider H-DI sequences with full-rank period. The following theorem and corollary provide a relationship between a period $\mathcal{G}$ and the residue set W for a (H, W)-DI sequence.

Theorem 4.3. *If f is an (H, W)-DI sequence with minimal period $\mathcal{G}$ of full rank, then*

$$w - v \in \mathcal{G} \tag{20}$$

holds for all $w, v \in W$.

Proof. Since $\mathcal{G}$ has rank N, it follows that the quotient $\mathbb{Z}^N/\mathcal{G} = \{L(t) = t + \mathcal{G} | t \in \mathbb{Z}^N\}$ is a finite commutative group under the addition $\oplus$ defined by $L(t_1) \oplus L(t_2) = L(t_1 + t_2)$.

By (i) of Lemma 4.2, the map

$$H_{\mathcal{G}} : \mathbb{Z}^N/\mathcal{G} \to \mathbb{Z}^N/\mathcal{G}$$

$$L(t) = t + \mathcal{G} \mapsto L(Ht) = Ht + \mathcal{G},$$

or

$$H_{\mathcal{G}}(t + \mathcal{G}) = Ht + \mathcal{G} \tag{21}$$

is well defined. Indeed, as $L(t+g) = L(t)$ for $g \in \mathcal{G}$, well-definedness of $H_{\mathcal{G}}$ requires that $L(H(t+g)) = L(Ht)$. This is the case, as $L(H(t+g)) = L(Ht + Hg) \overset{(17)}{=} L(Ht)$. Moreover, $H_{\mathcal{G}}$ forms a group automorphism such that for the map $\pi : \mathbb{Z}^N \to \mathbb{Z}^N/\mathcal{G}$,

$t \mapsto t + \mathcal{G}$, it holds that

$$H_{\mathcal{G}} \circ \pi = \pi \circ H\,.$$

Indeed, for any $t_1, t_2 \in \mathbb{Z}^N$, it holds that $H_{\mathcal{G}}(L(t_1) \oplus L(t_2)) = H_{\mathcal{G}}(L(t_1+t_2)) = L(H(t_1+t_2)) = L(Ht_1 + Ht_2) = L(Ht_1) \oplus L(Ht_2) = H_{\mathcal{G}}L(t_1) \oplus H_{\mathcal{G}}L(t_2)$. It remains to prove that the map $H_{\mathcal{G}}$ is bijective. Suppose there exist different $t_1 + \mathcal{G}$ and $t_2 + \mathcal{G}$ in $\mathbb{Z}^N/\mathcal{G}$ such that $H_{\mathcal{G}}(t_1+\mathcal{G}) = H_{\mathcal{G}}(t_2+\mathcal{G})$. It follows that $H(t_1) - H(t_2) \in \mathcal{G}$ and by (iii) of Lemma 4.2 we conclude that $t_1 - t_2 \in \mathcal{G}$ which gives a contradiction. Therefore $H_{\mathcal{G}}$ is injective and since $\mathbb{Z}^N/\mathcal{G}$ is finite, $H_{\mathcal{G}}$ is bijective.

Since the H-DI sequence $f : \mathbb{Z}^N \to \mathbb{N}$ is periodic with period $\mathcal{G}$, it is completely determined by a map $g : \mathbb{Z}^N/\mathcal{G} \to \mathbb{N}$: $g(t+\mathcal{G}) = f(t)$. That is, f is the periodic continuation of g, and all properties of f induce similar properties for g and vice-versa. Thus, the property of f being $(H,$-DI implies that

$$g(Ht + w + \mathcal{G}) = g(t + \mathcal{G})\,. \tag{22}$$

Now consider the group generated by the maps

$$\begin{aligned} A_w &: \mathbb{Z}^N/\mathcal{G} \to \mathbb{Z}^N/\mathcal{G} \\ t+\mathcal{G} &\mapsto Ht + w + \mathcal{G} = H_{\mathcal{G}}(t+\mathcal{G}) + w\,. \end{aligned} \tag{23}$$

Then Eq. (22) is equivalent to

$$g(A_w(t+\mathcal{G})) = g(t+\mathcal{G}) \tag{24}$$

for all $w \in W$, all $t + \mathcal{G} \in \mathbb{Z}^N/\mathcal{G}$. Also, by invertibility of A_w, (24) yields

$$g(t+\mathcal{G}) = g(A_w^{-1}(t+\mathcal{G}))\,. \tag{25}$$

Observe that $g(A_v(A_w^{-1}(t+\mathcal{G}))) \overset{(23)}{=} g(A_v(H_{\mathcal{G}}^{-1}(t+\mathcal{G}-w)) \overset{(23)}{=} g(t+\mathcal{G}-w+v)$ and also $g(A_v(A_w^{-1}(t+\mathcal{G}))) \overset{(24)}{=} g(A_w^{-1}(t+\mathcal{G})) \overset{(25)}{=} g(t+\mathcal{G})$. This chain of equalities yields the equation $g(t+\mathcal{G}-w+v) = g(t+\mathcal{G})$, for all $t \in \mathbb{Z}^N$. As $\mathcal{G}$ is a minimal period, this implies that $w - v \in \mathcal{G}$. ■

For a residue set W, we form the set W^* as

$$W^* = \{w - v | w, v \in W\}\,.$$

The condition (20) as expressed in the previous theorem is then equivalent to

$$W^* \subset \mathcal{G}\,, \quad \text{or } \langle W^* \rangle \subseteq \mathcal{G}\,. \tag{26}$$

The next corollary shows that a minimal period $\mathcal{G}$ of an (H, W)-DI sequence does not only contain $\langle W^* \rangle$.

Corollary 4.4. *If f is an (H, W)-DI sequence, then a minimal full-rank period $\mathcal{G}$ of f satisfies*

$$\langle W^* \cup H(W^*) \cup \cdots \cup H^{N-1}(W^*) \rangle \subseteq \mathcal{G}\,. \tag{27}$$

Proof. By Theorem 4.3, we have that $\langle W^* \rangle \subseteq \mathcal{G}$. By point (i) of Lemma 4.2, a minimal period is H-invariant. Therefore $H^j(W^*)$ has to be in a minimal period for all $j \in \mathbb{N}$. By the Hamilton–Cayley Theorem we have $H^N = \sum_{j=0}^{N-1} \alpha_j H^j$, where the α_j are the integer coefficients of the characteristic polynomial of the matrix H. This demonstrates the assertion. ■

Remark

1. The corollary says that, for a given residue set W, $\langle W^* \cup H(W^*) \cup \cdots \cup H^{N-1}(W^*) \rangle$ is a possible minimal period, and not necessarily $\langle W^* \rangle$.
2. If $\langle W^* \rangle$ is not of full rank, (27) provides a full-rank subgroup which may be a possible period. As an illustration of this, consider again Example 2 with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W = \{(2, 2), (0, -1)\}$. Then $W^* = \{(0, 0), (2, 3), (-2, -3)\}$, $H(W^*) = \{(0, 0), (-1, 5), (1, -5)\}$, and $\langle W^* \rangle = \langle (2, 3) \rangle$; while $\langle W^* \cup H(W^*) \rangle = \langle (2, 3), (3, -2) \rangle$ is indeed the minimal period for a corresponding (H, W)-DI sequence, as shown in Fig. 4(d).

In analogy to the one-dimensional setting, we introduce the concepts of divisor, greatest common divisor and least common multiple of subgroups of $\mathbb{Z}^N$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are subgroups such that $\mathcal{G}_1 \supseteq \mathcal{G}_2$, then $\mathcal{G}_1$ is called a **divisor** of $\mathcal{G}_2$. For two subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$, two new subgroups can be formed which are called the **greatest common divisor** (gcd) and the **least common multiple** (lcm), respectively. They are defined as

$$\begin{aligned} \gcd(\mathcal{G}_1, \mathcal{G}_2) &= \{\alpha g_1 + \beta g_2 | \alpha, \beta \in \mathbb{Z}, \\ &\qquad g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2\} \\ \operatorname{lcm}(\mathcal{G}_1, \mathcal{G}_2) &= \mathcal{G}_1 \cap \mathcal{G}_2 \end{aligned}$$

In other words, the greatest common divisor is the smallest (w.r.t. inclusion) subgroup of $\mathbb{Z}^N$ which contains both $\mathcal{G}_1$ and $\mathcal{G}_2$. The least common multiple is the largest subgroup of $\mathbb{Z}^N$ which is contained in $\mathcal{G}_1$ and $\mathcal{G}_2$.

Corollary 4.4 then states that a minimal period $\mathcal{G}$ should be a divisor of $\langle W^* \cup H(W^*) \cup \cdots \cup H^{N-1}(W^*)\rangle$.

The following theorem provides an analogy for the one-dimensional case [Barbé & von Haeseler, 2000], stating that a possible minimal period θ of a p-DI sequence satisfies $\gcd(\theta, p) = 1$.

Theorem 4.5. *If f is an (H, W)-DI sequence of minimal full-rank period $\mathcal{G}$, then*

$$\gcd(\mathcal{G}, \mathcal{H}) = \mathbb{Z}^N ,$$

where $\mathcal{H} = \langle H\rangle$.

Proof. To prove that $\gcd(\mathcal{G}, \mathcal{H}) = \mathbb{Z}^N$ it suffices to demonstrate that $W \subset \gcd(\mathcal{G}, \mathcal{H})$, because W being a residue set for H implies that $W + H(\mathbb{Z}^N) = \mathbb{Z}^N$. For $g \in \mathcal{G}$ we have

$$g \overset{(1)}{=} \zeta(g) + H(\Xi(g)) ,$$

which yields $\zeta(g) = g - H(\Xi(g)) \in \gcd(\mathcal{G}, \mathcal{H})$. Now, for any $w \in W$, either $w \in \zeta(\mathcal{G})$ or $w \notin \zeta(\mathcal{G})$. The first case already implies $w \in \gcd(\mathcal{G}, \mathcal{H})$, while the second implies, by periodicity of f through Theorem 4.3, that $w - \zeta(g) \in \mathcal{G}$. Combined with $\zeta(g) \in \gcd(\mathcal{G}, \mathcal{H})$, it follows that $w \in \gcd(\mathcal{G}, \mathcal{H})$. ∎

Remark. If $\mathcal{G} = \langle G\rangle$ and $\mathcal{H} = \langle H\rangle$ are subgroups of $\mathbb{Z}^N$, then $\gcd(\mathcal{G}, \mathcal{H}) = \mathbb{Z}^N$ if and only if there exist $N \times N$ integer matrices A and B such that

$$GA + HB = I , \tag{28}$$

where I is the $N \times N$ unit matrix. The reason is that $\gcd(\mathcal{G}, \mathcal{H}) = \mathbb{Z}^N$ implies that all unit vectors $(1, 0, \ldots, 0)^t$, $(0, 1, \ldots, 0)^t$ up to $(0, 0, \ldots, 1)^t$ can be written as $\mathbb{Z}$-linear combinations of the column vectors in G and H, and this is what (28) says. Condition (28) is reminiscent of a well-known similar condition for two integers g and h to be relatively prime. Observe also, that G and H can be replaced by any matrices G' and H' which generate the same subgroups $\mathcal{G}$ and $\mathcal{H}$, respectively: i.e. $G' = GP$, and $H' = HQ$ with $P, Q \in Gl(N, \mathbb{Z})$. Also, the unit matrix I can be replaced by any $N \times N$ integer matrix in $Gl(N, \mathbb{Z})$.

Let us now consider the question of what possible periods may occur in the solutions of an (H, W)-DI problem. The necessary conditions for a group $\mathcal{G}$ to be a period can be split into two different kinds of conditions. The first kind only refers to the relation between $\mathcal{H}$ and $\mathcal{G}$, while the second refers to the residue set W under consideration.

The following lemma summarizes the necessary conditions for $\mathcal{G}$ to be a possible period of an H-DI sequence in terms of the generating matrices G and H of the groups $\mathcal{G}$ and $\mathcal{H}$, respectively.

Lemma 4.6. *A full-rank subgroup $\mathcal{G} = \langle G\rangle$ of $\mathbb{Z}^N$ is a possible minimal period for an H-DI sequence if $G \in \mathbb{Z}^{N\times N}$ satifies*

(i) $\tilde{H} = G^{-1}HG \in \mathbb{Z}^{N\times N}$
(ii) $\mathrm{lcm}(\mathcal{H}, \mathcal{G}) = \langle HG\rangle$
(iii) $\gcd(\mathcal{G}, \mathcal{H}) = \mathbb{Z}^N$.

Proof. The first assertion is a reinterpretation of point (i) in Lemma 4.2. In fact, for $\mathcal{G}$ to be a period we must have $H(\mathcal{G}) \subset \mathcal{G}$, i.e. $HG(\mathbb{Z}^N) \subset G(\mathbb{Z}^N)$ which is equivalent to $G^{-1}HG \in \mathbb{Z}^{N\times N}$.

The second assertion is another formulation of point (ii) of Lemma 4.2. Let $T \in \mathbb{Z}^{N\times N}$ be such that $\langle T\rangle = \mathrm{lcm}(\mathcal{H}, \mathcal{G})$. If $\mathcal{G}$ is a period, then by point (ii) of Lemma 4.2, we have that the condition $Ht \in \mathcal{G}$, meaning that $Ht = Gs \in \langle T\rangle$, implies that $t = Gt'$ for some $t' \in \mathbb{Z}^N$. Therefore $Ht = HGt' \in \langle T\rangle$. This shows that

$$\langle T\rangle \subseteq \langle HG\rangle . \tag{29}$$

Now let $u = HGu' \in \langle HG\rangle$. Since $\mathcal{G}$ is a period, we have $\tilde{H} = G^{-1}HG \in \mathbb{Z}^{N\times N}$ and therefore $u = HGu' = G\tilde{H}u'$, implying $u \in \mathcal{G}$. Thus we conclude that

$$\langle HG\rangle \subseteq \langle T\rangle . \tag{30}$$

Expressions (29) and (30) imply that $\langle T\rangle = \langle HG\rangle$.

The third assertion replicates Theorem 4.5. ∎

A subgroup $\mathcal{G} \subseteq \mathbb{Z}^N$ satisfying the conditions of Lemma 4.6 will be called an **H-compatible subgroup**. It is possible to perform a systematic determination of periods $\mathcal{G} = \langle G\rangle$ which are H-compatible (restricting the size of the fundamental region of these periods). Once a specific subgroup $\mathcal{G} = \langle G\rangle$ has been fixed, the question arises to determine residue sets which actually allow $\mathcal{G}$ to be a period of a corresponding (H, W)-DI sequence. The resulting procedure will of course exploit the condition $W^* \subset \mathcal{G}$ of Theorem 4.3. The next lemma prepares for a useful theorem in that

respect. It provides a general form for H-residue sets W that satisfy Theorem 4.3.

Lemma 4.7. *Let W be an H-residue set such that $W^* = \{w - v | w, v \in W\} \subset \mathcal{G}$, with $\mathcal{G}$ a full-rank H-compatible period. Then any other H-residue set V such that $V^* \subset \mathcal{G}$ is given by*

$$V = \{w + \nu + HGq_w | w \in W\} \tag{31}$$

where $\nu \in \mathbb{Z}^N$ and $q_v \in \mathbb{Z}^N$ for $w \in W$.

Proof. If W is an H-residue set, then any other residue set of H is given by

$$V = \{W + \mu + Hm_w | w \in W\}, \tag{32}$$

with $\mu \in \mathbb{Z}^N$and $m_w \in \mathbb{Z}^N$ for $w \in W$. The condition $V^* \subset \mathcal{G}$ requires that, for any v_1, $v_2 \in V$, we have

$$\begin{aligned} v_1 - v_2 &\overset{(32)}{=} w_1 + \mu + Hm_{w_1} - w_2 - \mu - Hm_{w_2} \\ &= w_1 - w_2 + H(m_{w_1} - m_{w_2}) \in \mathcal{G}. \end{aligned}$$

As $w_1 - w_2 \in \mathcal{G}$, it follows that $H(m_{w_1} - m_{w_2}) \in \mathcal{G}$, or equivalently, $H(m_{w_1} - m_{w_2}) = Gt$ for some $t \in \mathbb{Z}^N$. But, by point (ii) in Lemma 4.2, this implies that $m_{w_1} - m_{w_2} \in \mathcal{G}$. As this must be the case for any pair of vectors m_{w_1}, m_{w_2}, this requires that $m_w = Gq_w + c$, with q_w, $c \in \mathbb{Z}^N$, for all $w \in W$. Substituting this in (32), gives $v = w + \mu + Hc + HGq_w$. Letting $\nu = \mu + Hc$ produces (31). ■

Remark. Treating W, V as $N \times \Delta$-matrices with H-consistent numbering (see Sec. 2) requires that, in (31), $\nu = Hq$ for some $q \in \mathbb{Z}^N$. Then (31) can be rewritten as $V_{[]} = W_{[]} + Hq + HGQ$, with $Q \in \mathbb{Z}^{N \times \Delta}$ and $q \in \mathbb{Z}^N$. Or, when H-residue sets W satisfying $W^* \subset \mathcal{G} = \langle G \rangle$ are called **(H, G)-compatible residue sets**, then the set of all (H, G)-compatible residue sets, represented as matrices, is given by

$$\mathcal{W}_{(H,G)} = \{W_{[]} + Hq + HGQ | q \in Z^N, Q \in \mathbb{Z}^{N \times \Delta}\}, \tag{33}$$

where W is an arbitrary but fixed (H, G)-compatible residue set.

Theorem 4.8. *Let $\mathcal{G} = \langle G \rangle$ be an H-compatible full-rank period. A residue set W of H satisfies $W^* \subseteq \mathcal{G}$ if and only if W is of the form*

$$W = \nu + G(\tilde{W}), \tag{34}$$

where $\tilde{W}$ is a residue set of $\tilde{H} = G^{-1}HG$ and $\nu \in \mathbb{Z}^N$.

Proof

1. If $W^* \subseteq \mathcal{G}$, then W is of the form (34).

Let $\tilde{V}$ be a $\tilde{H}$-residue set. We show that $G(\tilde{V})$ is a residue set for H. First, let us assume that there exist $\tilde{v}$, $\tilde{w} \in \tilde{V}$, with $\tilde{w} \neq \tilde{v}$, such that $G(\tilde{w} - \tilde{v}) \in H(\mathbb{Z}^N)$. Then by point (ii) of Lemma 4.6, we have that $G(\tilde{w} - \tilde{v}) = HGt = G\tilde{H}t$ for some $t \in \mathbb{Z}^N$, and this implies that $\tilde{w} - \tilde{v} = \tilde{H}t$, a contradiction with the fact that $\tilde{w}$ and $\tilde{v}$ are distinct elements in the residue set $\tilde{V}$. Since $\mathcal{G}$ has full rank, it follows that the cardinality of $G(\tilde{V})$ is equal to the cardinality of $\tilde{V}$. Since $|\tilde{V}| = |\det \tilde{H}| = |\det H|$, it follows that $G(\tilde{V})$ is a H-residue set, and it clearly satisfies $(G(\tilde{V}))^* \subset \mathcal{G}$.

Replacing V in Lemma 4.7 by $G(\tilde{V})$ we obtain that all H-residue sets W satisfying $W^* \subset \mathcal{G}$ are of the form

$$\begin{aligned} &\{G\tilde{v} + HGq_{\tilde{v}} + \nu | \tilde{v} \in \tilde{V}\} \\ &\quad = \nu + \{G(\tilde{v} + G^{-1}HGq_{\tilde{v}}) | \tilde{v} \in \tilde{V}\} \\ &\quad = \nu + G(\{\tilde{v} + \tilde{H}q_{\tilde{v}} | \tilde{v} \in \tilde{V}\}). \end{aligned}$$

As $\tilde{V}$ is a $\tilde{H}$-residue set, so is $\tilde{W} = \{\tilde{v} + \tilde{H}q_{\tilde{v}} | \tilde{v} \in \tilde{V}\}$, and it follows indeed that W satisfies (34).

2. If W satisfies (34), then $W^* \in \mathcal{G}$, as $W^* = G(\tilde{W}^*)$. ■

Remark. Using the matrix representation of residue sets, (34) redefines the set $\mathcal{W}_{(H,G)}$ of all distinct (H, G)-compatible residue sets presented in (33), as

$$\mathcal{W}_{(H,G)} = \{Hq + HGQ + G\tilde{W}_{[]} | q \in \mathbb{Z}^N, Q \in \mathbb{Z}^{N \times \Delta}\}, \tag{35}$$

where $\tilde{W}$ is an arbitrary but fixed $\tilde{H}$-residue set. (Replacing $\tilde{W}$ by another $\tilde{H}$-residue set $\tilde{W}'$ will change Q into Q', where Q' is the integer matrix such that $\tilde{W}_{[]} \overset{(3)}{=} \tilde{W}'_{[]}\tilde{H}\tilde{Q}$, $Q' = Q + \tilde{Q}$.)

The following lemma shows that the condition on W for $\mathcal{G}$ to be a period of an (H, W)-DI sequence, as given in the previous theorem, already implies the fulfillment of the additional condition (iii) in Lemma 4.2. Thus there is no need to consider this as an additional condition to be checked for compatibility among H, G, W.

Lemma 4.9. *Let $W^* \in \mathcal{G}$, i.e. $W = \nu + G(\tilde{W})$, then $\Xi(W + \mathcal{G}) \subseteq \mathcal{G}$.*

Proof. With $\tilde{\Xi}$ and $\tilde{\zeta}$ the image part map and remainder part map for $(\tilde{H}, \tilde{W})$, we compute for $w \in W$ and $g \in G$ that

$$\begin{aligned} w + g &= \nu + G\tilde{w} + Gs \\ &= \nu + Gs' \\ &= \nu + G(\tilde{\zeta}(s') + \tilde{H}\tilde{\Xi}(s')) \\ &= \nu + G\tilde{\zeta}(s') + HG\tilde{\Xi}(s'). \end{aligned} \tag{36}$$

As $\nu + G\tilde{\zeta}(s') \in W$ (by (34)), it follows from (36) that $\nu + G(\tilde{\zeta})(s) = \zeta(w + g)$ and $\Xi(w + g) = G\tilde{\Xi}(s')$. The last equality is equivalent to $\Xi(W + \mathcal{G}) \subseteq \mathcal{G}$. ■

Remark. Observe that both $W^* \subset \langle G \rangle$ (26) and $W_{[]} \in \{Hq + HGQ + G\tilde{W}_{[]} | q \in Z^N, Q \in Z^{N \times \Delta}\}$ (35) express the condition for (H, G)-compatibility of W. The first condition is equivalent to $G^{-1}W^*_{[]}$ being an integer matrix, and this fact can be used to check whether a given W is (H, G)-compatible, or even to find in a systematic way G-matrices which are both H-compatible and satisfy the condition that $G^{-1}W^*_{[]}$ is an integer matrix (one could call such a G to be **(H, W)-compatible**). The second condition (35) is more useful to determine residue sets which are (H, G)-compatible.

Example 5

(1) Let $H = \begin{bmatrix} -3 & 3 \\ -1 & 0 \end{bmatrix}$. Then $\mathcal{G} = \left\langle G = \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} \right\rangle$ is H-compatible and is a possible minimal period for an H-DI sequence. To find a residue set W such that a corresponding (H, W)-DI sequence may have this $\mathcal{G}$ as minimal period, we use Theorem 4.8: $\tilde{H} = G^{-1}HG = \begin{bmatrix} 0 & -3 \\ 1 & -3 \end{bmatrix}$, and $\tilde{W} = \{(0, 0), (1, 0), (-1, 0)\}$ is a $\tilde{H}$-residue set. Then $W = G(\tilde{W}) = \{(0, 0), (2, 4), (-2, -4)\}$ is (H, G)-compatible. Figure 9(a) shows the (H, W)-DI sequence of maximal diversity. It is nonperiodic, but range-reduction gives a periodic solution with period $\mathcal{G}$. Figure 9(b) displays the periodic solution of maximal diversity. As mentioned before, divisors of $\mathcal{G}$, are also possible minimal periods of (H, W)-DI sequences. There are 10 divisors:

$$\mathcal{G} = \mathcal{G}_{10} = \left\langle \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} \right\rangle \quad \mathcal{G}_9 = \left\langle \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \right\rangle$$

$$\mathcal{G}_8 = \left\langle \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \right\rangle \quad \mathcal{G}_7 = \left\langle \begin{bmatrix} 1 & 6 \\ 2 & -2 \end{bmatrix} \right\rangle$$

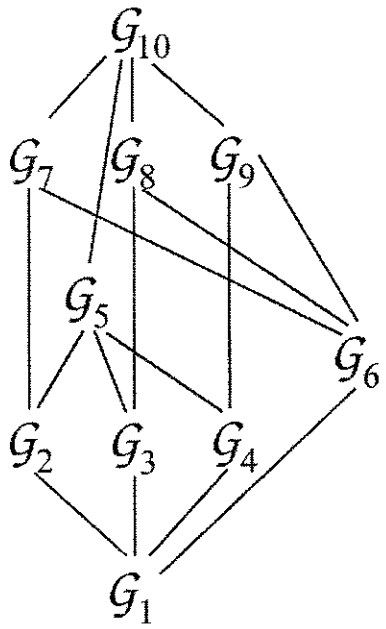


Fig. 8. The partial order of the divisors of the period $\mathcal{G}$ in Example 5. $\mathcal{G}_i$ is smaller than $\mathcal{G}_j$ means that $\mathcal{G}_i$ is a divisor of $\mathcal{G}_j$, i.e. $\mathcal{G}_i \supset \mathcal{G}_j$.

$$\mathcal{G}_6 = \left\langle \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \right\rangle \quad \mathcal{G}_5 = \left\langle \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \right\rangle$$

$$\mathcal{G}_4 = \left\langle \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \right\rangle \quad \mathcal{G}_3 = \left\langle \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\rangle$$

$$\mathcal{G}_2 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\rangle \quad \mathcal{G}_1 = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

The generating matrix of $\mathcal{G}_i$ as displayed above will be denoted by G_i. The partial order of this set of divisors is shown in Fig. 8.

However, $\mathcal{G}_9$, $\mathcal{G}_8$, $\mathcal{G}_7$, $\mathcal{G}_4$, $\mathcal{G}_3$, $\mathcal{G}_2$ are not H-compatible (as the corresponding matrix $\tilde{H}_i = G_i^{-1}HG_i$ is not an integer matrix). Thus only $\mathcal{G}_6$, $\mathcal{G}_5$, $\mathcal{G}_1$ remain to be considered as possible minimal periods. These are all (H, W)-compatible, as the corresponding $G_i^{-1}W^*_{[]}$ is an integer matrix. $\mathcal{G}_6$, $\mathcal{G}_5$, $\mathcal{G}_1$ are indeed actual minimal periods of (H, W)-DI sequences. Solutions with minimal periods $\mathcal{G}_6$ and $\mathcal{G}_5$ are shown in Figs. 9(c) and 9(d), respectively. The solution with period $\mathcal{G}_1$ is of course the trivial solution.

(2) The period $\mathcal{G}$ as considered under point (1) is also H_1-compatible, where $H_1 = \begin{bmatrix} 2 & -3 \\ 3 & -2 \end{bmatrix}$. Again, an H_1-residue set W_1 which may allow this period is given by $W_1 = G(\tilde{W}_1)$, where $\tilde{W} = \{(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)\}$ is a residue

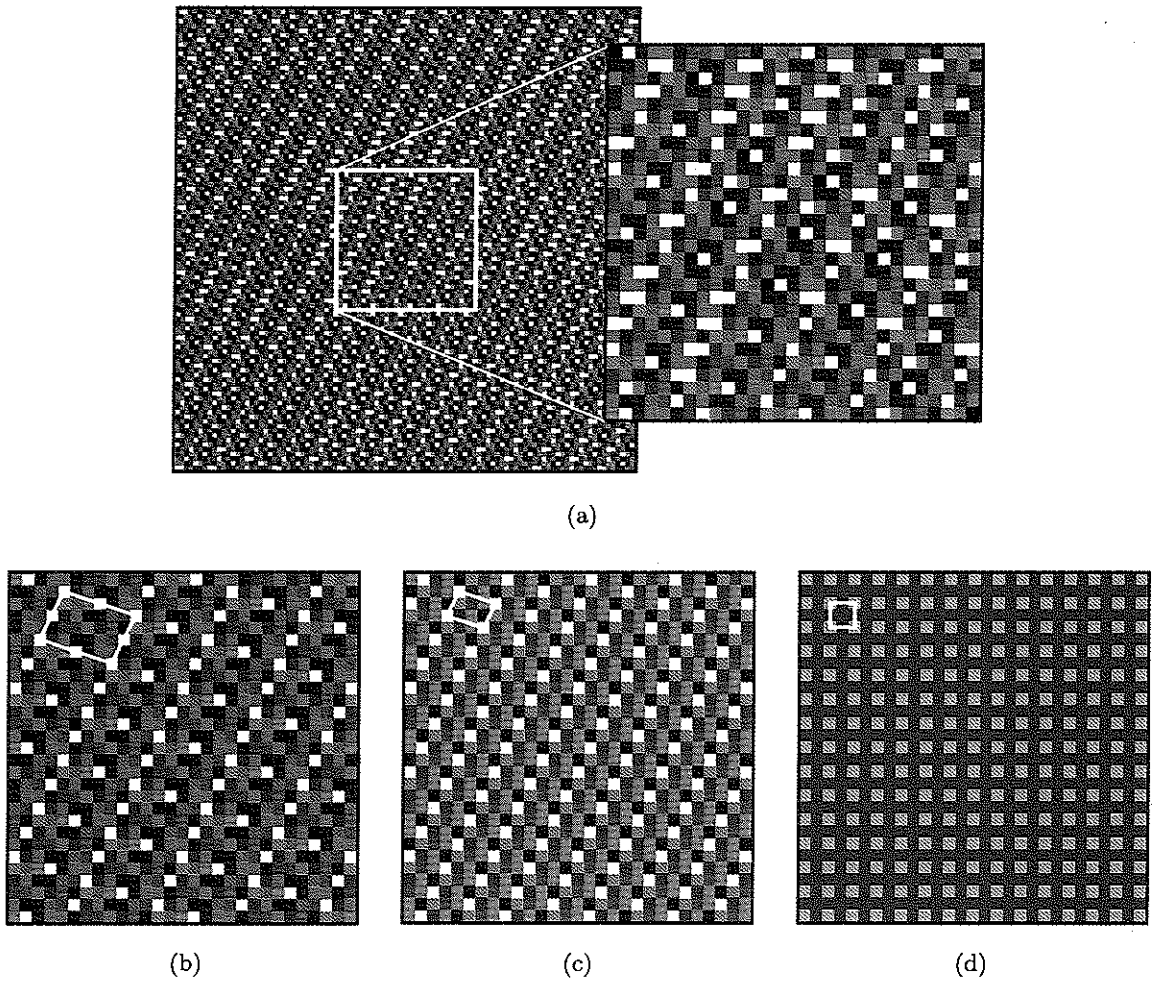


Fig. 9. Example 5(1). $H = \begin{bmatrix} -3 & 3 \\ -1 & 0 \end{bmatrix}$ and $W = \{(0, 0), (2, 4), (-2, -4)\}$. (a) The solution of maximal diversity is nonperiodic. (b) The periodic solution of maximal diversity with minimal period $\mathcal{G}$ (the period is indicated). (c) The periodic solution of maximal diversity with minimal period $\mathcal{G}_6$. (d) The periodic solution of maximal diversity with minimal period $\mathcal{G}_5$.

set for $\tilde{H}_1 = G^{-1}H_1G = \begin{bmatrix} -1 & 6 \\ -1 & 1 \end{bmatrix}$. This gives $W_1 = \{(-4, -8), (-2, -4), (0, 0), (2, 4)(4, 8)\}$. Figure 10(a) shows the corresponding (H_1, W_1)-DI sequence of maximal diversity: it is already periodic, and thus all solutions are periodic. Possible other minimal periods are again the divisors of $\mathcal{G}$, but now only $\mathcal{G}_{10}$, $\mathcal{G}_8$, $\mathcal{G}_6$, $\mathcal{G}_5$, $\mathcal{G}_3$, $\mathcal{G}_1$ are possible (the others are not H_1-compatible, because the corresponding $G^{-1}H_1G$ is not an integer matrix). Figures 10(a)–10(f) show the corresponding solutions of maximal diversity. Observe that Fig. 10(c) can be obtained from a range-reduction of Fig. 9(b). Also, Fig. 9(c) can be obtained from a range-reduction of Fig. 10(d). This shows that some periodic sequences at least can be both (H, W)-DI and (H', W')-DI, whereby $H \neq H'$ and $W \neq W'$ (even $|\det H| \neq |\det H'|$).

(3) By Theorem 4.8, $W_2 = W_1 + (2, 1) = \{(-2, 7), (0, -3), (2, 1), (4, 5), (6, 9)\}$ is also an (H_1, G)-compatible H_1-residue set [H_1 from point (2) above, and G from point (1) above]. However, unlike what happened under (1) and (2), $\mathcal{G}$ is no longer the minimal period for an (H_1, W_2)-DI sequence. The sequence of maximal diversity for this (H_1, W_2)-DI problem is already periodic with period $\mathcal{G}_6$, and is actually pattern-equivalent to the pattern shown in Fig. 10(c). Although $\mathcal{G}_{10}$, $\mathcal{G}_8$, $\mathcal{G}_6$, $\mathcal{G}_5$, $\mathcal{G}_3$, $\mathcal{G}_1$ satisfy all conditions for being a minimal period of an (H_1, W_2)-DI sequence, only $\mathcal{G}_6$ and $\mathcal{G}_1$ can actually be realized. This illustrates the fact that the

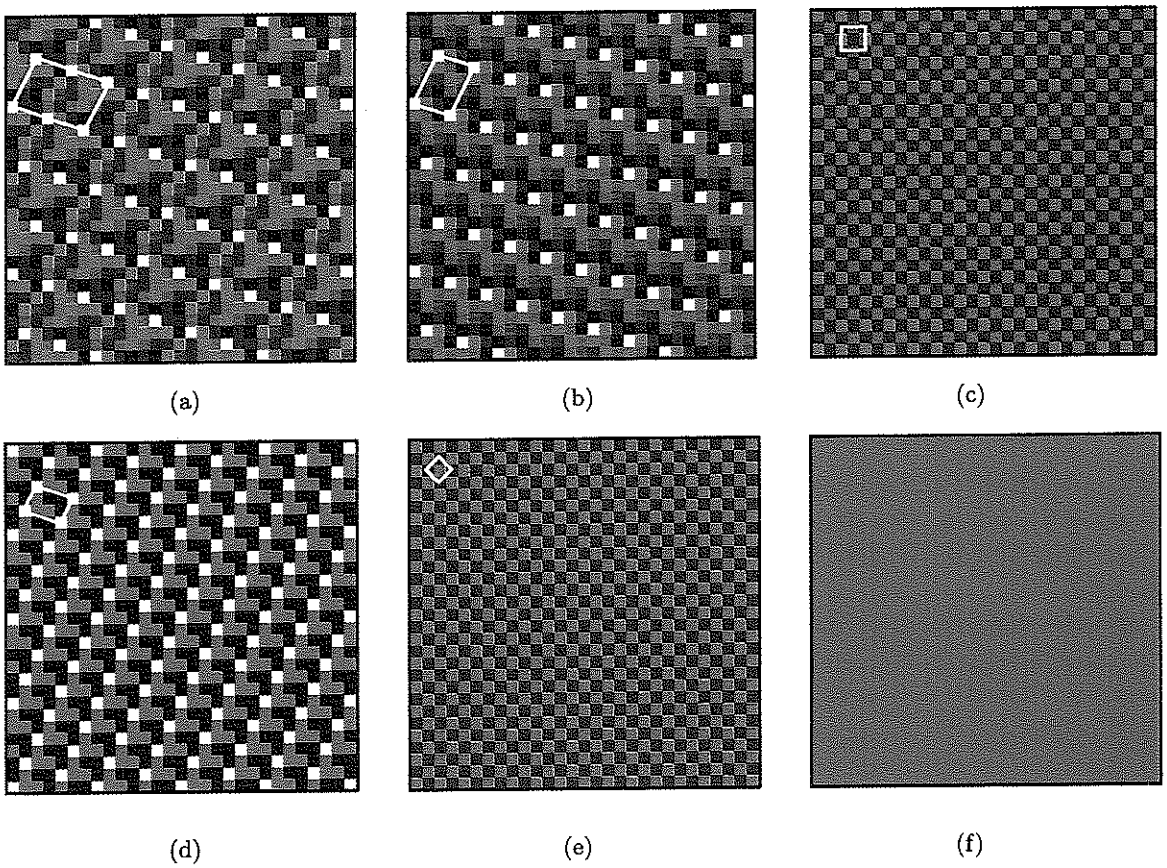


Fig. 10. Example 5(2). $H_1 = \begin{bmatrix} 2 & -3 \\ 3 & -2 \end{bmatrix}$, and H_1-residue set $W_1 = \{(-4, -8), (-2, -4), (0, 0), (2, 4), (4, 8)\}$. (a) The solution of maximal diversity is periodic with minimal period $\mathcal{G}$ (the period is indicated). (b–f) The periodic solutions of maximal diversity with minimal periods $\mathcal{G}_8$, $\mathcal{G}_6$, $\mathcal{G}_5$, $\mathcal{G}_3$, $\mathcal{G}_1$, respectively.

condition on $\mathcal{G}$ to be the minimal period of an (H, W)-DI sequence when $\mathcal{G}$ is (H, W)-compatible, is only a necessary condition.

The next theorem shows that an H-compatible group $\mathcal{G}$ is indeed a minimal period of some H-DI sequence.

Theorem 4.10. *If $\mathcal{G} = \langle G \rangle$ is a subgroup of full rank which is H-compatible, then there exists an H-DI sequence f with minimal period $\mathcal{G}$. In fact, it is an $(H, G(\tilde{W}))$-sequence, where $\tilde{W}$ is a residue set for $\tilde{H} = G^{-1}HG$.*

Proof. As established in the proof of Theorem 4.8, $W = G(\tilde{W})$ is a residue set of H such that $\mathcal{G}$ is a possible minimal period of an (H, W)-DI sequence. We will show that there actually is an (H, W)-DI sequence with minimal period $\mathcal{G}$. Following the arguments in the proof of Theorem 4.3, a periodic H-DI sequence f is given by the periodic continuation of a map $g : \mathbb{Z}^N/\mathcal{G} \to \mathbb{N}$. As we want f to be an (H, W)-DI sequence this requires (22) to hold: i.e.

$$g(Ht + w + \mathcal{G}) = g(t + \mathcal{G}) \tag{37}$$

for all $t \in \mathbb{Z}^N$, $w \in W$. As $w = G\tilde{w} \in \mathcal{G}$, this is equivalent to

$$g(Ht + \mathcal{G}) = g(t + \mathcal{G}) . \tag{38}$$

This last condition is satisfied by defining g as follows:

$$g(t + \mathcal{G}) = \begin{cases} 1 & \text{if } t \in \mathcal{G} \\ 0 & \text{otherwise.} \end{cases}$$

Then the periodic continuation f of g, which is an (H, W)-DI sequence, takes the value 1 only at elements of $\mathcal{G}$. This shows that no smaller period than $\mathcal{G}$ is possible for this sequence f. ■

Remark. First notice that the (H, W)-DI sequence constructed in Theorem 4.10 is not necessarily the only one with minimal period $\mathcal{G}$. If $\nu \not\equiv 0 \bmod \mathcal{G}$ and if $W = \nu + G(\tilde{W})$, then condition (37) in the proof of the above theorem becomes $g(Ht + w + \nu + \mathcal{G}) = g(t + \mathcal{G})$, which is equivalent to $g(Ht + \nu + \mathcal{G}) = g(t + \mathcal{G})$. If there exists a $t_0 \in \mathbb{Z}^N$ such that $Ht_0 + \nu + \mathcal{G} = t_0 + \mathcal{G}$, which is the case when $t_0 = -(H - I)^{-1}\nu \in \mathbb{Z}^N$, then

$$g(t + \mathcal{G}) = \begin{cases} 1 & \text{if } t - t_0 \in \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

defines a map such that the above condition, $g(Ht + \nu + \mathcal{G}) = g(t + \mathcal{G})$, holds. The periodic continuation of g is then again an (H, W)-DI sequence with minimal period $\mathcal{G}$.

As already demonstrated by Theorem 4.9 in [Barbé & von Haeseler, 2000], the question whether a given residue set $W = \nu + G(\tilde{W})$ does allow a (H, W)-DI sequence with an actual minimal period $\mathcal{G}$, depends on the group action of the group generated by the map $A : \mathbb{Z}^N/\mathcal{G} \to \mathbb{Z}^N/\mathcal{G} : t \mapsto Ht + \nu$.

We can summarize our results in the following theorem.

Theorem 4.11. *A full-rank group $\mathcal{G} = \langle G \rangle$ is a minimal period for an H-DI sequence if and only if G is H-compatible.*

5. Additional Examples and Observations

In order to give an idea about the diversity of H-DI patterns, we display some additional examples which are somehow morphologically generic.

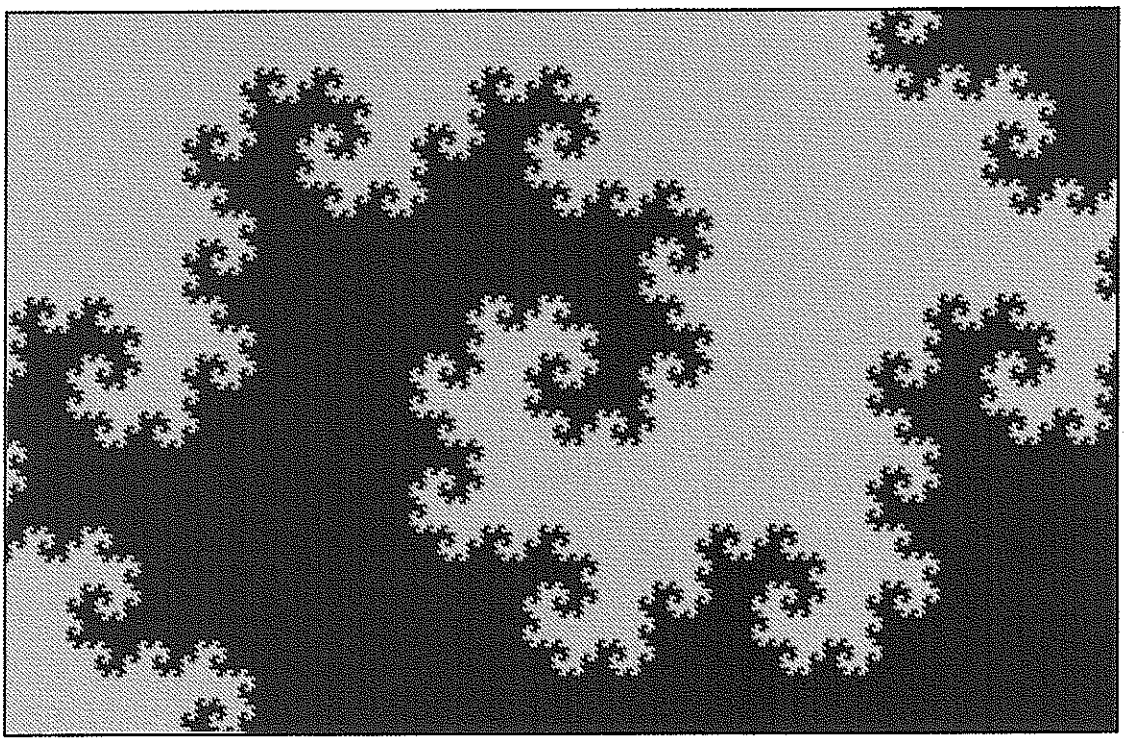
Figure 11(a) displays the same fractal-like spiral as the one shown in Fig. 4(a), but now without the possibility for a nontrivial periodic solution. It corresponds to the solution of maximal diversity for $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W = \{(0, 0), (0, 1)\}$. As $|\det(H - I)| = 1$, all $(H, W + \gamma)$-DI solutions, $\gamma \in \mathbb{Z}^2$, display the same pattern of maximal diversity. Figure 11(b) displays another factal-like spiral. The spiraling is related to the rotation induced by H. Spiraling fractals covering the whole plane seem novel, although it should be no surprise to find local fractal-like structures at the boundaries of these spirals which remind of structures appearing in [Bandt, 1991]. Figures 12–16 display the pattern of maximal diversity for some additional (H, W)-DI sequences. Figures 13(a) and 14(a) are both associated with an H-matrix inducing reflection along the 45°-diagonal. Figure 13(a) shows reflection symmetry along this diagonal, while Fig. 14(a) shows some invariance under both reflection and rescaling. In other cases, the relationship between the pattern and the underlying H-matrix might be less obvious, but keeping in mind that the content of the square generated by the column vectors of the matrix $U = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ is mapped into the parallelogram generated by HU usually helps to spot the self-similarity.

Fractal-like solutions of maximal diversity which produce periodic solutions by range-reduction, were already presented in Figs. 4–7.

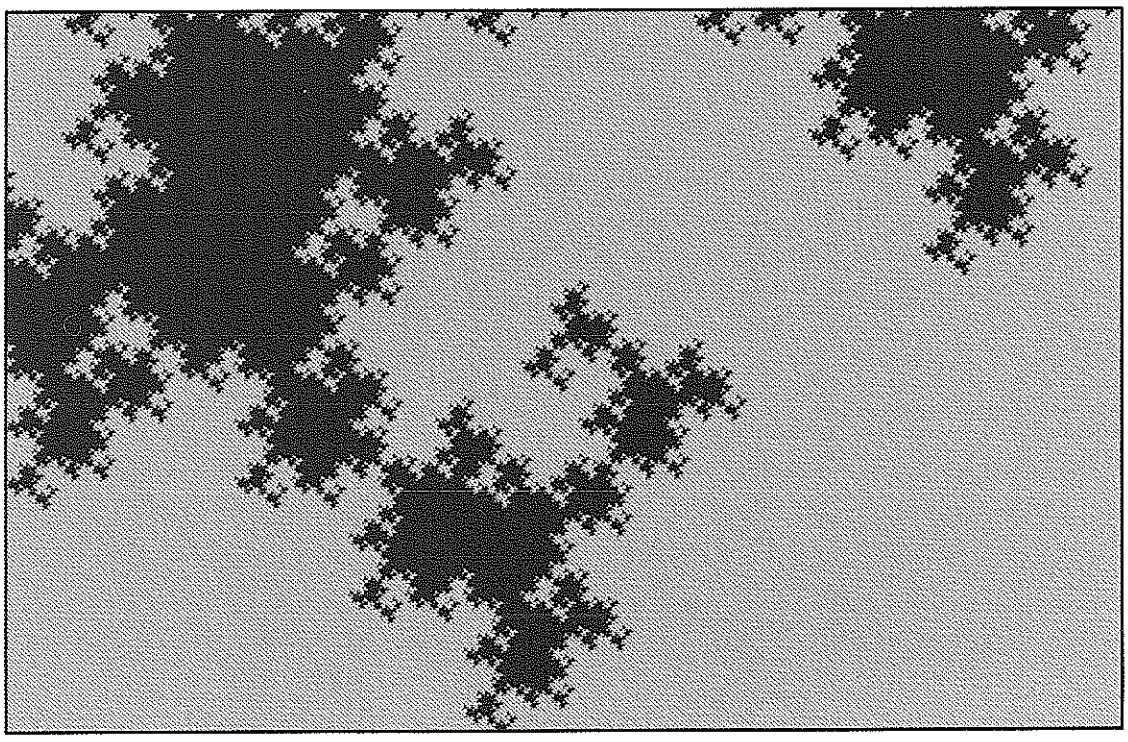
It sometimes happens that an (H, W)-DI problem has a solution of maximal diversity which is the trivial solution. If $0 \in W$, it was shown in [Barbé & von Haeseler, 2000] that, for the one-dimensional case $H = p$, $p \in \mathbb{Z}$, that W forms a *complete digit* set for $H\,(= p)$, meaning that any $s \in \mathbb{Z}$ can be represented in a unique way as $s = \sum_{j=0}^{K} w_j p^j$ for some $K \in \mathbb{Z}$, $w_K \neq 0$, where $w_j \in W$. This can be generalized to the N-dimensional case: if the H-residue set W contains 0, then the solution of maximal diversity (H, W)-DI is trivial if and only if W is a complete digit-set, i.e. if any $s \in \mathbb{Z}^N$ can be written in a unique way as $s = \sum_{j=0}^{K} H^j w_j$. An example of this is $H = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$, $W = \{(0, 0), (1, 0), (-1, 0), (0, 1), (0, -1)\}$. Strangely, for all $W + \gamma$, $\gamma \in \mathbb{Z}^2$, the solution of maximal diversity is the trivial one.

The solution of maximal diversity of some (H, W)-DI problems are periodic, meaning that all solutions are periodic. In analogy with the one-dimensional case, the residue-sets containing 0 for which this occurs are called *quasi-complete digit sets.* An example is obtained for $H = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ (same H as above), but $W = \{(0, 0), (-1, 2), (1, -2), (2, 1), (-2, -1)\}$ (actually all $W + \gamma$, $\gamma \in \mathbb{Z}^2$ produce an overall periodic solution of maximal diversity). Like for complete digits sets, the question of determining all quasi-complete digit sets is still open.

Whereas all examples involving some possibility for nontrivial periodic solutions [see Figs. 4–7,

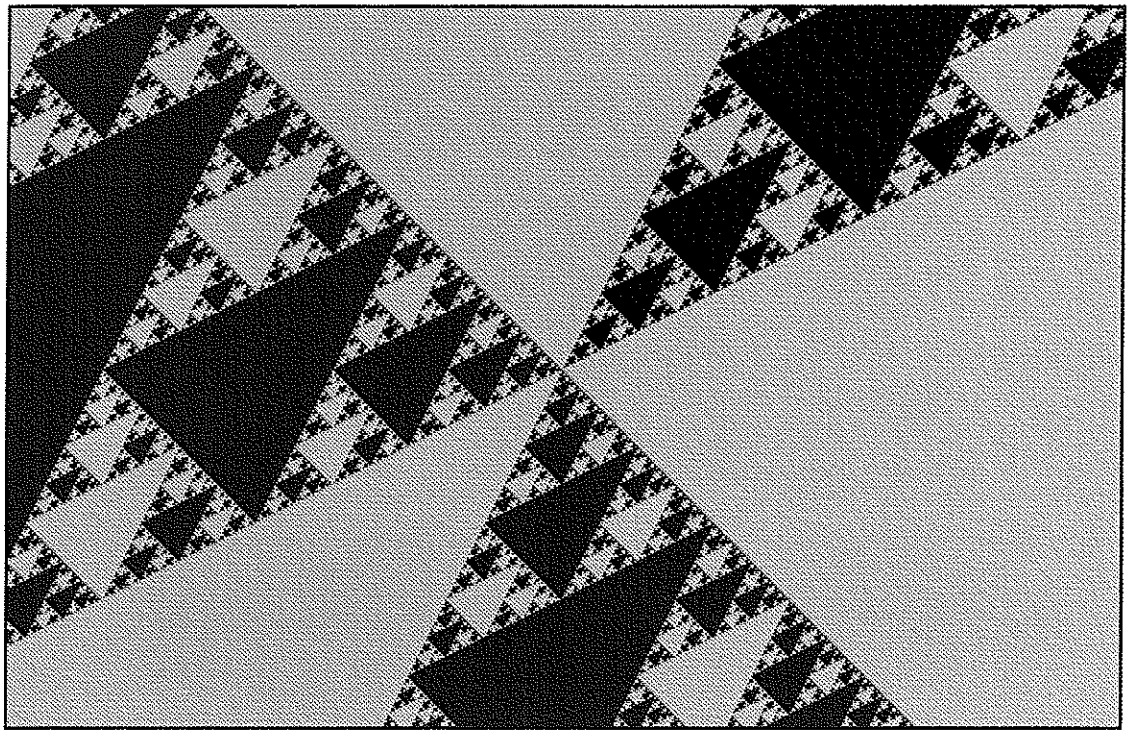


(a)

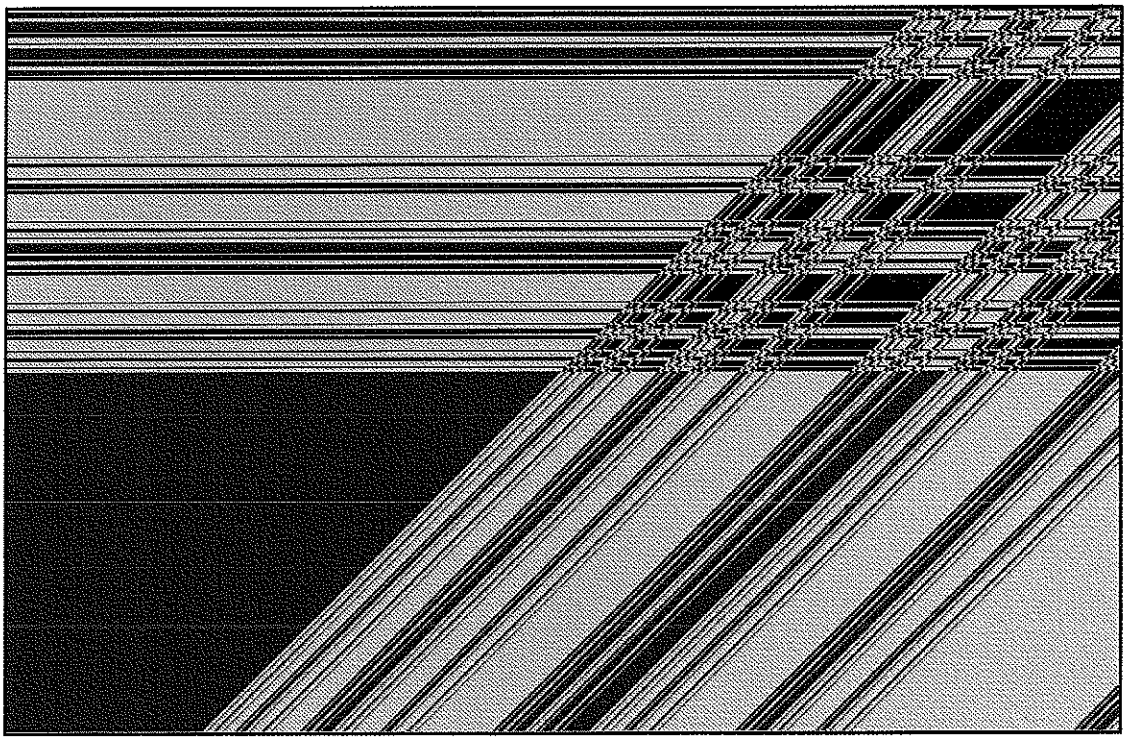


(b)

Fig. 11. (a) Pattern of maximal diversity for the (H, W)-DI problems with $H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $W \in \{\{(0, 0), (0, 1)\} + \gamma | \gamma \in \mathbb{Z}^2\}$. (b) The solution of maximal diversity for $H = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $W = \{(0, 0), (2, 1), (1, 2), (1, 1)\}$.

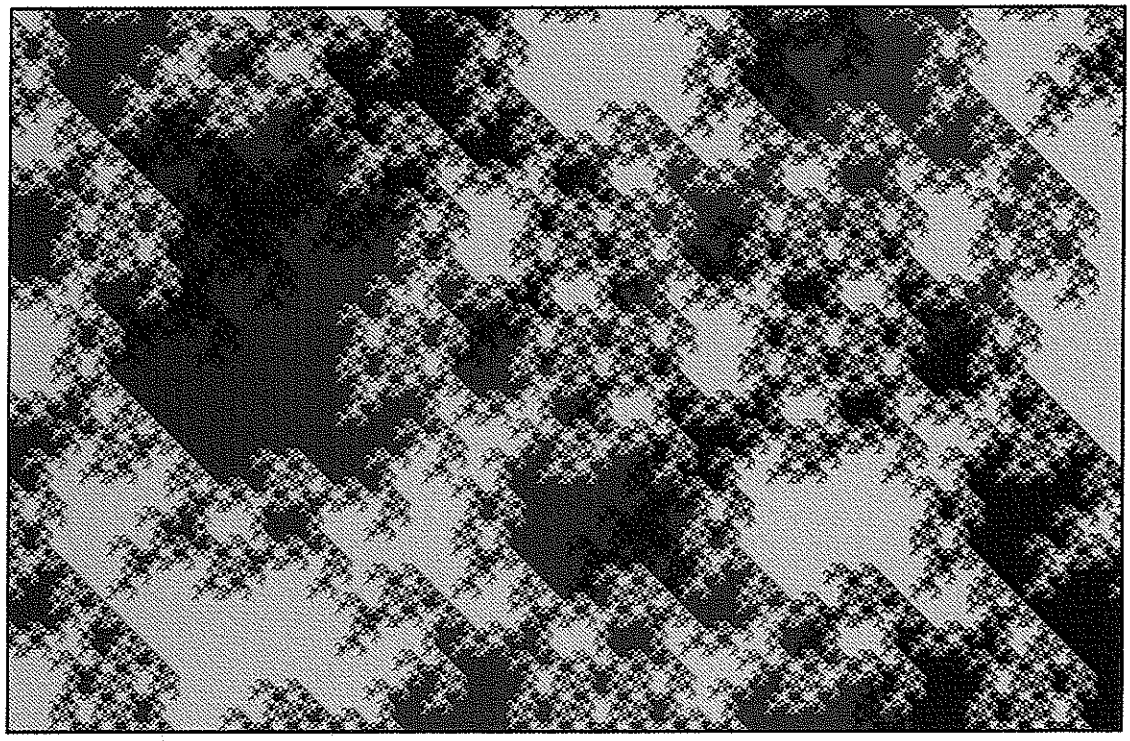


(a)

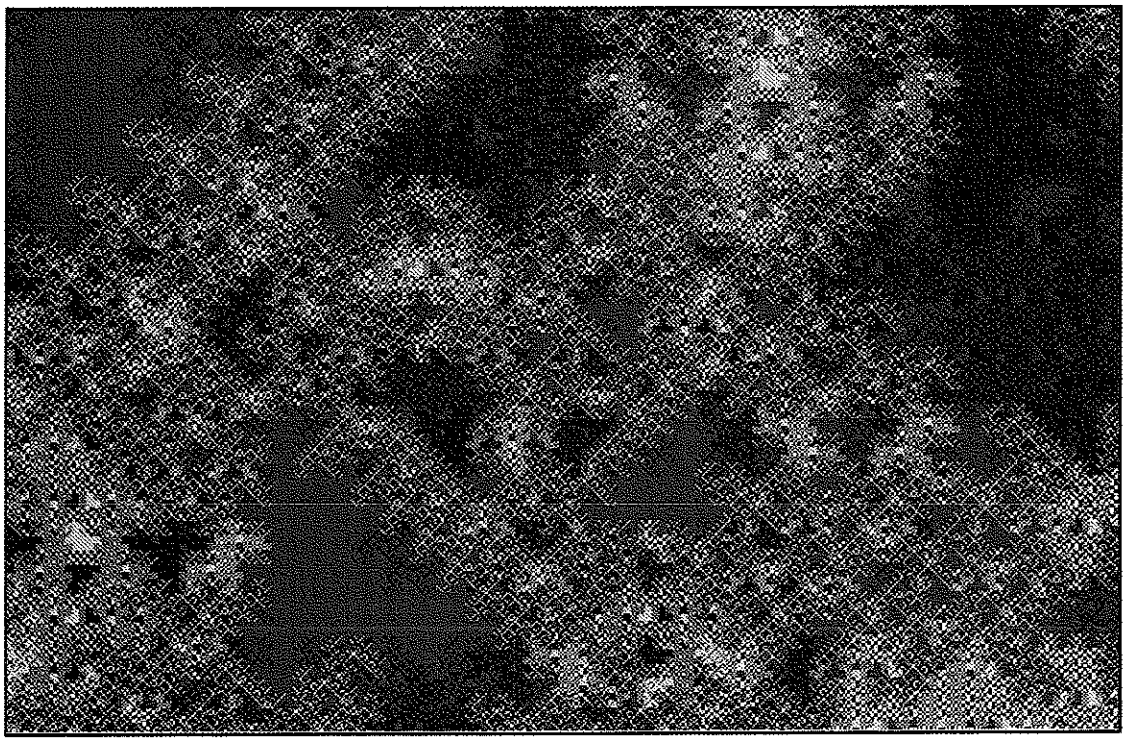


(b)

Fig. 12. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $W = \{(0, 0), (1, 1), (1, 2), (2, 1)\}$. (b) Solution of maximal diversity for $H = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $W = \{(0, 0), (1, 0), (5, 0)\}$.

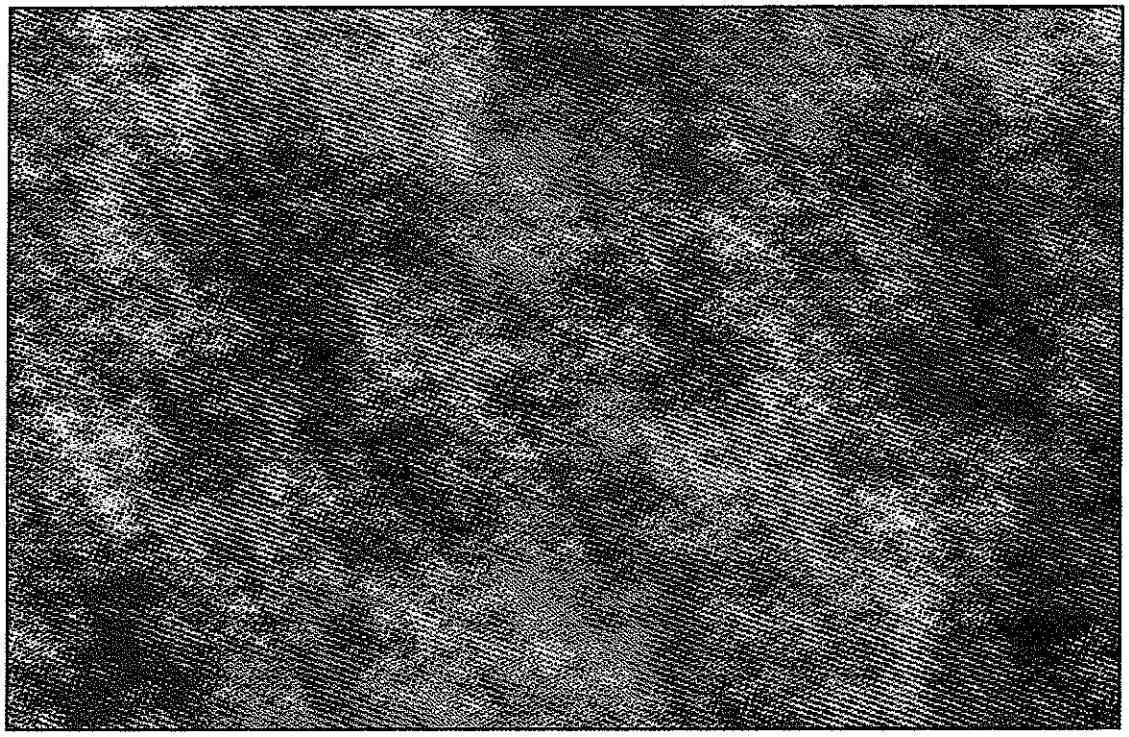


(a)

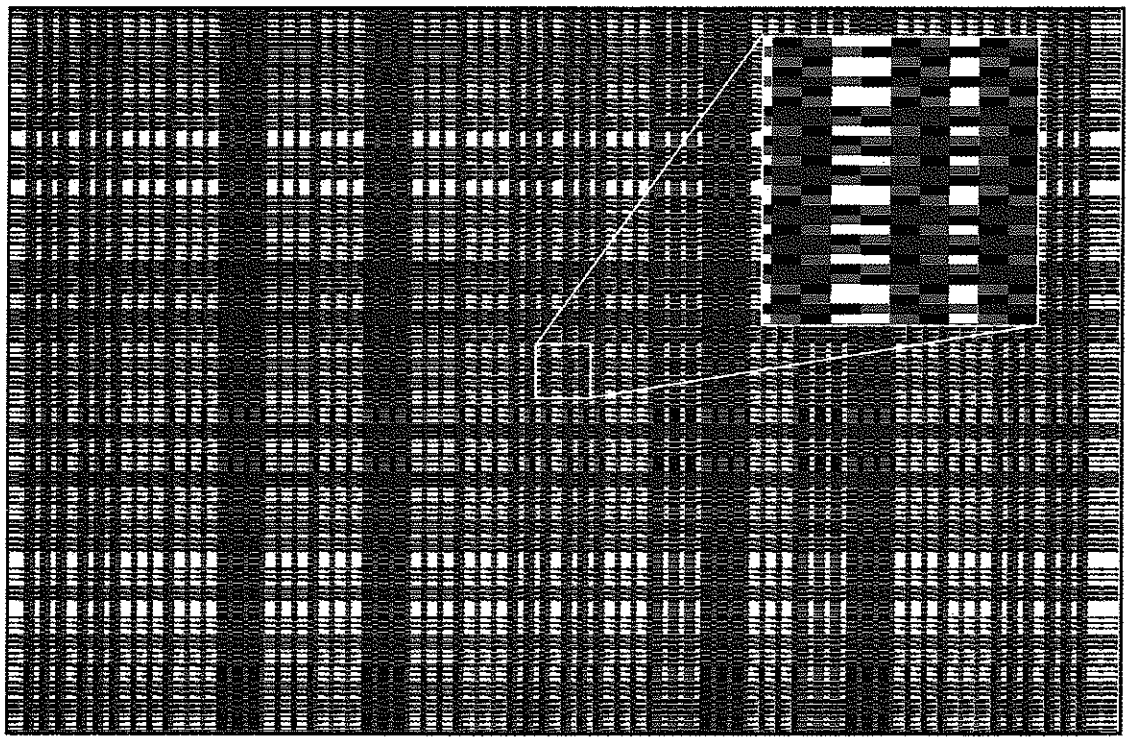


(b)

Fig. 13. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ and $W = \{(0, 0), (-1, -1), (1, 0), (2, -1)\}$. (b) Solution of maximal diversity for $H = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $W = \{(0, 0), (4, -1), (5, 0), (1, -1)\}$.

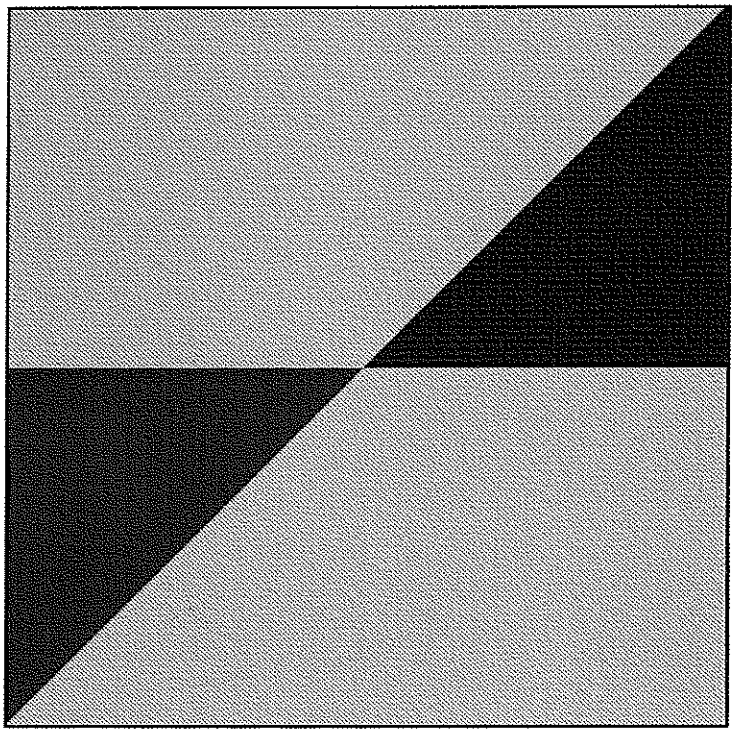


(a)

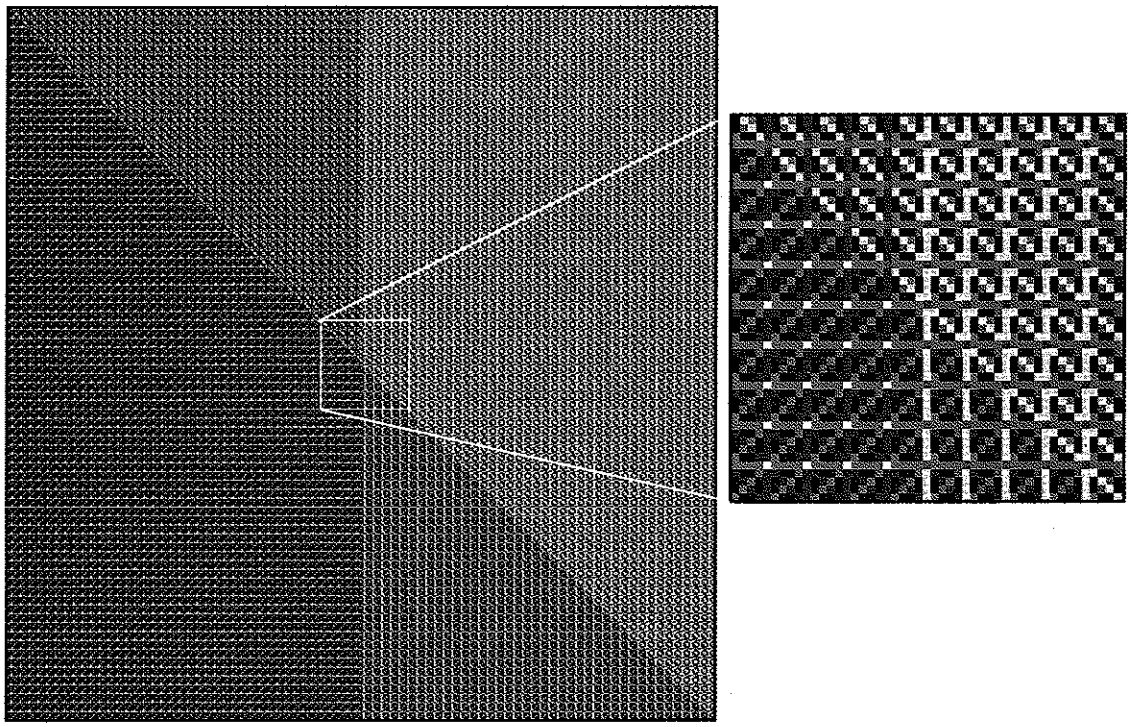


(b)

Fig. 14. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$ and $W = \{(3, 1), (-1, 2), (1, -2), (2, -1), (-2, -1)\}$. (b) Solution of maximal diversity for $H = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ and $W = \{(0, 1), (0, 17), (0, 81), (1, 1), (1, 17), (1, 81), (2, 1), (2, 17), (2, 81)\}$.

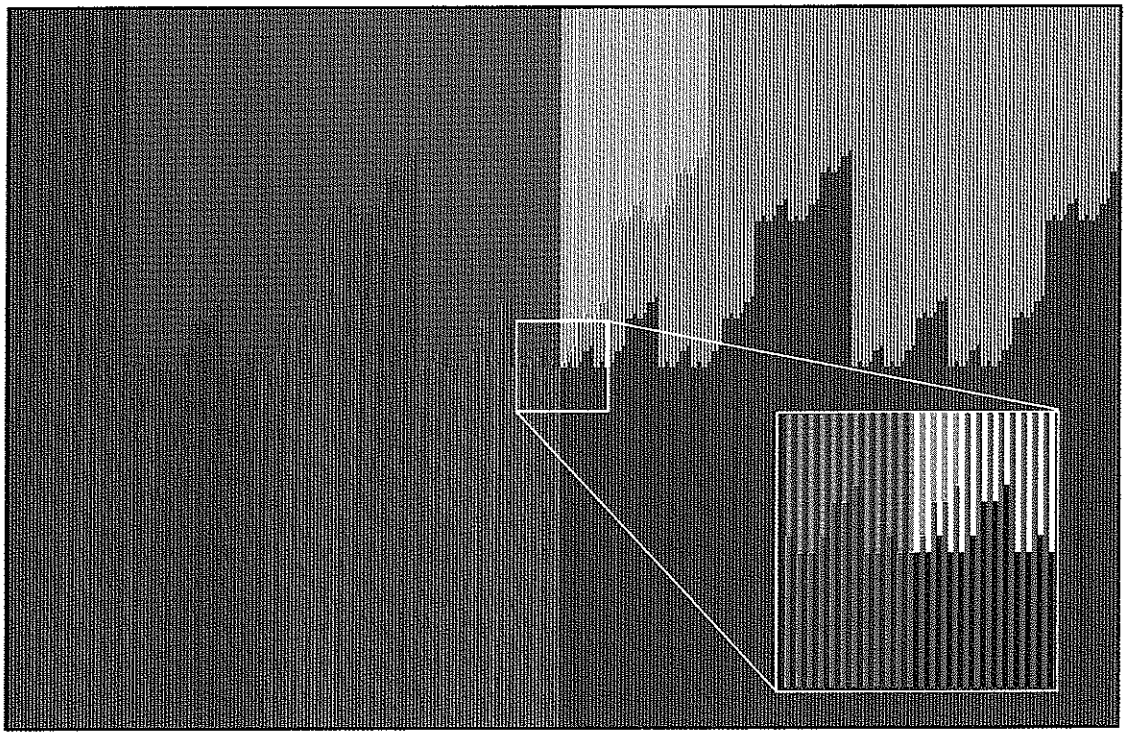


(a)

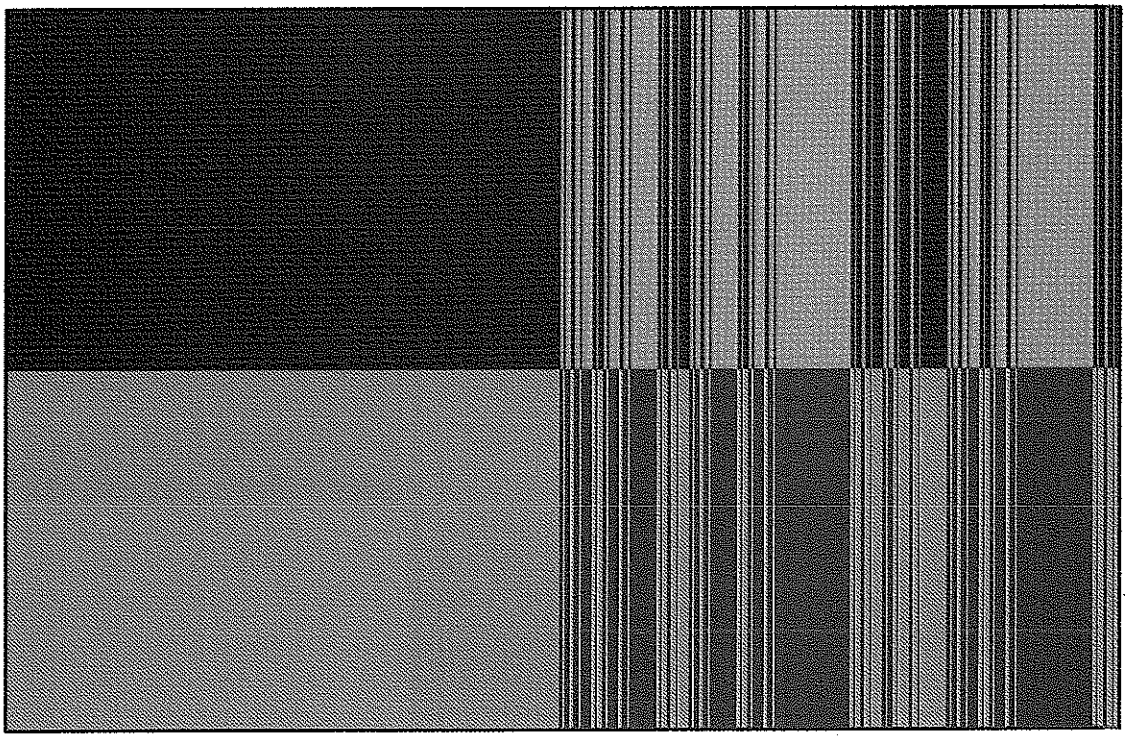


(b)

Fig. 15. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $W = \{(0, 0), (1, 0)\}$ (also same pattern for all shifts of this residue set). (b) Solution of maximal diversity for $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $W = \{(0, 1), (0, 5)\}$.

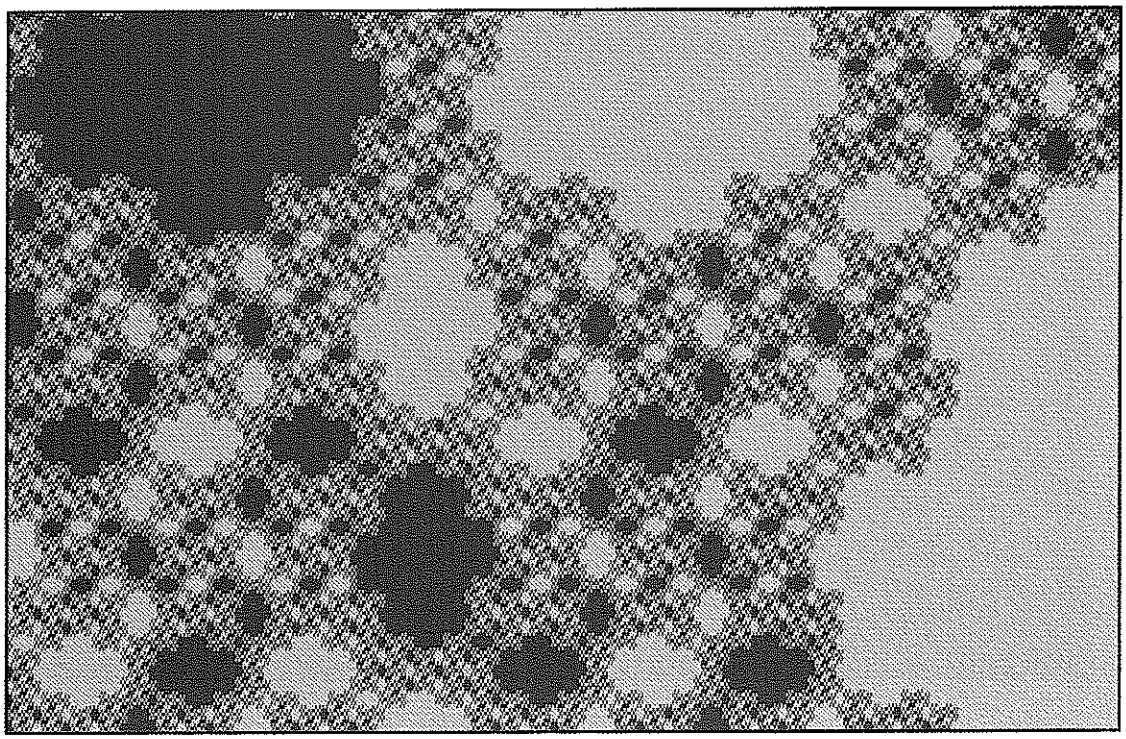


(a)

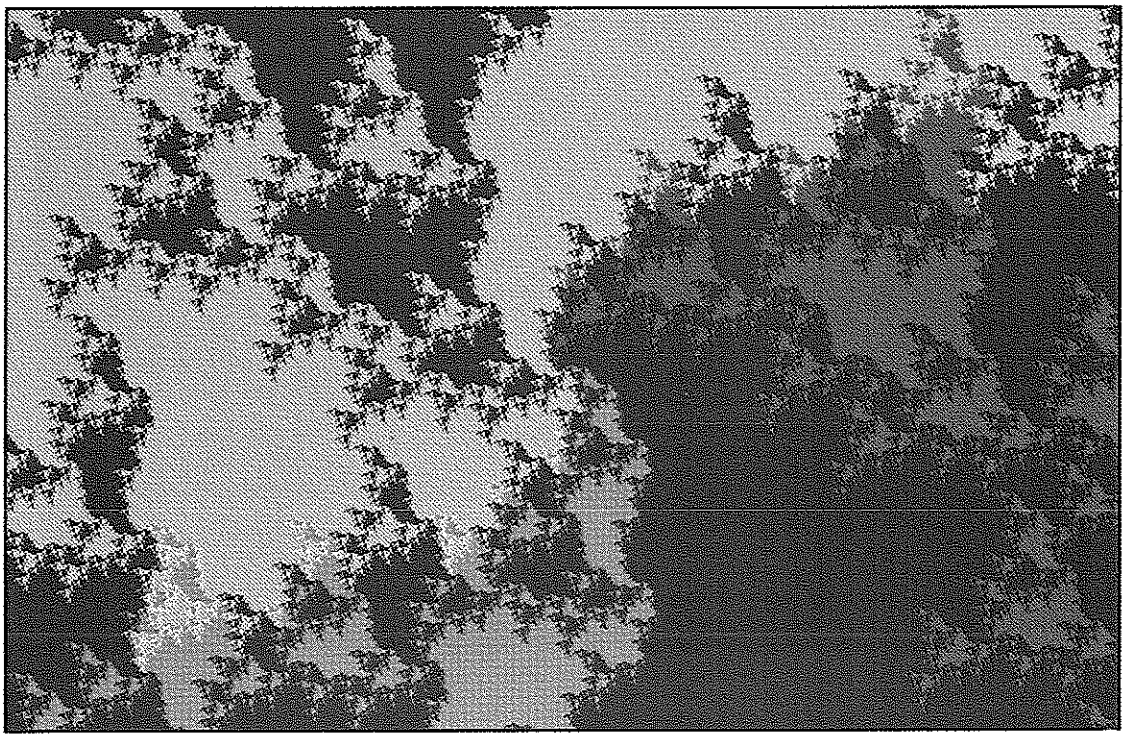


(b)

Fig. 16. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $W = \{(0, 0), (0, 1), (0, 2), (4, 3), (4, 4), (4, 5), (2, 0), (2, 1), (2, 2)\}$. (b) Solution of maximal diversity for $H = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $W = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1)(1, 2), (5, 0), (5, 1)(5, 2)\}$.

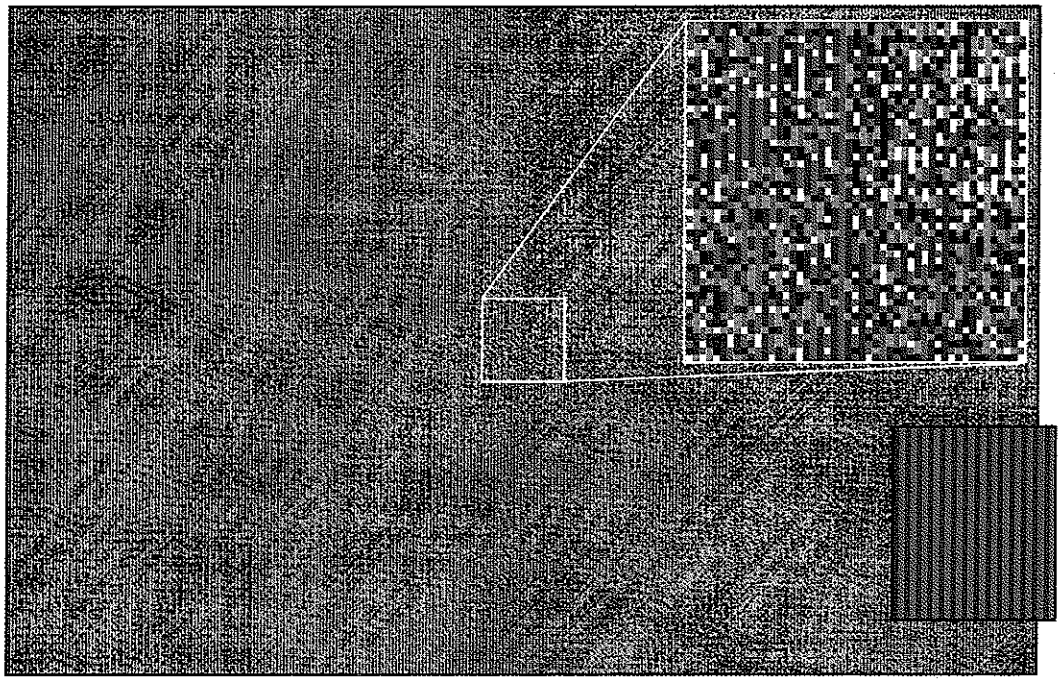


(a)

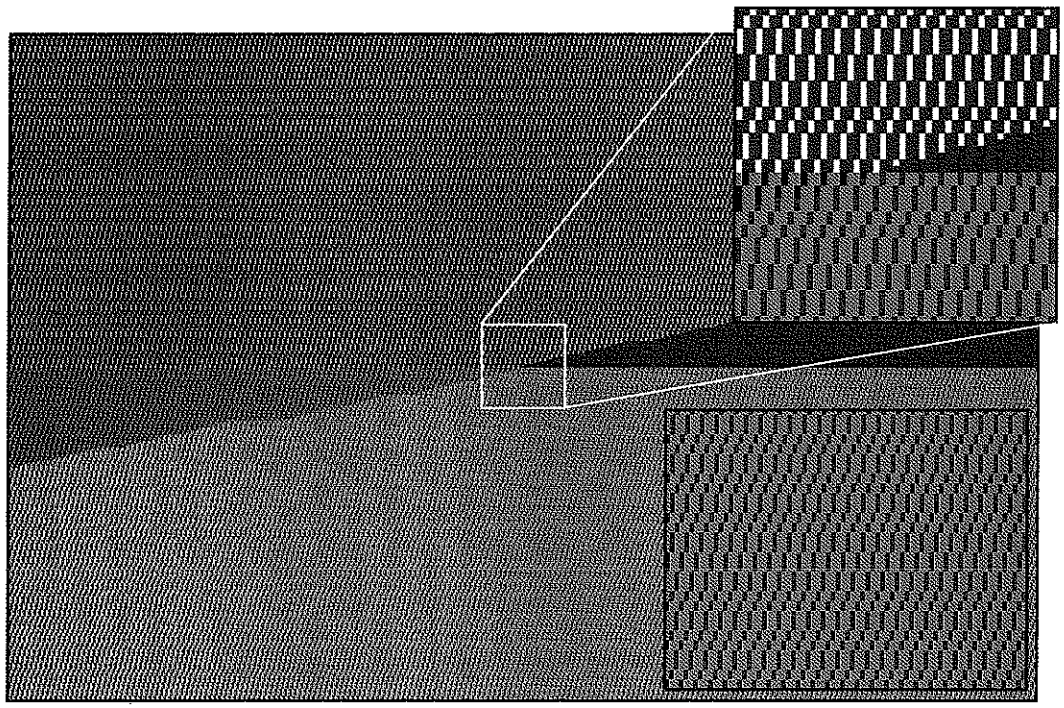


(b)

Fig. 17. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and $W = \{(0, 0), (0, -1), (1, 2), (1, 1)\}$. (b) Solution of maximal diversity for $H = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $W = \{(0, 0), (1, 1), (2, 0), (2, 1), (3, -1)\}$.



(a)



(b)

Fig. 18. (a) Solution of maximal diversity for the (H, W)-DI problem with $H = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $W = \{(0, 0), (0, -5), (4, 0)\}$. The green–red insert is the corresponding periodic solution of maximal diversity. (b) Solution of maximal diversity for $H = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $W = \{(0, 0), (3, 0), (3, 1), (6, 1)\}$. The corresponding periodic solution of maximal diversity (with period not of full rank) is shown in the green–red insert. It has period 3 in horizontal direction, while it is nonperiodic in any other direction. The only solution with full-rank period is the trivial solution.

15(b)] already displayed some fully periodic pixel-subset, this is not necessarily the case, as shown in Fig. 18(a). Figure 18(b) shows a DI-sequence which has a one-dimensional period (a non-full-rank period).

Acknowledgments

The authors thank G. Skordev for discussion and comments. This work was supported by the Concerted Action Project GOA-Mefisto of the Flemish Community and by the Belgian Program on Interuniversity Attraction Poles of the Belgian Prime Minister's Office for Science, Technology and Culture (IUAP P4-02). F. von Haeseler thanks the DFG (Deutsche Forschungsgemeinschaft) for financial support, and the Department of Electrical Engineering of the K. U. Leuven for its hospitality.

References

Bandt, C. [1991] "Self-similar sets 5: Integer matrices and fractal tilings of $\mathbb{R}^N$," *Proc. Am. Math. Soc.* **112**, 549–562.

Barbé, A., von Haeseler, F., Peitgen, H.-O. & Skordev, G. [1995] "Coarse-graining invariant orbits of one-dimensional two-state linear cellular automata," *Int. J. Bifurcation and Chaos* **5**, 1611–1631.

Barbé, A. [1996] "Coarse-graining invariant orbits of one-dimensional $\mathbb{Z}_p$-linear cellular automata," *Int. J. Bifurcation and Chaos* **6**, 2237–2297.

Barbé, A. [1997] "Complex order from disorder and from simple order in coarse-graining invariant orbits of certain two-dimensional linear cellular automata," *Int. J. Bifurcation and Chaos* **7**, 1451–1496.

Barbé, A. & von Haeseler, F. [2000] "Symmetries of decimation invariant sequences and digit sets," SISTA/Cosic report nr. 00-31, Department of Electrical Engineering, Katholieke Universiteit Leuven, *Theor. Comput. Sc.*, to appear.

Barbé, A. & Skordev, G. [2001] "Decimation invariant sequences and their automaticity," *Theor. Comput. Sc.* **259**, 379–403.

Lind, D. [1982] "Dynamical properties of quasihyperbolic toral automorphisms," *Ergod. Th. Dyn. Syst.* **2**, 49–68.

Lüneburg, H. [1989] *Tools and Fundamental Constructions of Combinatorial Mathematics* (BI-Wissenschaftsverlag, Mannheim).

Senechal, M. [1995] *Quasicrystals and Geometry* (Cambridge University Press).