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RESCALED EVOLUTION SETS OF LINEAR CELLULAR AUTOMATA ON A CYLINDER

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Rescaled evolution sets of linear cellular automata on a lattice $\mathbb{Z} \oplus G$, where G is a finite Abelian group, with states in the finite field $\mathbb{F}_q$ are defined and investigated.

Keywords: Cellular automata; finite fields; fractals; rescaling; scaling-sequences; selfsimilarity.

1. Introduction

Cellular automata (CA) are special dynamical systems being discrete in space and in time. The space is the set of maps, the so-called *configurations*, from a point-lattice L in a finite set S, the *state set*. A cellular automaton is a special map from the set of configurations, denoted as Σ and called *configuration space*, into itself. This map and its iterates provide a dynamical system with discrete time.

The lattice L under consideration is usually $\mathbb{Z}$ or $\mathbb{Z}^N$; but every finitely generated Abelian group can serve as a lattice. The elements of the lattice are also called *cells*. We will denote the elements of the configuration space Σ by $\underline{c}$. A cellular automaton is a map $A : \Sigma \to \Sigma$ such that the state of the cell l of $A(\underline{c})$ is given by a local rule R, as explained later.

The *orbit* (w.r.t. a cellular automaton A) of an initial configuration $\underline{c}$ is the set $\{A^t(\underline{c}) \,|\, t \in \mathbb{N}\}$, where we agree that $A^0(\underline{c}) = \underline{c}$ and $A^{t+1} = A \circ A^t$ are the iterates of A.

In order to define linear cellular automata (LCA), the state set S has to have an additional structure, namely that of a finite Abelian group, or finite commutative ring, or finite field. Then the configuration space, i.e. the maps from L to S, has also the structure of an additive group. The addition is defined pointwise, i.e. for $\underline{c}_1, \underline{c}_2 \in \Sigma$ the configuration $\underline{c}_1 + \underline{c}_2$ is defined as $(\underline{c}_1 + \underline{c}_2)(l) = \underline{c}_1(l) + \underline{c}_2(l)$ for $l \in L$.

A linear cellular automaton is a cellular automaton A such that the map $A : \Sigma \to \Sigma$ is a group homomorphism, i.e. $A(\underline{c}_1 + \underline{c}_2) = A(\underline{c}_1) + A(\underline{c}_2)$. This is equivalent to: There exists an $R \in \Sigma$ such that R has finite support, i.e. the set $\mathrm{supp}(R) = \{l \,|\, l \in L \text{ and } R(l) \neq 0\}$ is a finite set, such that $A(\underline{c}) = R * \underline{c}$ holds for all $\underline{c} \in \Sigma$, where $R * \underline{c}$ denotes the convolution defined as

$$(R * \underline{c})(l) = \sum_{s \in \mathrm{supp}(R)} \underline{c}(l - s) R(s) \,.$$

The map R, or the local rule is also called a (finite) mask, e.g. [Garzon, 1995; Wolfram, 1994].

In this paper, we consider lattices of the form $\mathbb{Z} \oplus G$, where $(G, +)$ is a finite Abelian group, and linear cellular automata with state set being a finite field $\mathbb{F}_q$ of characteristic p, a prime number, see [Allouche *et al.*, 2001; Allouche & Skordev, 2001; Aso & Honda, 1984]. Linear cellular automata on the lattice $\mathbb{Z}_m$ and values in $\mathbb{F}_2$ were considered in [Martin *et al.*, 1984].

If G is the cyclic group $\mathbb{Z}_m = \{0, \ldots, m-1\}$, the lattice $\mathbb{Z} \oplus \mathbb{Z}_m$ may be considered as a cylindrical tube of infinite length. This is illustrated in Fig. 1.

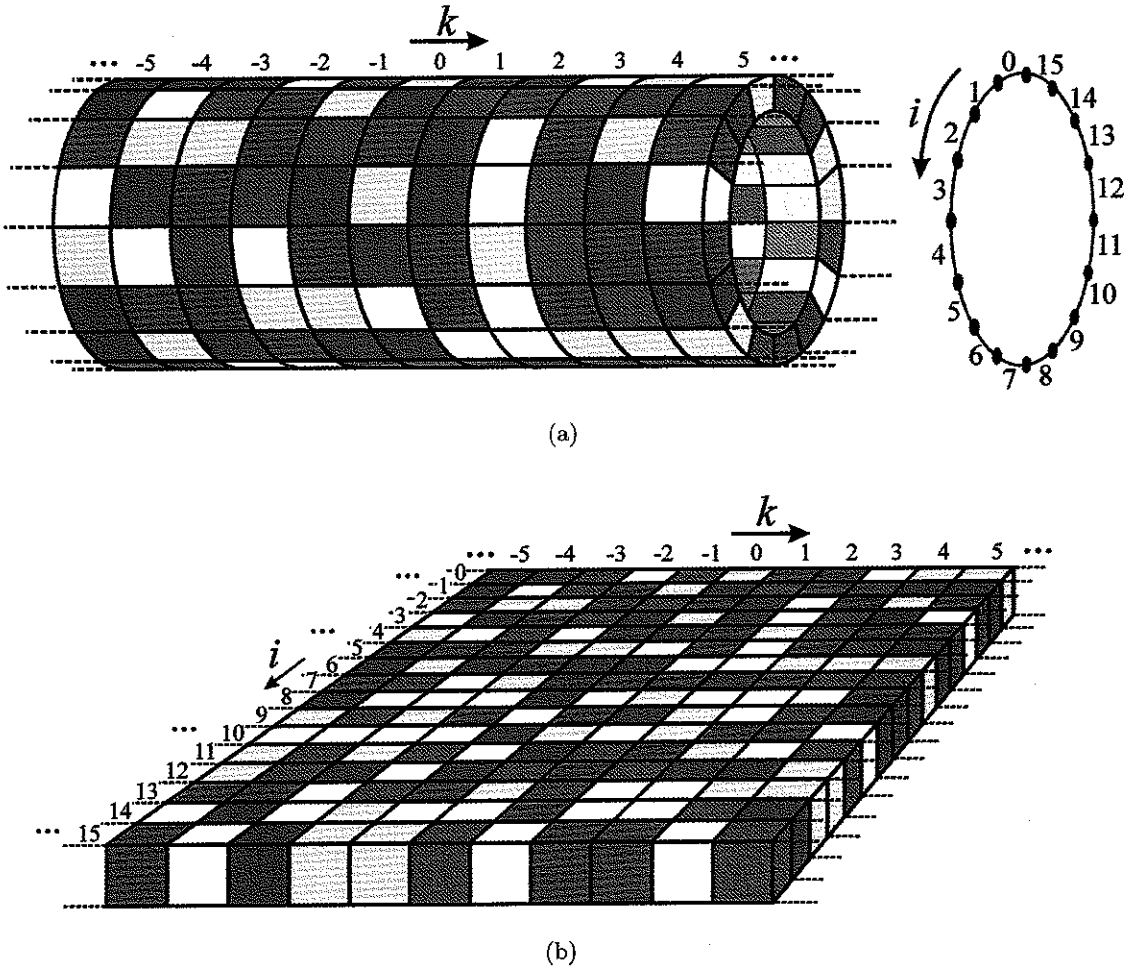


Fig. 1. (a) The cylindrical configuration-space of a cellular automaton defined on $\mathbb{Z} \oplus \mathbb{Z}_m$, with $m = 16$. Cell-states are represented by colors. White cells are in state 0. (b) Unfolded representation of the CA's configuration.

We are particularly interested in the long term behavior of the LCA defined in the above way. In order to achieve this goal we consider the graphical representation of the orbit together with a proper rescaling procedure. As already shown by [Willson, 1984, 1992] for the lattice $\mathbb{Z}$, this line of approach leads to fractal sets or rescaled evolution sets which resemble the long term evolution of the LCA. In several papers Willson's approach has been generalized even to nonlinear cellular automata, see [v. Haeseler *et al.*, 1992a, 1992b, 1993, 1995a, 2001; Takahashi, 1992] and the literature cited there.

The aim of this note is to show the existence of rescaled evolution sets also for lattices of the form $\mathbb{Z} \oplus G$, where G is a finite Abelian group. In the course of this note it will become clear that all reasoning presented is equally possible for lattices of the form $\mathbb{Z}^l \oplus G$.

The paper is organized as follows. In the second part we will provide the reader with the necessary definitions and technical details. In the third section we shall discuss the case $\mathbb{Z} \oplus \mathbb{Z}_m$, i.e. G is a cyclic group of order m, and p, the characteristic of the field, is relatively prime to m. We shall give criteria which will ensure the existence of rescaled evolution sets.

The third section is devoted to the case $G = \mathbb{Z}_m$ and p divides m. Again we will present criteria for the existence of rescaled evolution sets.

Finally, we investigate the general situation, i.e. $\mathbb{Z} \oplus G$, G finite Abelian group. As it will turn out, the understanding of the cyclic case $G = \mathbb{Z}_m$ easily leads to an understanding of the general case. The presentation of the general case is therefore brief.

In the Appendix we prove a result which is necessary for our purposes but not directly related to linear cellular automata.

2. Preliminaries, Definitions

We denote by $\mathbb{N}$ the natural numbers $\{0, 1, \ldots\}$, by $\mathbb{Z}$ the integers. A finite Abelian group is denoted by G and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} = \{0, \ldots, m-1\}$ denotes the cyclic group of order m, i.e. the integers modulo $m \geq 2$. By $|G|$ we denote the order of the group G. By $\mathbb{F}_q$ we denote the field with $q = p^\alpha$ elements for a given prime number p [Gallian, 1994; Liedl & Niederreiter, 1984].

With $\mathbb{F}_q[[x, x^{-1}]]$ we denote the F_q-vectorspace of Laurent series, i.e.

$$F_q[[x, x^{-1}]] = \left\{ f(x) = \sum_{j \in \mathbb{Z}} f_j x^j \;\middle\|\; f_j \in \mathbb{F}_q \right\}.$$

With $\mathbb{F}_q[x, x^{-1}]$ we denote the $\mathbb{F}_q$-vectorspace of Laurent polynomials, i.e. $f(x) \in \mathbb{F}_q[[x, x^{-1}]]$ is a Laurent polynomial if all but finitely many coefficients f_j are equal to zero.

2.1. *Linear cellular automata*

As already mentioned in the Introduction, the configuration space is $\Sigma = \mathbb{F}_q^{\mathbb{Z}\oplus G}$, and a configuration $\underline{c} \in \Sigma$ is a map from the lattice $\mathbb{Z} \oplus G$ to $\mathbb{F}_q$, i.e. $\underline{c} : \mathbb{Z} \oplus G \to \mathbb{F}_q$.

Since a map is determined by its values we may also think of $\underline{c}$ as a sequence $(c_{k,g})_{(k,g)\in\mathbb{Z}\oplus G}$, where $c_{k,g} = \underline{c}(k, g)$.

Instead of regarding $\underline{c}$ as a sequence we may consider $\underline{c}$ as a formal Laurent series, denoted then as $c(x, y)$ and written as

$$c(x, y) = \sum_{(k,g)\in\mathbb{Z}\oplus G} c_{k,g} x^k y^g ,$$

where $c_{k,g} = \underline{c}(k, g)$. Taking the algebraic structures into account we may consider $c(x, y)$ as an element of the $\mathbb{F}_q[\mathbb{Z} \oplus G]$-module $\mathbb{F}_q(\mathbb{Z} \oplus G) = \{\sum_{\gamma\in\mathbb{Z}\oplus G} f_\gamma x^\gamma \mid f_\gamma \in \mathbb{F}_q\}$, where $\mathbb{F}_q[\mathbb{Z} \oplus G]$ denotes the group ring of $\mathbb{Z} \oplus G$ over $\mathbb{F}_q$, see [Kargapolov & Merzljakov, 1979].

Furthermore, we may also regard $\underline{c}$ as a formal Laurent series $c(x)$ over $\mathbb{Z}$ with coefficients in $\mathbb{F}_q^G$. We then write

$$c(x) = \sum_{k\in\mathbb{Z}} \rho_k x^k , \tag{1}$$

where $\rho_k \in \mathbb{F}_q^G$ with $\rho_k(g) = \underline{c}(k, g)$,

Finally, the most important interpretation of $\underline{c}$ is the following. We consider $\underline{c}$ as an element of $\mathbb{F}_q[[x, x^{-1}]]^G$, i.e. $\underline{c}$ is regarded as a $|G|$-dimensional vector with entries in $\mathbb{F}_q[[x, x^{-1}]]$ and we write

$$\mathbf{C}(x) = (c_g(x))_{g\in G} ,$$

where $c_g(x) = \sum_{k\in\mathbb{Z}} \rho_k(g) x^k$.

As a next step, we consider the local rule R for the linear cellular automaton A with regard to the different representations of $\underline{c}$. As already mentioned, the local rule R is a configuration $R = (r_{k,g})_{(k,g)\in\mathbb{Z}\oplus G}$ such that there exist integers $d_1 \leq d_2$ with

$$\mathrm{supp}(R) \subset \{k \mid d_1 \leq k \leq d_2\} \times G .$$

The LCA $A : \Sigma \to \Sigma$ is defined as the convolution of $\underline{c}$ and R, i.e.

$$A(\underline{c})(k, g) = \sum_{h\in G} \sum_{l=d_1}^{d_2} r_{l,h} c_{k-l,g-h} \tag{2}$$

for all $\underline{c}$ and all $(k, g) \in \mathbb{Z} \oplus G$. Note that the operations $k - l$, $g - h$ have to be performed in the groups $\mathbb{Z}$ and G, respectively.

Equation (2), viewed in the framework of the $\mathbb{F}_q[\mathbb{Z} \oplus G]$-module $\mathbb{F}_q(\mathbb{Z} \oplus G)$, is nothing else but multiplication of $c(x, y)$ with the Laurent polynomial $R(x, y)$. In other words, if we regard $\underline{c}$ as an element of $\mathbb{F}_q(\mathbb{Z} \oplus G)$ and R as an element of $\mathbb{F}_q[\mathbb{Z} \oplus G]$, i.e. we write $c(x, y)$ for $\underline{c}$ and $R(x, y)$ for R, then $A(\underline{c})$ as an element of $\mathbb{F}_q(\mathbb{Z} \oplus G)$ is given by $R(x, y)c(x, y)$, i.e.

$$A(\underline{c})(x, y) = R(x, y)c(x, y) .$$

Finally, if $\underline{c}$ is considered as $\mathbf{C}(x)$, then the automaton A is modeled by a matrix multiplication. An analysis of the multiplication of $R(x, y)c(x, y)$ shows that there exists a matrix $\mathbf{R}(x) \in \mathbb{F}_q[x, x^{-1}]^{G\times G}$ and $|G|$ Laurent polynomials $r_g(x) = \sum_{k\in\mathbb{Z}} r_{k,g} x^k$ such that

$$\mathbf{A}(\underline{c})(x) = \mathbf{R}(x)\mathbf{C}(x) , \tag{3}$$

and $\mathbf{R}(x) = (r_{g-h}(x))_{g,h\in G}$. In other words, $A(\underline{c})$ regarded as an element of $\mathbb{F}_q[[x, x^{-1}]]^G$ and represented by $\mathbf{A}(\underline{c})(x)$ is given by the above product. As we will see, the structure of the matrix $\mathbf{R}(x)$ is determined by the structure of the group G.

Example. If $G = \mathbb{Z}_m$, then the matrix $\mathbf{R}(x)$ is of the form

$$\mathbf{R}(x) = \begin{pmatrix} r_0(x) & r_{m-1}(x) & r_{m-2}(x) & \cdots & r_1(x) \\ r_1(x) & r_0(x) & r_{m-1}(x) & \cdots & r_2(x) \\ r_2(x) & r_1(x) & r_0(x) & \cdots & r_3(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m-1}(x) & r_{m-2}(x) & r_{m-3}(x) & \cdots & r_0(x) \end{pmatrix} \tag{4}$$

where

$$r_i(x) = \sum_{k=d_1}^{d_2} r_{k,i} x^k , \tag{5}$$

for $i = 0, \ldots, m-1$. Note that the matrix $\mathbf{R}(x)$ is a circulant matrix, see [Davis, 1974].

We conclude by fixing notations related to the orbit $\{A^t(\underline{c}) \,|\, t \in \mathbb{N}\}$ of an initial configuration $\underline{c}$ with respect to A. If $\underline{c}$ is considered as a sequence, then the configuration $A^t(\underline{c})$ at time t is represented by $(c^{(t)}_{k,g})_{(k,g)\in\mathbb{Z}\oplus G}$. If $\underline{c}$ is interpreted as an element in $\mathbb{F}_q(\mathbb{Z}\oplus G)$, this configuration is denoted by $A^t(\underline{c})(x, y) = \sum_{k,g\in\mathbb{Z}\oplus G} c^{(t)}_{k,g} x^k y^g = R(x, y)^t c(x, y)$. Finally, if $\underline{c}$ is regarded as an element of $\mathbb{F}_q[[x, x^{-1}]]^G$, the configuration at time t is denoted by the $\mathbb{F}_q^G$-vector $\mathbf{A^t}(\underline{c})(x)$, with components $(\mathbf{A^t}(\underline{c})_g(x)$, $g \in G$ and we have $\mathbf{A^t}(\underline{c})(x) = \mathbf{R}(x)^t \mathbf{C}(x)$.

2.2. *Graphical representation and rescaled evolution set of the orbits of LCA on $\mathbb{Z} \oplus G$*

In this section we introduce the notion of a graphical representation. In [v. Haeseler *et al.*, 1993, 1995a, 2001; Takahashi 1992; Willson, 1984, 1992] graphical representations are considered for orbits of one-dimensional cellular automata on the point lattice $\mathbb{Z}$ with states in a finite set. The orbit, which represents the evolution of the CA's configuration over time is then defined on the two-dimensional point lattice $\mathbb{Z} \times \mathbb{N}$, and the corresponding graphical representation of this orbit is represented as a subset of the plane $\mathbb{R}^2$.

In the case at hand, the orbit is defined on the three-dimensional point lattice $\mathbb{Z} \times G \times \mathbb{N}$, (as illustrated in Fig. 2(a) for $G = \mathbb{Z}_m$), and its graphical representation is done in $(\mathbb{R}^2)^G$, i.e. in $|G|$ slices with a two-dimensional graphical representation in $\mathbb{R}^2$, as illustrated in Figs. 2(b) and 2(c) for $G = \mathbb{Z}_m$.

If $\underline{c}$ is an initial configuration of the automaton A with local rule R and if $t \in \mathbb{N}$, then the point-set

$$Y_t(\underline{c}, R) = \{[k, k+1] \times \{g\} \times [s, s+1] \,|\, A^s(\underline{c})(k, g) \neq 0,\ (k, g) \in \mathbb{Z} \oplus G,\ 0 \leq s < t\} \subset (\mathbb{R}^2)^G \tag{6}$$

is called *the graphical representation*, up to time t, of the orbit of $\underline{c}$ w.r.t. R. Fixing $g \in G$ in Eq. (6) gives the *graphical representation of slice* g, and is denoted by $Y_t(\underline{c}, R)_g$. Figure 3 displays some examples for $G = \mathbb{Z}_3$.

Consider the space $\mathcal{H}(\mathbb{R}^2)$ of all nonempty compact subsets of the plane $\mathbb{R}^2$. In this space we introduce the Hausdorff metric $h(.,.)$, induced by the maximum norm $\| \ \|_\infty$ in $\mathbb{R}^2$, see [Edgar, 1990, p. 66]. The distance $h(A, B)$ between $A, B \in \mathcal{H}(\mathbb{R}^2)$ is defined as

$$h(A, B) = \inf\{\varepsilon > 0 \,|\, B \subset O_\varepsilon(A),\ A \subset O_\varepsilon(B)\},$$

where the ε-collar $O_\varepsilon(A)$ (and similar $O_\varepsilon(B)$) is defined as

$$O_\varepsilon(A) = \{M \in \mathbb{R}^2 \,|\, \text{ there is } M_1 \in A \text{ with } \|M - M_1\|_\infty < \varepsilon\}$$

and $\|M - M_1\|_\infty = \max\{|x - x_1|, |y - y_1|\}$, with $M = (x, y)$, and $M_1 = (x_1, y_1)$. Equipped with this distance, $\mathcal{H}(\mathbb{R}^2)$ becomes a complete metric space.

For $g \in G$ and an initial configuration $\underline{c}$ with finite nonempty support, the sequence

$$\left(\frac{1}{p^t} Y_{p^t}(\underline{c}, R)_g\right)_{t\in\mathbb{N}}$$

is a sequence of compact subsets. If this sequence is a Cauchy sequence w.r.t. the Hausdorff distance, then we call the limit the *rescaled evolution set*, and denote the limit as $\mathcal{A}_g(\underline{c}, R)$.

Remark 1. The rescaled evolution sets when the rule $R(x, y) = 0$ do not exist. If $R(x, y) \neq 0$ and $R(x, y)^t = 0$ for some $t \in \mathbb{N}$, then the rescaled evolution sets are all equal to $\{(0, 0)\}$.

2.3. *k-Fermat and weak k-Fermat polynomials*

In this section we introduce the important notion of k-Fermat Laurent polynomials and weak

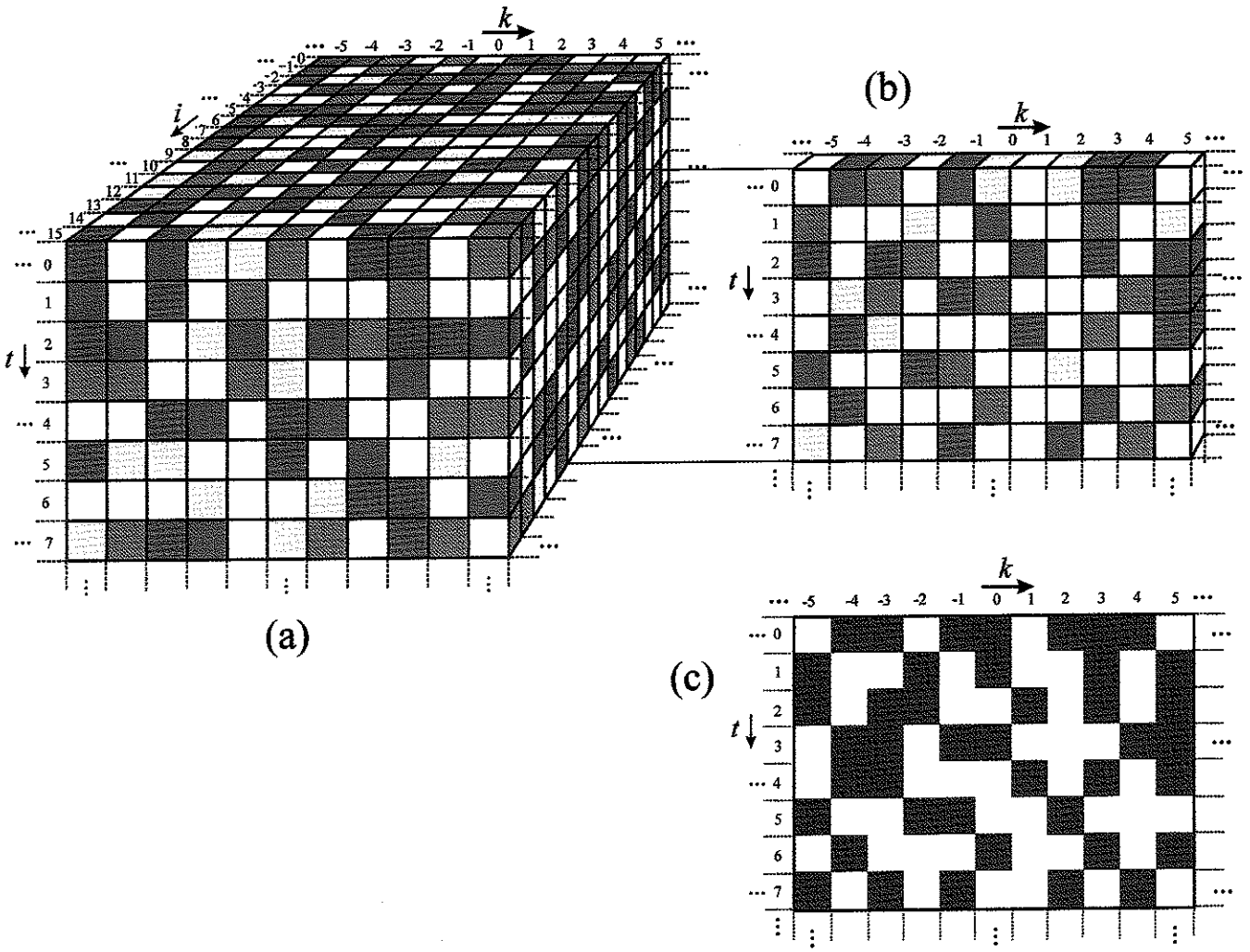


Fig. 2. (a) Three-dimensional representation of the orbit of a CA on $\mathbb{Z}\oplus\mathbb{Z}_{16}$. It represents the evolution of a CA's configuration over time t. (b) Extracting an orbit slice (slice # 10). Replacing all nonwhite cells, i.e. cells with nonzero state, by cells of the same color (e.g. blue), and restricting the representation in the plane $\mathbb{R}^2$, as displayed in (c), forms the graphical representation of the orbit slice. The set of graphical representations of all m orbit slices forms the graphical representation of the complete orbit.

k-Fermat Laurent polynomials, see [Allouche *et al.*, 1996; v. Haeseler *et al.*, 1993]. Let $\mathcal{K}$ be a finite commutative ring (with 1) and let $k \in \mathbb{Z}$, $k \geq 2$. The Laurent polynomial $\alpha(x) \in \mathcal{K}[x, x^{-1}]$ is called *k-Fermat* Laurent polynomial (or is a Laurent polynomial with k-Fermat property) if

$$\alpha(x)^k = \alpha(x^k)\,.$$

Every Laurent polynomial with coefficients in $\mathbb{F}_q$ is q-Fermat polynomial, see e.g. [Lidl & Niederreiter, 1984].

A Laurent polynomial $\alpha(x) \in \mathcal{K}[x, x^{-1}]$ is called *weak k-Fermat* if for

$$\alpha(x) = \sum_i \alpha_i x^i$$

the kth power is given by

$$\alpha(x)^k = \sum_i a_i x^{ki}\,,$$

and $\alpha_i \neq 0$ if and only if $a_i \neq 0$.

Obviously, every k-Fermat Laurent polynomial is a weak k-Fermat Laurent polynomial.

Every Laurent polynomial $\alpha(x) \in \mathbb{F}_q[x, x^{-1}]$, $q = p^\alpha$ and $\alpha \geq 1$ is a weak p-Fermat. In fact, for $\alpha = 1$ it is p-Fermat.

Given a weak k-Fermat Laurent polynomial $\alpha(x) \in \mathbb{F}_q[x, x^{-1}]$, then $\alpha(x)$ defines a

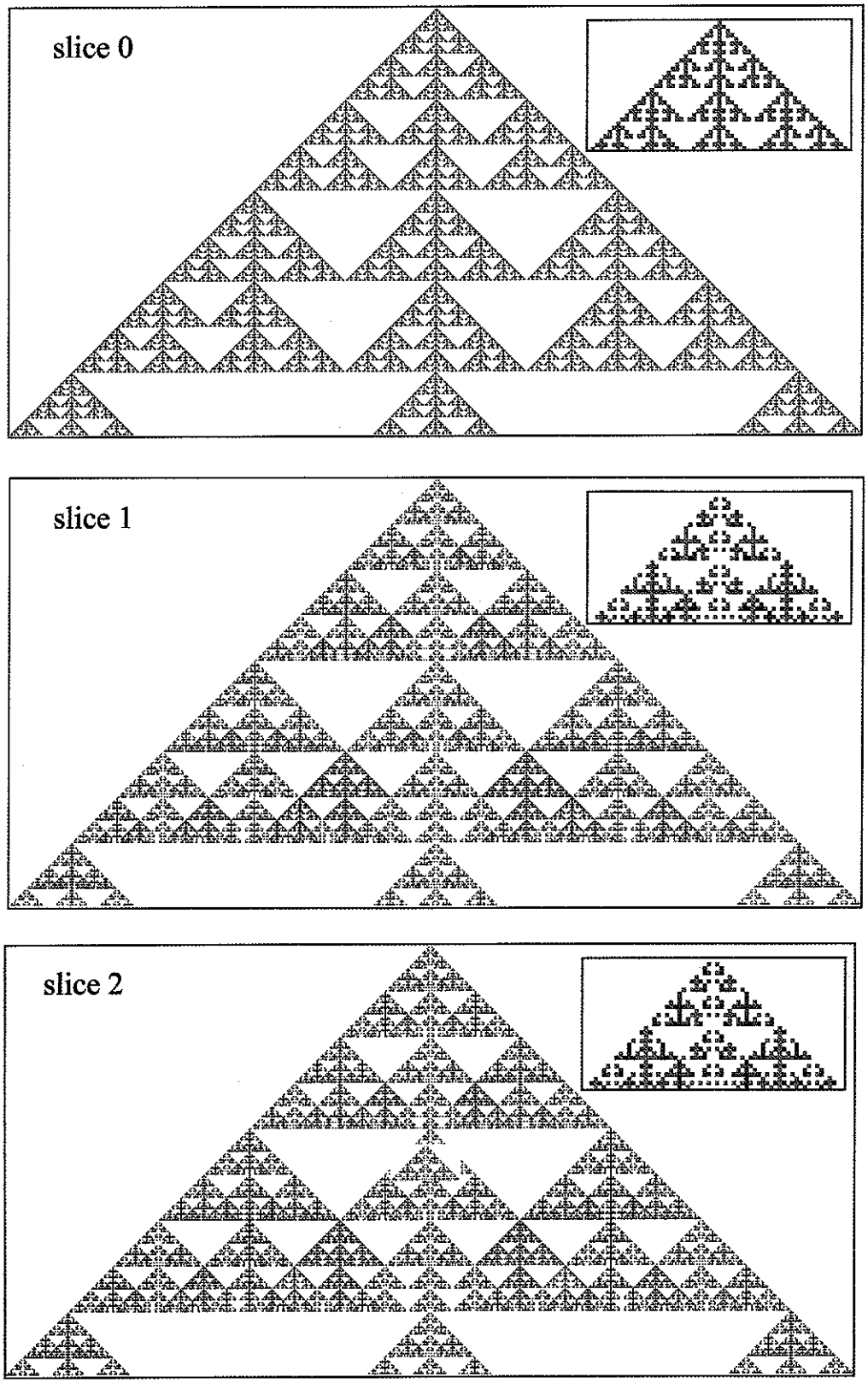
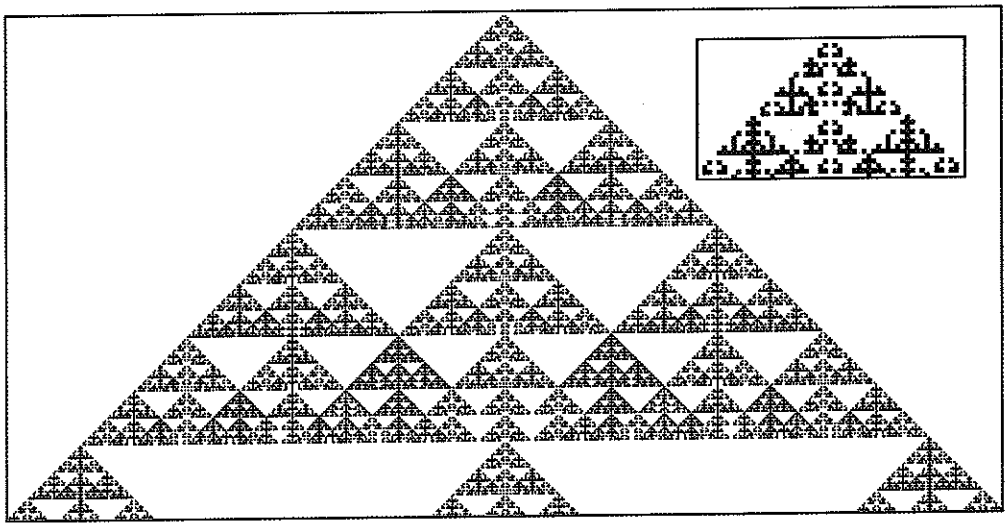
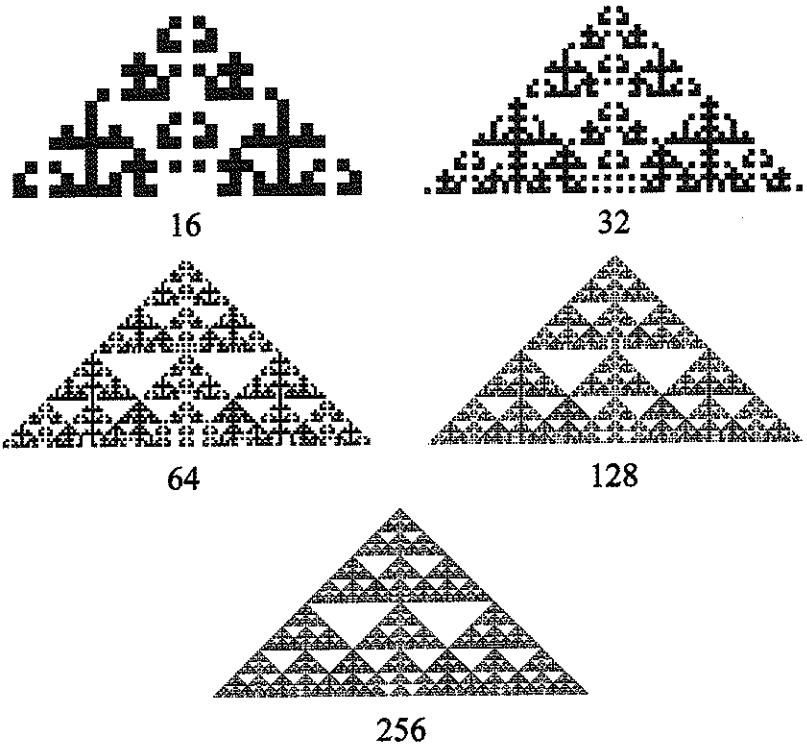


Fig. 3. The "colored" graphical representations $Y_{300}(1, R)_i$, $i = 0, 1, 2$ of the three orbit slices for a CA on $\mathbb{Z} \oplus \mathbb{Z}_3$ with values in $\mathbb{F}_4 = \{0, 1, \zeta, \zeta^2\}$, with $\zeta^2 = 1 + \zeta$. The local rule is defined by the matrix $R = \begin{pmatrix} r_{-1,0} & r_{0,0} & r_{1,0} \\ r_{-1,1} & r_{0,1} & r_{1,1} \\ r_{-1,2} & r_{0,2} & r_{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \zeta & 0 & \zeta \\ 1 & 0 & 1 \end{pmatrix}$. The initial configuration is $c(x, y) = 1$. The inserts are enlargements of the top region of the patterns. Color code: white $= 0$, green $= 1$, red $= \zeta$, blue $= \zeta^2$.



(a)



(b)

Fig. 4. (a) The (mono-colored) graphical representation $Y_{300}(1, R)_1$ of orbit slice 1 of the previous figure. (b) Rescaling of the graphical representations $Y_t(1, R)_1$ for $t = 16, 32, 64, 128, 256$, i.e. the orbit slice under (a) is cut off at $t = 15, 31, 63, 127, 255$, after which the corresponding patterns are rescaled to the same size. With the graphical resolution available in this reproduction, the final pattern ($t = 256$) is almost indistinguishable from the limit $\mathcal{A}_1(1, R)$ (the rescaled evolution set).

one-dimensional linear cellular automaton on $\mathbb{F}_q[[x, x^{-1}]]$, whereby the local rule is given by $\alpha(x)$, and the orbit of an initial configuration $\beta(x)$ by $\alpha(x)^t\beta(x)$. The orbit of the initial configuration $\beta(x) = 1$ is then given by $H_\alpha(x, z) = \sum_{t=0}^{\infty} \alpha(x)^t z^t$, which is formally equal to the rational function

$$H_\alpha(x, z) = \frac{1}{1 - z\alpha(x)}.$$

Theorem 4.5 in [v. Haeseler *et al.*, 1992b] states that the rescaled evolution set of this orbit, denoted by $\mathcal{A}(\alpha(x))$, exists and that the scaling sequence is

$(p^t)_{t\in\mathbb{N}}$. By abuse of language, we sometimes refer to the rescaled evolution set $\mathcal{A}(\alpha)$ as the *rescaled evolution set of the Laurent polynomial* $\alpha(x)$ or as the *rescaled evolution set of the rational function* H_α. Moreover, as a consequence of Lemma 3.16 in part II of [v. Haeseler *et al.*, 2001] we note the following.

Lemma 1. *Let* $\alpha(x)$, $\beta(x) \in \mathbb{F}_q[x, x^{-1}]$. *If* $\beta(x) \neq 0$, *then the rational function*

$$\frac{\beta(x)}{1 - z\alpha(x)}$$

has scaling sequence $(p^t)_{t\in\mathbb{N}}$ *and the rescaled evolution set is equal to* $\mathcal{A}(\alpha)$.

2.4. *Automatic sequences*

We introduce the notion of automatic sequences and refer to [Allouche & Mendès France, 1995; v. Haeseler, 2003; v. Haeseler & Petersen, 1998; Petersen, 1997; Salon, 1986, 1987] for more details.

Let $\mathcal{K}$ be a finite set. A two-dimensional sequence $\underline{a}$ is a map $\underline{a} : \mathbb{Z}^2 \to \mathcal{K}$.

For $k \geq 2$ a natural number and $(u, v) \in \{0, \ldots, k-1\}^2$, we define the maps $D_{u,v} : \mathcal{K}^{\mathbb{Z}^2} \to \mathcal{K}^{\mathbb{Z}^2}$ as

$$D_{u,v}(\underline{a})(l_1, l_2) = \underline{a}(kl_1 + u, kl_2 + v)\,.$$

The maps $D_{u,v}$ are called k-decimations and $D_{u,v}(\underline{a})$ is called k-decimation of $\underline{a}$.

Considering the sequence $\underline{a}$ as a formal Laurent series, i.e.

$$a(x, y) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \underline{a}(l_1, l_2)x^{l_1}y^{l_2}\,,$$

then we have

$$D_{u,v}(\underline{a})(x, y, z) = \sum_{(l_1,l_2)\in\mathbb{Z}^2} \underline{a}(kl_1 + u, kl_2 + v)x^{l_1}y^{l_2}$$

as the formal series for the decimation. The sequence $\underline{a}$ is called k-automatic if the set of all iterated k-decimations

$$\{D_{u_s,v_s} \circ \cdots \circ D_{u_0,v_0}(\underline{a}) \,|\, s \in \mathbb{N}\}$$

is finite. This set, augmented by $\underline{a}$, is called the *k-kernel* of $\underline{a}$ and is denoted by $\ker_k(\underline{a})$. The sequence $\underline{a}$ above is a two-dimensional sequence, but in the same spirit, one can also consider k-automatic sequences in any dimension.

We say that the Laurent series of a k-automatic sequence is k-automatic.

Remark 2

1. All rational functions (in x, y) with coefficients in the field $\mathbb{F}_q$, $q = p^\alpha$, are p-automatic [Allouche & Mendès France, 1995; Salon, 1986, 1987].
2. The rational functions of one variable with coefficients in a finite commutative ring are k-automatic for all k, since their power series are ultimately periodic sequences. For rational functions of more than one variable this is usually not the case, see [v. Haeseler & Petersen, 1998].

3. The Case $\mathbb{Z} \oplus \mathbb{Z}_m$ with m and p Relatively Prime

In this section we discuss the situation of the existence of rescaled evolution sets of orbit slices and the dependence on the initial condition of rescaled evolution for an LCA defined on the Abelian group $\mathbb{Z}\oplus\mathbb{Z}_m$ under the hypothesis that the order m of $\mathbb{Z}_m$ and the characteristic p of $\mathbb{F}_q$ are relatively prime. In the next section we shall discuss how to approach the case p divides m.

The key for describing the behavior of the linear cellular automaton on $\mathbb{F}_q^{\mathbb{Z}\oplus\mathbb{Z}_m}$ is an explicit formula for the powers of the $m\times m$ polynomial matrix $\mathbf{R}(x)$ in Eq. (4). As already mentioned, $\mathbf{R}(x)$ is a circulant matrix with coefficients in $\mathbb{F}_q[x, x^{-1}]$, therefore we can write

$$\mathbf{R}(x) = \sum_{j=0}^{m-1} r_j(x)P^j\,, \tag{7}$$

where $r_j(x) \in \mathbb{F}_q[x, x^{-1}]$ and P is the $m \times m$ permutation matrix defined as

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8}$$

Note that P itself is a circulant matrix such that $P^m = \mathrm{Id}$. The first result allows us to describe the

powers of the matrix $\mathbf{R}(x)$ more easily. It states that there exists a field extension of $\mathbb{F}_q$, namely the mth cyclotomic field over $\mathbb{F}_q$ (see [Lidl & Niederreiter, 1984]), such that the matrix $\mathbf{R}(x)$ is equivalent to a diagonal matrix.

Theorem 1. *Let p and m be relatively prime. If $\mathbb{F}_q^{(m)}$ denotes the mth cyclotomic field over $\mathbb{F}_q$, then there exists a $\zeta \in \mathbb{F}_q^{(m)}$ and a matrix $U \in (\mathbb{F}_q^{(m)})^{m\times m}$ such that*

$$\mathbf{R}(x) = U \begin{pmatrix} \alpha_0(x) & 0 & \cdots & 0 \\ 0 & \alpha_1(x) & \cdots & 0 \\ \vdots & & \ddots & \cdots \\ 0 & \cdots & 0 & \alpha_{m-1}(x) \end{pmatrix} U^{-1}.$$

Moreover, if ζ is a primitive root of unity in $\mathbb{F}_q^{(m)}$, then

$$U = (\zeta^{ij})_{i,j=0,\ldots,m-1}, \tag{9}$$

and

$$\begin{pmatrix} \alpha_0(x) \\ \vdots \\ \alpha_{m-1}(x) \end{pmatrix} = U \begin{pmatrix} r_0(x) \\ \vdots \\ r_{m-1}(x) \end{pmatrix}.$$

Proof. One easily computes that $\det(P - \lambda \mathrm{Id}) = \lambda^m - 1$. The splitting field for the polynomial $x^m - 1$ is the mth cyclotomic field over $\mathbb{F}_q$, see e.g. [Lidl & Niederreiter, 1984], and it is denoted as $\mathbb{F}_q^{(m)}$. The roots of unity in $\mathbb{F}_q^{(m)}$ are generated by ζ, i.e. they are $1, \zeta, \zeta^2, \ldots, \zeta^{m-1}$, with $\zeta^m = 1$. Following [Davis, 1974] we form the matrices $U = (\zeta^{ij})_{i,j=0,\ldots,m-1}$ and $U^* = (\zeta^{m-ij})_{i,j=0,\ldots,m-1}$. Then one computes that $UU^* = m\,\mathrm{Id}$. Since m and p are relatively prime we have that $U^{-1} = m^{-1}U^*$. Moreover, one has

$$P = U \,\mathrm{diag}(\zeta^0, \zeta, \ldots, \zeta^{m-1}) U^{-1},$$

where $\mathrm{diag}(a_0, \ldots, a_{m-1})$ denotes a diagonal matrix whose entries on the diagonal are specified.

Since $\mathbf{R}(x) = \sum r_j(x) P^j$, we have that

$$\mathbf{R}(x) = U \left(\sum_{j=0}^{m-1} r_j(x) \,\mathrm{diag}(\zeta^0, \ldots, \zeta^{m-1})^j \right) U^{-1}.$$

Moreover, if we set

$$\alpha_i(x) = \sum_{j=0}^{m-1} r_j(x) \zeta^{ji} \tag{10}$$

for $i = 0, \ldots, m-1$, we obtain $\mathbf{R}(x) = U(\sum_{j=0}^{m-1} \mathrm{diag}(\alpha_0(x), \ldots, \alpha_{m-1}(x)) U^{-1}$. This also proves the relation between the vectors formed by α_i and r_i. ■

Example. Consider an automaton on $\mathbb{Z} \oplus \mathbb{Z}_3$, i.e. $m = 3$, with states in $\mathbb{F}_q$ and $\gcd(3, q) = 1$. Then the matrix U, with elements in the field extension $\mathbb{F}_q^{(3)}$, is given by

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 \\ 1 & \zeta^2 & \zeta \end{pmatrix},$$

and the matrix $\mathbf{R}(x) = r_0(x) + r_1(x)P + r_2(x)P^2$ is then equivalent to the diagonal matrix

$$\begin{pmatrix} r_0(x) + r_1(x) + r_2(x) & 0 & 0 \\ 0 & r_0(x) + \zeta r_1(x) + \zeta^2 r_2(x) & 0 \\ 0 & 0 & r_0(x) + \zeta^2 r_1(x) + \zeta r_2(x) \end{pmatrix}.$$

This example shall be continued later.

Theorem 1 now provides an explicit formula for the powers of $\mathbf{R}$. Indeed, we have

$$\mathbf{R}(x)^n = U \,\mathrm{diag}(\alpha_0(x)^n, \ldots, \alpha_{m-1}(x)^n) U^{-1}$$

for all $n \in \mathbb{N}$.

As a consequence of Theorem 1 we obtain an explicit formula for the series $F_{\mathbf{R}}(x, z) = \sum_{t=0}^{\infty} \mathbf{R}(\mathbf{x})^t z^t$ and also another formula for $\mathbf{R}(x)^n$.

For given $a_i \in \mathbb{F}_q^{(m)}$, $i = 0, \ldots, m-1$, we define the circulant $m \times m$-matrix $\mathrm{circ}(a_0, \ldots, a_1)$ as

$$\mathrm{circ}(a_0, \ldots, a_{m-1}) = \begin{pmatrix} a_0 & a_{m-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & & \ddots & \vdots \\ a_{m-1} & a_{m-2} & \cdots & a_0 \end{pmatrix}.$$

Lemma 2. *The series $F_{\mathbf{R}}(x, z)$ is of the form*

$$F_{\mathbf{R}}(x,z) = \sum_{t=0}^{\infty} \mathbf{R}(x)^t z^t$$
$$= U \operatorname{diag}\left(\frac{1}{1-z\alpha_0(x)}, \ldots, \frac{1}{1-z\alpha_{m-1}(x)}\right) U^{-1}.$$

Moreover, if $G_j = m^{-1}\operatorname{circ}(1, \zeta^j, \zeta^{2j}, \ldots \zeta^{(m-1)j})$ *are circulant matrices for* $j = 0, \ldots, m-1$, *then*

$$F_{\mathbf{R}}(x, z) = \frac{1}{1-z\alpha_0(x)} G_0 + \frac{1}{1-z\alpha_1(x)} G_1 + \cdots + \frac{1}{1-z\alpha_{m-1}(x)} G_{m-1}.$$

Proof. We have that $\sum \mathbf{R}(x)^t z^t = \sum U \operatorname{diag}(\alpha_0(x)^t, \ldots, \alpha_{m-1}(x)^t) z^t U^{-1}$, and using the fact that $\sum a(x)^t z^t = 1/(1-za(x))$ for a polynomial $a(x)$ we obtain the first assertion.

Performing the matrix multiplications and collecting the terms $1/(1-z\alpha_j(x))$ prove the second assertion. ■

Example (continued). For the example that was previously considered, we find that G_i in the above lemma are given by $G_0 = 1/3\operatorname{circ}(1, \zeta^0, \zeta^0) = \operatorname{circ}(1, 1, 1)$, $G_1 = 1/3\operatorname{circ}(1, \zeta, \zeta^2)$, $G_2 = 1/3\operatorname{circ}(1, \zeta^2, \zeta^4)$. This gives

$$F_{\mathbf{R}}(x, z) = \frac{3^{-1}}{1-z\alpha_0(x)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{3^{-1}}{1-z\alpha_1(x)} \begin{pmatrix} 1 & \zeta^2 & \zeta \\ \zeta & 1 & \zeta^2 \\ \zeta^2 & \zeta & 1 \end{pmatrix} + \frac{3^{-1}}{1-z\alpha_2(x)} \begin{pmatrix} 1 & \zeta & \zeta^2 \\ \zeta^2 & 1 & \zeta \\ \zeta & \zeta^2 & 1 \end{pmatrix}, \quad (11)$$

where $\alpha_0(x) = r_0(x)+r_1(x)+r_2(x)$, $\alpha_1(x) = r_0(x)+\zeta r_1(x)+\zeta^2 r_2(x)$, $\alpha_2(x) = r_0(x)+\zeta^2 r_1(x)+\zeta r_2(x)$.

As a consequence of Lemma 2, we have the following result on the existence of rescaled evolution sets for the orbit slices.

Theorem 2. (*LCA on* $\mathbb{Z} \oplus \mathbb{Z}_m$ *with states in* $\mathbb{F}_q$, $q = p^n$, p *prime*, m *and* p *relatively prime*). *Let* $c(x) = \sum_{k\in\mathbb{Z}} \rho_k x^k$, *see Eq.* (*1*), *represent an initial configuration* $\underline{c}$ *such that* $\rho_k \equiv 0$ *for all* $k \neq 0$ *and* $\rho_0 \not\equiv 0$. *If the rational functions* $H_{\alpha_j}(x, z) = 1/(1-z\alpha_j(x))$ *are* $\mathbb{F}_q^{(m)}$*-linearly independent, i.e. if* $\sum_{j=0}^{m-1} \lambda_j H_{\alpha_j} = 0$ *with* $\lambda_j \in \mathbb{F}_q^{(m)}$ *implies that* $\lambda_j = 0$ *for all* j, *then for every* $j \in \mathbb{Z}_m$ *the sequence*

$$\left(\frac{1}{p^t} Y_{p^t}(\underline{c}, R)_j\right)_{t\in\mathbb{N}}$$

is a Cauchy sequence w.r.t. the Hausdorff distance.

Proof. If ρ_0 is considered as an element of $\mathbb{F}_q^m$, then one has

$$\mathbf{A^t}(\underline{c})(x) = \mathbf{R}(x)^t \rho_0,$$

and

$$\sum_{t=0}^{\infty} z^t \mathbf{A^t}(\underline{c})(x) = \sum_{t=0}^{\infty} z^t \mathbf{R}(x)^t \rho_0 = F_{\mathbf{R}}(x, z)\rho_0. \quad (12)$$

Using the expression for $F_{\mathbf{R}}(x, z)$ in Lemma 2 gives

$$\sum_{t=0}^{\infty} z^t \mathbf{A^t}(\underline{c})(x) = \sum_{j=0}^{m-1} \frac{1}{1-z\alpha_j(x)} F_j \rho_0. \quad (13)$$

Due to the structure of the matrices F_j and due to the fact that ρ_0 is not the zero vector it follows that at least one of the summands is different from zero. In fact, if one component of $F_j\rho_0$ is different from zero, then all components are different from zero. Together with the linear independence of the rational functions H_{α_j} it follows that the sum is not equal to 0 and moreover, no component is equal to zero.

From Eq. (12), it follows that the jth component of $\sum_{t=0}^{\infty} z^t \mathbf{A^t}(\underline{c})(x)$, which gives the expression for the jth orbit slice, can be expressed as

$$\sum_{i=0}^{m-1} \lambda_{j,i} \frac{1}{1-z\alpha_i(x)} = \sum_{(i,t)\in\mathbb{Z}\times\mathbb{N}} \gamma_{i,t} x^i z^t, \quad (14)$$

where $\lambda_{j,i}$ and $\gamma_{i,t} \in \mathbb{F}_q^{(m)}$. This expression is not zero. Since this component is a rational function, it is p-automatic. As a consequence, the double sequence $\sum_{(i,t)\in\mathbb{Z}\times\mathbb{N}} \overline{\gamma}_{i,t} x^i z^t$, where $\overline{\gamma}_{i,t} = 1$ if $\gamma_{i,t} \neq 0$ and $\overline{\gamma}_{i,t} = 0$ otherwise, is also p-automatic. The weak p-Fermat property of the polynomials $\alpha_i(x)$, $i = 0, \ldots, m-1$, implies that $D_{0,0}(\sum \overline{\gamma}_{i,t} x^z y^t) = \sum \overline{\gamma}_{i,t} x^i z^t$. Now Theorem 4.7 in [Petersen, 1997] states that a p-automatic two-dimensional sequence $S = (s_{i,t})_{(i,t)\in\mathbb{Z}\times\mathbb{N}}$ with $D_{(0,0)}(S) = S$ has scaling sequence $(p^t)_{t\in\mathbb{N}}$. Therefore the double sequence $(\overline{\gamma}_{i,t})_{(i,t)\in\mathbb{Z}\times\mathbb{N}}$ has scaling sequence $(p^t)_{t\in\mathbb{N}}$.

Since the graphical representations of $\sum \overline{\gamma}_{i,t} x^i z^t$ and $\sum \gamma_{i,t} x^i z^t$ coincide, we have that $(p^t)_{t\in\mathbb{N}}$ is also scaling sequence for the last one. ■

Since $\mathbf{R}(x)$ is a sum of circulant matrices, $\mathbf{R}(x)$ commutes with P, i.e. $\mathbf{R}(x)P = P\mathbf{R}(x)$. This implies that if the initial configuration vector ρ_0 in Theorem 2 is replaced by $\rho_0' = P\rho_0$, i.e. the components of ρ_0' are those of ρ_0, but cyclically permuted, then the rescaled evolution sets $\mathcal{A}_i(\underline{c} = \rho_0', R(x, y))$ are permuted in the same way as the coefficients of ρ_0.

By Theorem 9 in the Appendix we get a fairly clear picture of the rescaled evolution sets.

Lemma 3. *(LCA on $\mathbb{Z} \oplus \mathbb{Z}_m$ with states in $\mathbb{F}_q$, $q = p^n$, p prime, m and p relatively prime) Let $\mathbf{R}(x)$ be conjugate (w.r.t. the mth cyclotomic field over $\mathbb{F}_q$) to $\operatorname{diag}(\alpha_0(x), \ldots, \alpha_{m-1}(x))$. Let the initial configuration $c(x) = \sum_{k\in\mathbb{Z}} \rho_k x^k$ be such that $\rho_k \equiv \underline{0}$ for $k \neq 0$ and $\rho_0 \not\equiv 0$.*

If the rational functions $1/(1 - z\alpha_j(x))$, $j = 0, \ldots, m-1$, are $\mathbb{F}_q^{(m)}$-linearly independent, then there exist $\lambda_{i,j} \in \mathbb{F}_q^{(m)}$, $i, j = 0, \ldots, m-1$, such that

$$\mathcal{A}(\underline{c}, R)_i = \bigcup_{\lambda_{i,j}\neq 0} \mathcal{A}\left(\frac{1}{1 - z\alpha_j(x)}\right).$$

holds for all $i \in \mathbb{Z}_m$.

Proof. Consider ρ_0 as an element of $\mathbb{F}_q^m$. By the proof of Theorem 2 there exist $\lambda_{i,j}$ such that the jth component of $\sum_t \mathbf{A}^t(\underline{c})(x)$ is given as

$$\sum_{i=0}^{m-1} \lambda_{j,i} \frac{1}{1 - z\alpha_i(x)}.$$

Since the rational functions $1/(1 - z\alpha_i(x))$ are linearly independent, the assertion follows from Theorem 9 in the Appendix. ■

Examples

1. We first consider an example of a CA on $\mathbb{Z} \oplus \mathbb{Z}_3$ with states in $\mathbb{F}_2$ (i.e. $m = 3$, $q = p = 2$). The splitting field of $x^3 - 1$ over $\mathbb{F}_2$ is $\mathbb{F}_2[\zeta]$, with $\zeta^2 = 1 + \zeta$. That is, the 3rd cyclotomic field over $\mathbb{F}_2$ equals $\mathbb{F}_4$.

   The local rule of the CA is given by the matrix

$$R = \begin{pmatrix} r_{-2,0} & r_{-1,0} & r_{0,0} & r_{1,0} & r_{2,0} \\ r_{-2,1} & r_{-1,1} & r_{0,1} & r_{1,1} & r_{2,1} \\ r_{-2,2} & r_{-1,2} & r_{0,2} & r_{1,2} & r_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

   with coefficients in $\mathbb{F}_2$. This gives, see Eq. (5), the polynomials

$$r_0(x) = x^{-2} + x^{-1} + x^2,$$
$$r_1(x) = x^{-1} + 1,$$
$$r_2(x) = x^{-1} + x + x^2,$$

   with coefficients in $\mathbb{F}_2$.

   From Eq. (10) we calculate, keeping in mind that from here on all calculations are done in $\mathbb{F}_2^{(3)} = \mathbb{F}_4$, that

$$\alpha_0(x) = x^{-2} + x^{-1} + 1 + x,$$
$$\alpha_1(x) = x^{-2} + \zeta + \zeta^2 x + \zeta x^2,$$
$$\alpha_2(x) = x^{-2} + \zeta^2 + \zeta x + \zeta^2 x^2.$$

   The orbit slices are determined by the components of $F_{\mathbf{R}}(x, z)\rho_0$, whereby $F_{\mathbf{R}}(x, z)$ is given by Eq. (11). With the initial configuration $\rho_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, this gives for slice $i = 0$:

$$A_0(\underline{c}, R)(x, z) = \frac{1}{1 - z(x^{-2} + x^{-1} + 1 + x)} + \frac{1}{1 - z(x^{-2} + \zeta + \zeta^2 x + \zeta x^2)} + \frac{1}{1 - z(x^{-2} + \zeta^2 + \zeta x + \zeta^2 x^2)}, \tag{15}$$

   for slice 1:

$$A_1(\underline{c}, R)(x, z) = \frac{1}{1 - z(x^{-2} + x^{-1} + 1 + x)} + \frac{\zeta}{1 - z(x^{-2} + \zeta + \zeta^2 x + \zeta x^2)} + \frac{\zeta^2}{1 - z(x^{-2} + \zeta^2 + \zeta x + \zeta^2 x^2)}, \tag{16}$$

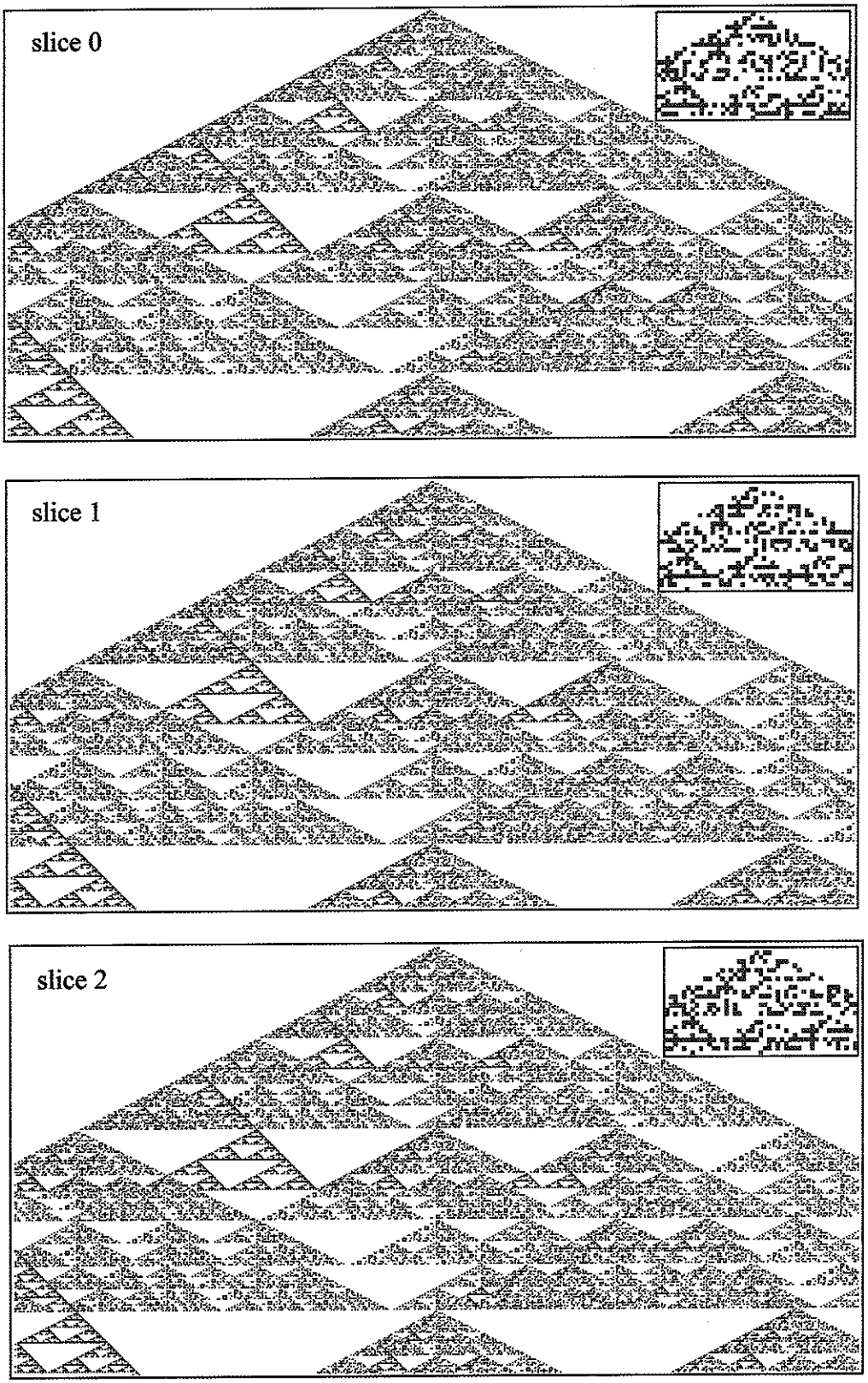
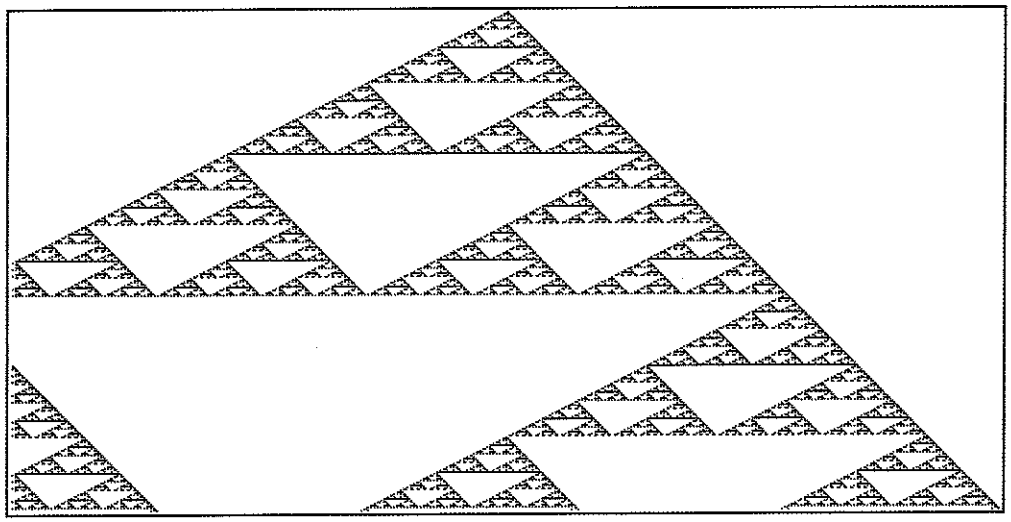
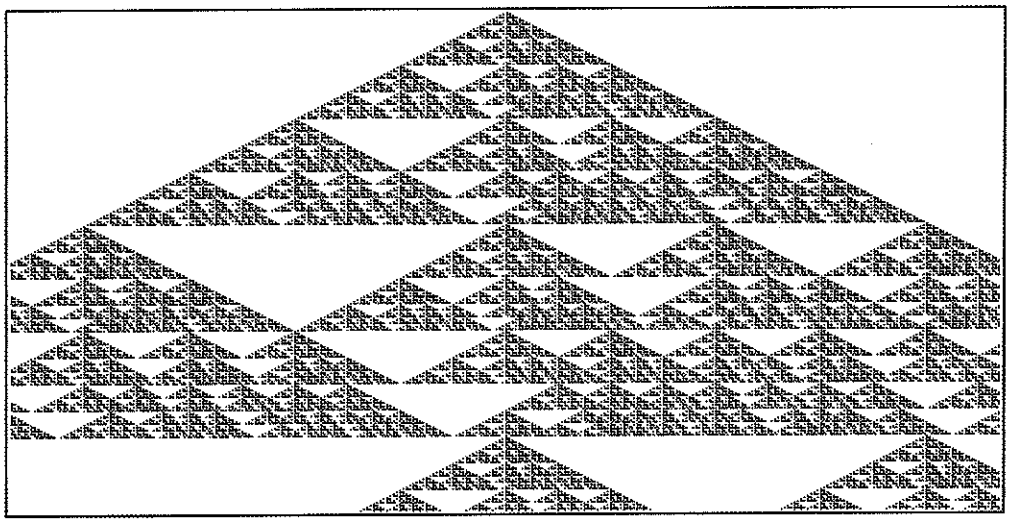


Fig. 5. The graphical representation of the three slices for the CA over $\mathbb{Z}\oplus\mathbb{Z}_3$ with states in $\mathbb{F}_2$. The local rule is defined by the matrix $R = \begin{pmatrix} r_{-2,0} & r_{-1,0} & r_{0,0} & r_{1,0} & r_{2,0} \\ r_{-2,1} & r_{-1,1} & r_{0,1} & r_{1,1} & r_{2,1} \\ r_{-2,2} & r_{-1,2} & r_{0,2} & r_{1,2} & r_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$; the initial configuration is $\rho_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The three slices are different (compare the inserts), but all have the same rescaled evolution set (this is why the three patterns become indistinguishable when seen from a distance).



(a)



(b)

Fig. 6. (a) The orbit of $1/(1 - z\alpha_0(x))$, where $\alpha_0(x) = x^{-2} + x^{-1} + 1 + x \in \mathbb{F}_2^{(3)} = \mathbb{F}_4$. (b) The orbit of $1/(1 - z\alpha_i(x))$ for $i = 1, 2$, where $\alpha_1(x) = x^{-2} + \zeta + \zeta^2 x + \zeta x^2$ and $\alpha_2(x) = x^{-2} + \zeta^2 x + \zeta x + \zeta^2 x^2$. Observe that the slices in Fig. 5 reflect a certain composition of both patterns above.

and for slice 2:

$$A_2(\underline{c}, R)(x, z) = \frac{1}{1 - z(x^{-2} + x^{-1} + 1 + x)} + \frac{\zeta^2}{1 - z(x^{-2} + \zeta + \zeta^2 x + \zeta x^2)} + \frac{\zeta}{1 - z(x^{-2} + \zeta^2 + \zeta x + \zeta^2 x^2)}. \tag{17}$$

Although these rational expressions are in $\mathbb{F}_4$, taking together the last two terms in each of them produces a rational expression (and corresponding Laurent-series) with coefficients only in the subfield $\mathbb{F}_2$, the "real" world in which the CA under consideration lives. Thus, for example, we find for slice 0:

$$A_0(\underline{c}, R)(x, z) = \frac{1}{1 - z(x^{-2} + x^{-1} + 1 + x)} + \frac{1 + z(x^{-2} + x)}{1 + z(1 + x + x^2) + z^2(x^{-4} + x^{-2} + x^{-1} + x + x^2 + x^3 + x^4)}.$$

Figure 5 displays the graphical representation of the orbit slices, while Fig. 6 displays the graphical representations of the orbits of the rational functions which appear as distinct terms in the expressions (15)–(17). That Lemma 3 applies is clearly illustrated in these figures: these orbit slices have scaling sequence $(2^n)_{n\in\mathbb{N}}$, and the rescaled evolution sets for a given slice i, displayed in Fig. 5, is the union of the rescaled evolution sets of the components in Eq. (13) which are displayed in Fig. 6. Observe that the rational functions $1/(1-z\alpha_1(x))$ and $1/(1-z\alpha_2(x))$ are $\mathbb{F}_2[\zeta]$-linearly independent and satisfy

$$Y_t(\alpha_1(x)) = Y_t(\alpha_2(x))$$

for all $t \in \mathbb{N}$, i.e. their graphical representations are identical. This explains why the rescaled evolution set of the orbit slices seems to be the union of only two different rescaled evolution sets, instead of the union of three different rescaled evolution sets as suggested by Lemma 3.

2. Consider again a CA on $\mathbb{Z}\oplus\mathbb{Z}_3$, but now with states in $\mathbb{F}_4$ (i.e. $m=3$, $p=2$). The extension field to be considered is the 3-d cyclotomic field of $\mathbb{F}_4$, namely $\mathbb{F}_4^{(3)}$, which happens to equal $\mathbb{F}_4$ itself. Consider the CA with rule

$$R = \begin{pmatrix} r_{-1,0} & r_{0,0} & r_{1,0} \\ r_{-1,1} & r_{0,1} & r_{1,1} \\ r_{-1,2} & r_{0,2} & r_{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \zeta & 0 & \zeta \\ 1 & 0 & 1 \end{pmatrix},$$

giving

$$r_0(x) = x^{-1} + 1 + x, \quad r_1(x) = \zeta x^{-1} + \zeta x\,,$$
$$r_2(x) = x^{-1} + x\,,$$

with coefficients in $\mathbb{F}_4$. Then one has, by Eq. (10) and calculations in $\mathbb{F}_4$, that

$$\alpha_0(x) = \zeta x^{-1} + 1 + \zeta x\,,$$
$$\alpha_1(x) = x^{-1} + 1 + x\,,$$
$$\alpha_2(x) = \zeta x^{-1} + 1 + \zeta x\,.$$

Observe that $\alpha_0(x) = \alpha_2(x)$, meaning that also here there are apparently only two sub-patterns present in the orbit slices. With the initial configuration $\rho_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and using Eq. (11), the following components of $F_{\mathbf{R}}(x, z)\rho_0$, giving the expressions for the orbit slices, are derived

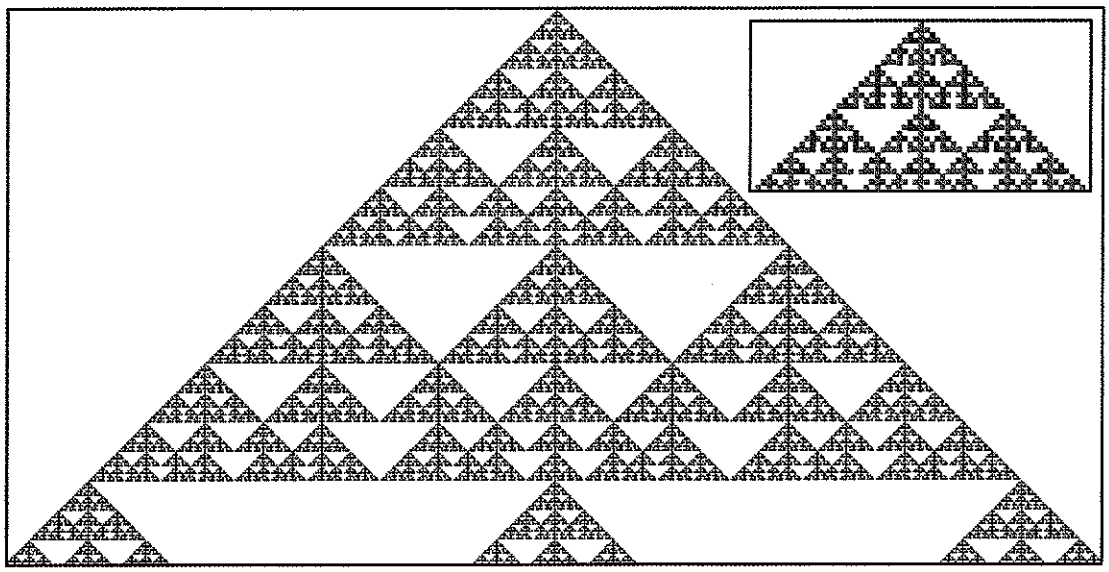
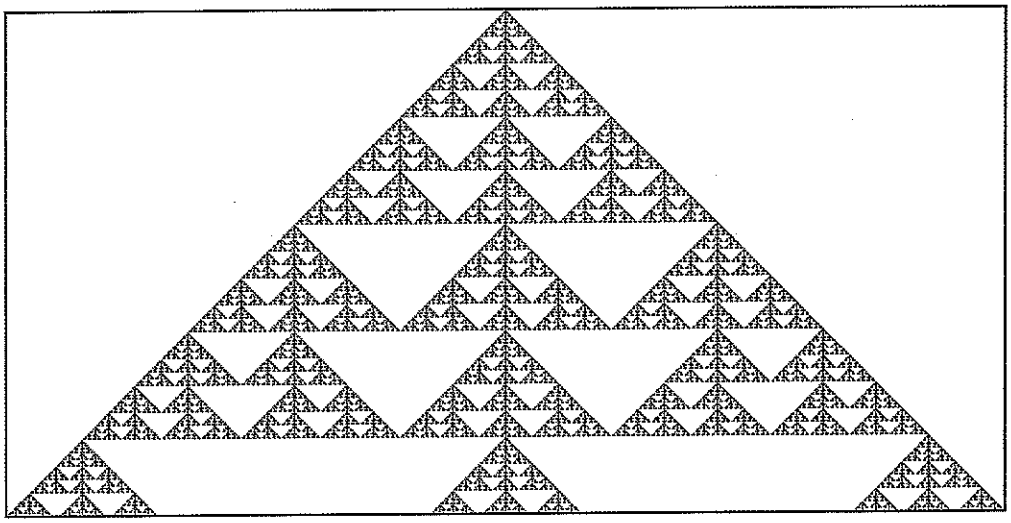
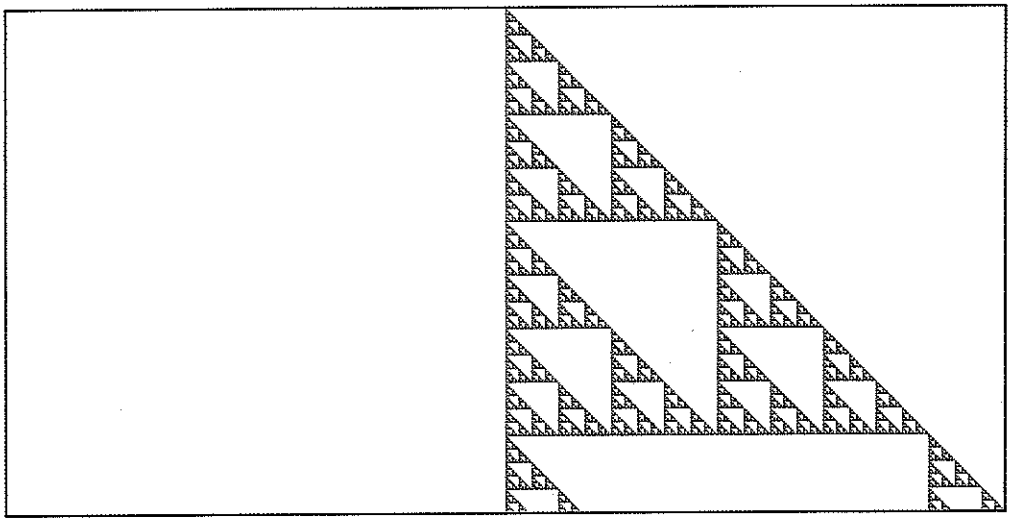


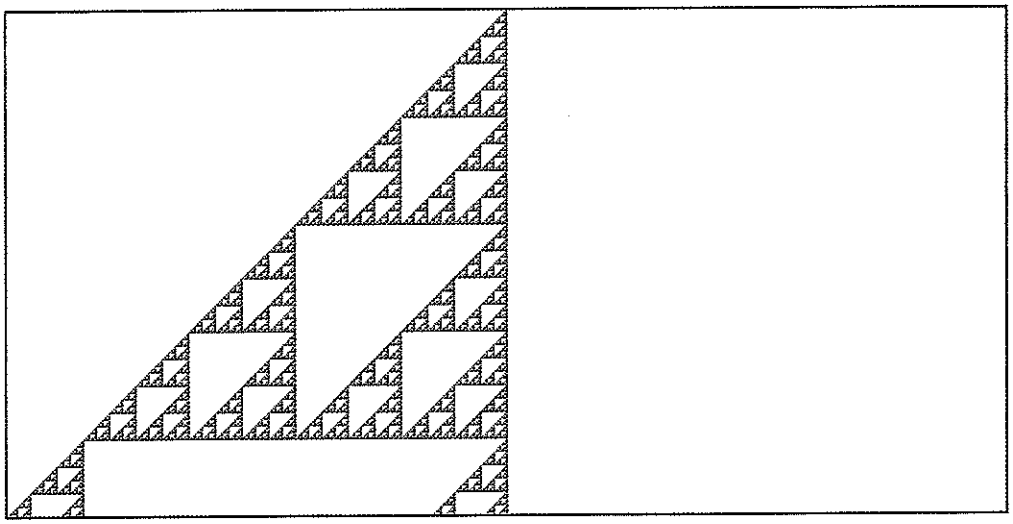
Fig. 7. The orbit of $1/(1-z\alpha_0(x))$, where $\alpha_0(x) = \zeta x^{-1}+1+\zeta x \in \mathbb{F}_2^{(3)} = \mathbb{F}_4$, corresponds to orbit slice 1 for the CA on $\mathbb{Z}\oplus\mathbb{Z}_3$ with state values in $\mathbb{F}_4$ and local rule defined by the matrix $R = \begin{pmatrix} r_{-1,0} & r_{0,0} & r_{1,0} \\ r_{-1,1} & r_{0,1} & r_{1,1} \\ r_{-1,2} & r_{0,2} & r_{1,2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \zeta & 0 & \zeta \\ 1 & 0 & 1 \end{pmatrix}$ and initial configuration $\rho_0 = \begin{pmatrix} \zeta^2 \\ 1 \\ 0 \end{pmatrix}$.



(a)



(b)



(c)

Fig. 8. (a–c) are the respective orbits associated with the polynomials $\alpha_0(x) = \zeta x^{-1} + \zeta^2 + x$, $\alpha_1(x) = \zeta^2 + x$, $\alpha_2(x) = x^{-1} + \zeta$ in $\mathbb{F}_4$. They form the components of the orbit slices presented in Figs. 9 and 10.

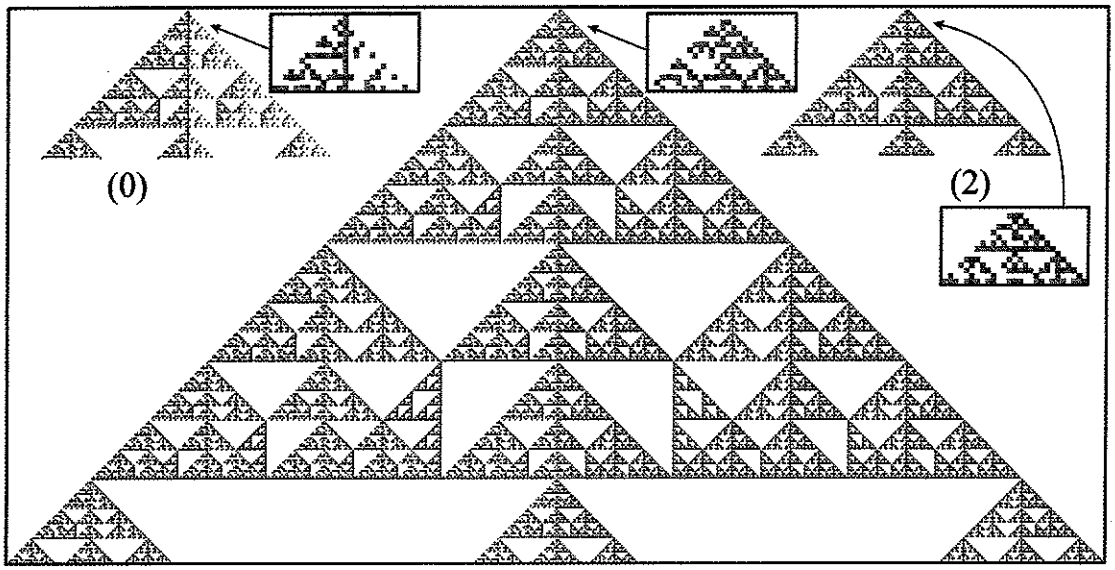


Fig. 9. The orbit slice 1 of the CA on $\mathbb{Z}\oplus\mathbb{Z}_3$ with state values in $\mathbb{F}_4$ and local rule defined by the matrix $R = \begin{pmatrix} r_{-1,0} & r_{0,0} & r_{1,0} \\ r_{-1,1} & r_{0,1} & r_{1,1} \\ r_{-1,2} & r_{0,2} & r_{1,2} \end{pmatrix} = \begin{pmatrix} \zeta^2 & \zeta & 0 \\ 0 & \zeta & \zeta \\ 1 & \zeta^2 & \zeta^2 \end{pmatrix}$ and initial configuration $\rho_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The inserts (0) and (2) respectively show the top of the orbit slices 0 and 2, which, as can be seen have the same rescaled evolution set as slice 1, although these orbit slices differ in detail.

for slice 0:

$$\frac{1}{1-z\alpha_0(x)} + \frac{1}{1-z\alpha_1(x)} + \frac{1}{1-z\alpha_2(x)} = \frac{1}{1-z\alpha_1(x)}, \quad (18)$$

for slice 1:

$$\frac{1}{1-z\alpha_0(x)} + \frac{\zeta}{1-z\alpha_1(x)} + \frac{\zeta^2}{1-z\alpha_2(x)} = \frac{\zeta}{1-z\alpha_0(x)} + \frac{\zeta}{1-z\alpha_1(x)}, \quad (19)$$

for slice 2:

$$\frac{1}{1-z\alpha_0(x)} + \frac{\zeta^2}{1-z\alpha_1(x)} + \frac{\zeta}{1-z\alpha_2(x)} = \frac{\zeta^2}{1-z\alpha_0(x)} + \frac{\zeta^2}{1-z\alpha_1(x)}. \quad (20)$$

These expressions also show that the cell states appearing in slice 2 are actually ζ times the states of the corresponding cell states in slice 1. The corresponding orbit slices are displayed in Figs. 3(a)–3(c). Observe that slice 0 corresponds to the orbit associated with the polynomial $\alpha_1(x)$. For the initial configuration $\rho_0 = \begin{pmatrix} \zeta^2 \\ 1 \\ 0 \end{pmatrix}$, it is found that orbit slice 2 is overall zero, while orbit slices 0 and 1 are given by $\zeta/(1-z\alpha_0(x))$ and $1/(1-z\alpha_0(x))$, respectively. Orbit slice 1 is displayed in Fig. 7. Observe that the orbit slices 1 and 2 in Figs. 3(b) and 3(c) are the proper linear combinations of the orbits displayed in Fig. 3(a) (corresponding to $1/(1-z\alpha_1(x))$) and Fig. 7 (corresponding to $1/(1-z\alpha_0(x))$), as given by Eqs. (18) and (19). Due to Lemma 3, the corresponding rescaled evolution sets for Figs. 3(b) and 3(c) are the union of the rescaled evolution sets associated with Figs. 3(a) and 7.

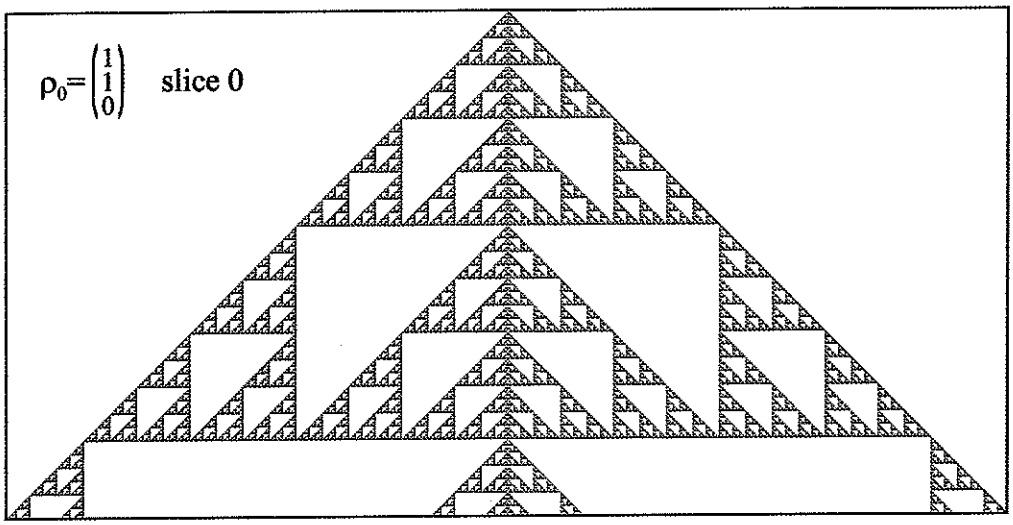
3. This is another example on $\mathbb{Z}\oplus\mathbb{Z}_3$ with states in $\mathbb{F}_4$, as in Example 2, but now with a local rule specified by

$$r_0(x) = \zeta^2 x^{-1} + \zeta, \quad r_1(x) = \zeta + \zeta x,$$
$$r_2(x) = x^{-1} + \zeta^2 + \zeta^2 x.$$

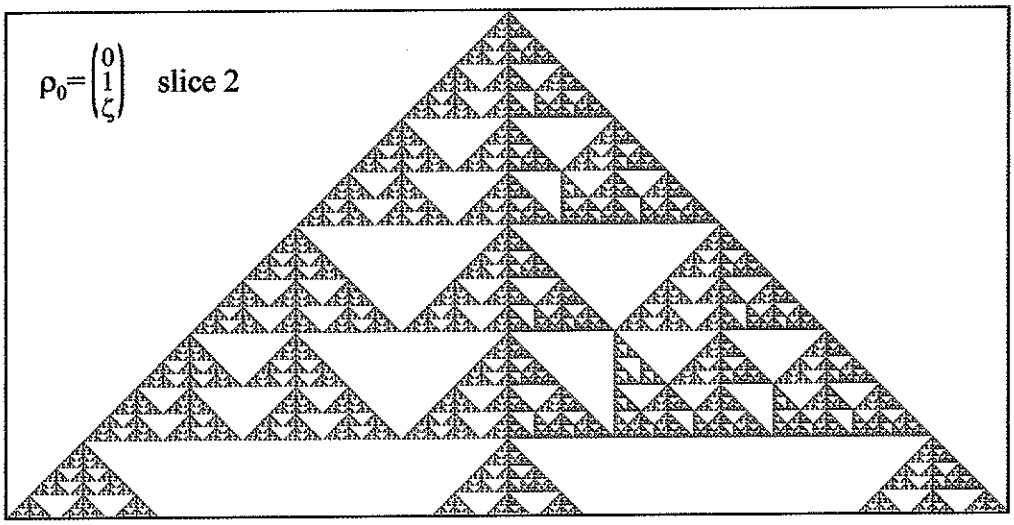
The corresponding polynomials α_i are

$$\alpha_0(x) = \zeta x^{-1} + \zeta^2 + x, \quad \alpha_1(x) = \zeta^2 + x,$$
$$\alpha_2(x) = x^{-1} + \zeta.$$

All graphical representations of the orbits associated with these polynomials α_i are different:



(a)



(b)

Fig. 10. (a) Orbit slice 0 of the CA on $\mathbb{Z}\oplus\mathbb{Z}_3$ with state values in $\mathbb{F}_4$ and local rule defined by the same matrix as in Fig. 9 but now with initial configuration $\rho_0 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ (orbit slices 1 and 2, although different in detail, are similar, i.e. have the same rescaled evolution set). (b) Orbit slice 2 for the initial configuration $\rho_0 = \begin{pmatrix}0\\1\\\zeta\end{pmatrix}$ (orbit slices 0 and 2 are similar).

they are shown in Fig. 8, and so we expect to find orbit slices where these three components are actually present. When the initial configuration is $\rho_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, the orbit slices do indeed contain these three patterns, as shown in Fig. 9. When the initial configuration is $\rho_0 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, all orbit slices are similar in nature to orbit slice 0 represented in Fig. 10(a), and contain the two components from Figs. 8(b) and 8(c). When $\rho_0 = \begin{pmatrix}0\\1\\\zeta\end{pmatrix}$, all orbit slices are of the same type as orbit slice 2 represented in Fig. 10(b).

We conclude this section with an explanation of the fact that in many instances the limit sets are all the same in the slices, as it is the case in the example shown in Fig. 5. Exceptions from this rule will only occur for a well tuned interplay between the local rule and the initial configuration, as illustrated in Example 2 above.

Theorem 3. *Let m and p be relatively prime and let $\mathbf{R}(x)$ be conjugate (w.r.t. the mth cyclotomic field over $\mathbb{F}_q$) to $\mathrm{diag}(\alpha_0(x), \ldots, \alpha_{m-1}(x))$. If the rational functions $1/(1 - z\alpha_j(x))$, $j = 0, \ldots, m-1$, are $\mathbb{F}_q^{(m)}$-linearly independent and if $c(x)$ satisfies the requirements of Lemma 3, then all orbit slices have the same rescaled evolution set, i.e.*

$$\mathcal{A}(\underline{c}, R)_j = \mathcal{A}(\underline{c}, R)_0$$

for all $j = 0, \ldots, m-1$.

Proof. As a consequence of Lemma 2, there exist $\tau_j \in \mathbb{F}_q^{(m)}$, $j = 0, \ldots, m-1$, such that

$$A_0(\underline{c}, R)(x, z) = \sum_{i=0}^{m-1} \frac{\tau_i}{1 - z\alpha_i(x)}$$

is the orbit in slice 0. Furthermore, for the slices in $j = 1, \ldots, m-1$, we have

$$A_j(\underline{c}, R)(x, z) = \sum_{i=0}^{m-1} \frac{\tau_i \zeta^{(m-j)i}}{1 - z\alpha_i(x)} .$$

By Theorem 9 in the Appendix, we conclude that

$$\mathcal{A}\left(\sum_{i=0}^{m-1} \frac{\tau_i \zeta^{(m-j)i}}{1 - z\alpha_i(x)}\right) = \bigcup_{\{\tau_i \,|\, \tau_i \neq 0,\, i=0,\ldots,m-1\}} \mathcal{A}\left(\frac{1}{1 - z\alpha_i(x)}\right) ,$$

i.e. all rescaled evolution sets are equal. ■

4. The Case $\mathbb{Z} \oplus \mathbb{Z}_m$ where p Divides m

In this section we discuss the case $m = p^\gamma m'$ where m' and p are relatively prime and $\gamma \in \mathbb{N} \setminus \{0\}$. The approach is similar to the case studied in Sec. 3. However, the diagonalization of the $m \times m$-permutation matrix P, given in Eq. (8), is slightly more involved in this case.

As before we have that $\mathbf{R}(x) = \sum_{j=0}^{m} r_j(x) P^j$, with $r_j(x) \in \mathbb{F}_q(x)$ the Laurent polynomials defining the local rule of the CA.

Lemma 4. *Let $m = p^\gamma m'$. There exists an $m \times m$-matrix W with coefficients in $\mathbb{F}_q^{(m')}$, the m'-cyclotomic field over $\mathbb{F}_q$, such that*

$$P^{p^\gamma} = W \,\mathrm{diag}(1, \ldots, 1, \zeta, \ldots, \zeta, \ldots, \zeta^{m'-1}, \ldots, \zeta^{m'-1}) W^{-1} ,$$

where each entry ζ^j is repeated p^γ-times and ζ generates the roots of 1 in $\mathbb{F}_q^{(m')}$. Moreover,

$$W = (w_{ip^\gamma + r, jp^\gamma + s})_{i,j \in \{0,1,\ldots,m'-1\},\, r,s \in \{0,1,\ldots,p^\gamma - 1\}} ,$$

with $w_{ip^\gamma + r, jp^\gamma + s} = \zeta^{ij}$ when $r = s$, and $w_{ip^\gamma + r, jp^\gamma + s} = 0$ when $r \neq s$.

Proof. Let e_j, $j = 0, \ldots, m-1$ denote the standard basis of $\mathbb{F}_q^m$. For $s = 0, \ldots, p^\gamma - 1$ we define the subspaces V_s generated by $(e_{lp^\gamma + s}, l = 0, \ldots, m' - 1)$. Every V_s is of dimension m' and, more important, every V_s is an invariant subspace of P^{p^γ}. Moreover, the matrix (w.r.t. the above basis of V_s) of the map $P^{p^\gamma} : V_s \to V_s$ is the permutation matrix of size m'. Since m' and p are relatively prime, it follows that P^{p^γ} restricted to V_s is conjugate (over $\mathbb{F}_q^{(m')}$) to $\mathrm{diag}(1, \zeta, \ldots, \zeta^{m'-1})$, with conjugating matrix $U = (\zeta^{ij})_{i,j \in \{0,1,\ldots,m'-1\}}$. Since $\mathbb{F}_q^m$ is the direct sum of the spaces V_s, $s = 0, \ldots, p^\gamma - 1$, the assertion follows. ■

As a consequence, we note

Corollary 1. *There exist m' Laurent polynomials $\alpha_i(x) \in \mathbb{F}_q^{(m')}[x, x^{-1}]$ and $m \times m$-matrices F_j with coefficients in $\mathbb{F}_q^{(m')}$ such that*

$$\sum_{t=0}^{\infty} \mathbf{R}(x)^{tp^\gamma} z^{tp^\gamma} = \sum_{j=0}^{m'-1} \frac{1}{1 - z^{p^\gamma} \alpha_j(x)^{p^\gamma}} F_j . \tag{21}$$

Proof. We have

$$\mathbf{R}(x)^{p^\gamma} = \left(\sum_{j=0}^{m-1} r_j(x) P^j\right)^{p^\gamma} = \sum_{j=0}^{m-1} r_j(x)^{p^\gamma} P^{jp^\gamma} . \tag{22}$$

By Lemma 4, the P^{jp^γ} are equivalent to a diagonal matrix:

$$P^{jp^\gamma} = W \,\mathrm{diag}(1, \ldots, 1, \zeta, \ldots, \zeta, \ldots, \zeta^{m'-1}, \ldots, \zeta^{m'-1})^j W^{-1} .$$

Introducing this in Eq. (22), gives

$$\mathbf{R}(x)^{p^\gamma} = W \,\mathrm{diag}(\beta_0(x), \ldots, \beta_0(x), \beta_1(x), \ldots, \beta_1(x), \ldots, \beta_{m'-1}(x), \ldots, \beta_{m'-1}(x)) W^{-1} ,$$

where each entry $\beta_j(x)$ is repeated p^γ times, and where

$$\beta_j(x) = \sum_{i=0}^{m-1} r_i(x)^{p^\gamma} \zeta^{ij} \tag{23}$$

are polynomials with coefficients in $\mathbb{F}_q^{(m')}$. Because we are working in a field of characteristic p, the right-hand side of Eq. (23) can also be written as $(\sum_{i=0}^{m-1} r_i(x)\zeta^{ij})^{p^\gamma}$, and thus we have

$$\beta_j(x) = \alpha_j(x)^{p^\gamma},$$

with

$$\alpha_j(x) = \sum_{i=0}^{m-1} r_i(x)\zeta^{ij}. \tag{24}$$

Using the identity $\sum_{j=0}^{\infty} \alpha(x)^t z^t = 1/(1 - z\alpha(x))$ one obtains

$$\begin{aligned}
&\sum_{t=0}^{\infty} \mathbf{R}(x)^{tp^\gamma} z^{tp^\gamma} \\
&\quad = W \operatorname{diag}\Bigg(\frac{1}{1 - z^{p^\gamma}\alpha_0(x)^{p^\gamma}}, \dots, \\
&\qquad \frac{1}{1 - z^{p^\gamma}\alpha_0(x)^{p^\gamma}}, \dots, \frac{1}{1 - z^{p^\gamma}\alpha_{m'-1}(x)^{p^\gamma}}, \dots, \\
&\qquad \frac{1}{1 - z^{p^\gamma}\alpha_{m'-1}(x)^{p^\gamma}} \Bigg) W^{-1}.
\end{aligned} \tag{25}$$

Performing the multiplications in this expression, and collecting the terms in $1/(1 - z^{p^\gamma}\alpha_i(x)^{p^\gamma})$, produces the result presented in Eq. (21). ■

Equation (21) relates to the CA's orbit for times which are multiples of p^γ. The full orbit is then given by

Corollary 2

$$\begin{aligned}
F_{\mathbf{R}}(x, z) &= \sum_{t=0}^{\infty} \mathbf{R}(x)^t z^t \\
&= \sum_{j=0}^{m'-1} \frac{1}{1 - z^{p^\gamma}\alpha_j(x)^{p^\gamma}} F_j(\mathrm{Id} + z\mathbf{R}(x) + z^2\mathbf{R}(x)^2 \\
&\quad + \cdots + z^{p^\gamma - 1}\mathbf{R}(x)^{p^\gamma - 1}).
\end{aligned}$$

Combining the above results gives a description of the slices as a power series.

Theorem 4. *Let $c(x) = \sum_{k\in\mathbb{Z}} \rho_k x^k$ be an initial configuration $\underline{c}$ such that $\rho_k \equiv 0$ for $k \neq 0$ and $\rho_0 \not\equiv 0$. There exist $\lambda_{i,j} \in \mathbb{F}_q^{(m')}$ and Laurent polynomials $h_{i,j}(x) \in \mathbb{F}_q^{(m')}[x, x^{-1}]$ (depending on $\underline{c}$) such that the jth component of $\sum_{t=0}^{\infty} \mathbf{A}^{\mathbf{t}}(\underline{c})(x)z^t$, i.e. the expression for the jth orbit slice, is given as*

$$\begin{aligned}
&\sum_{i=0}^{m'-1} \frac{1}{1 - z^{p^\gamma}\alpha_i(x)^{p^\gamma}} (\lambda_{i,j} + zh_{1,j}(x) + z^2 h_{2,j}(x) \\
&\qquad + \cdots + z^{p^\gamma - 1} h_{p^\gamma - 1, j}(x)).
\end{aligned}$$

Therefore each slice is a rational function and thus it is p-automatic. If the rational function is different from zero, then the result in [Barbé & v. Haeseler, 2003] implies that there exists a scaling sequence of the form $(p^{kt})_{t\in\mathbb{N}}$, where $k \in \mathbb{N}$ is chosen properly. However, as a consequence of Lemma 5, we shall see that $(p^t)_{t\in\mathbb{N}}$ is the appropriate scaling sequence.

Example. Consider an LCA on $\mathbb{Z}\oplus Z_6$, with states in $\mathbb{F}_3 = \{0, 1, 2\}$, with addition and multiplication modulo 3. Thus $m = 6$, $q = p^1 = 3$, $m' = 2$, and the field-extension $\mathbb{F}_3^{(2)}$ is $\mathbb{F}_3$ itself. The second root of unity is $\zeta = 2$. Then, according to Lemma 4, it holds that

$$P^3 = W \operatorname{diag}(1, 1, 1, 2, 2, 2) W^{-1},$$

with

$$W = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$$W^{-1} = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Observe how W and W^{-1} are related to the matrices $U = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and $U^{-1} = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ defined in Eq. (9). Let the local rule be defined

by the matrix

$$R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix},$$

which corresponds to the polynomials

$$\begin{aligned} r_0(x) &= x^{-1} + x \\ r_1(x) &= x^{-1} + 1 \\ r_2(x) &= 0 \\ r_3(x) &= 0 \\ r_4(x) &= x^{-1} + 2x \\ r_5(x) &= 1 + 2x\,. \end{aligned}$$

The polynomials $\alpha_j(x)$ and $\beta_j(x)$, $j = 0, 1$, as defined by Eqs. (24) and (23) are

$$\begin{aligned} \alpha_0(x) &= \sum_{i=0}^{5} r_i(x) = 2 + 2x \\ \beta_0(x) &= \alpha_0(x)^3 = 2 + 2x^3 \\ \alpha_1(x) &= \sum_{i=0}^{5} r_i(x) 2^i = x^{-1} + 1 + x \\ \beta_1(x) &= \alpha_1(x)^3 = x^{-3} + 1 + x^3\,. \end{aligned}$$

Then, according to Eq. (25), we calculate that

$$\sum_{t=0}^{\infty} \mathbf{R}(x)^{3t} z^{3t} = \frac{1}{1 - z^3\beta_0(x)} \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix} + \frac{1}{1 - z^3\beta_1(x)} \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix},$$

which corresponds to the form given in Eq. (21), where the matrices in the expression above correspond to F_0 and F_1, respectively. Observe that these matrices are circulant. Consider now the initial configuration ρ_0, then the evolution of the CA at times $3t$, $t = 0, 1, 2, \ldots$ in orbit slice j is given by the jth component of $(\sum_{t=0}^{\infty} \mathbf{R}(x)^{3t} z^{3t})\rho_0$. For $\rho_0 = (1\ 1\ 0\ 0\ 0\ 0)^T$ (T denoting transposition), we find for this evolution the following rational expressions:

for slices 0 and 1: $\dfrac{2}{1 - z^3\beta_0(x)} + \dfrac{2}{1 - z^3\beta_1(x)}$

for slice 2: 0

for slices 3 and 4: $\dfrac{2}{1 - z^3\beta_0(x)} + \dfrac{1}{1 - z^3\beta_1(x)}$

for slice 5: 0.

This shows that for the orbit slices 2 and 5, we will find zero at all times which are a multiple of 3. It does not show that these orbit slices are completely 0, as can be observed from Fig. 11 which shows the graphical representation of all the orbit slices under the given initial configuration. The complete evolution for slice j all times t is governed by the jth component of

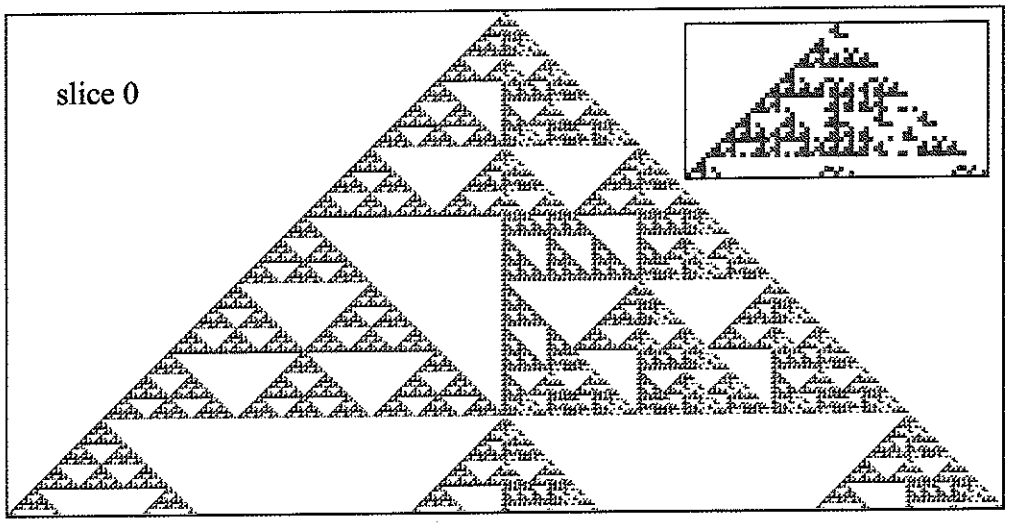
$$F_{\mathbf{R}}(x, z) = \left(\frac{1}{1 - z^3\beta_0(x)} F_0 + \frac{1}{1 - z^3\beta_1(x)} F_1 \right) \times (\mathrm{Id} + z\mathbf{R}(x) + z^2\mathbf{R}(x)^2)\rho_0\,.$$

As expected, all slices, though different in detail, are similar and reflect the presence of the two components which correspond to the orbits generated by $\alpha_0(x)$ and $\alpha_1(x)$. All slices have scaling sequence $(3^n)_{n\in\mathbb{N}}$, and have the same rescaled evolution set, which is the union of the rescaled evolution sets of the $\alpha_0(x)$- and $\alpha_1(x)$-related orbits. The initial configuration $\rho_0 = (1\ 0\ 0\ 1\ 0\ 0)^T$ extracts the α_0-related component, and is shown in Fig. 12(a), while for $\rho_0 = (1\ 0\ 0\ 2\ 0\ 0)^T$, the α_2-related component is extracted, as shown in Fig. 12(b).

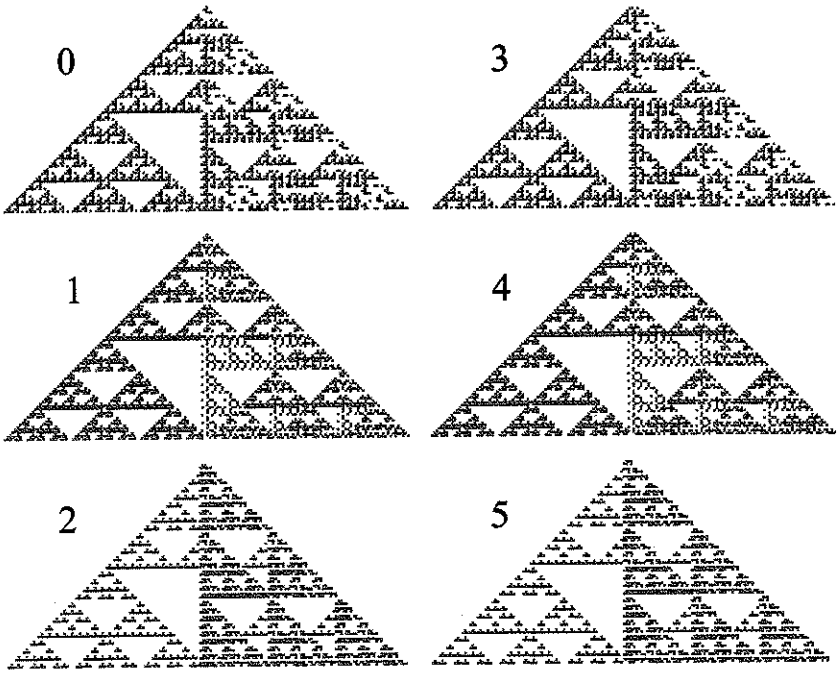
Lemma 5. *Let $\alpha(x)$ be a Laurent polynomial with coefficients in $\mathbb{F}_q$ and let $h_j(x)$, $j = 0, \ldots, p^\gamma - 1$ be Laurent polynomials with $h_0(x) \neq 0$. The rational function*

$$H(x, z) = \frac{1}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}} (h_0(x) + zh_1(x) + \cdots + z^{p^\gamma - 1} h_{p^\gamma - 1}(x))$$

has scaling sequence $(p^t)_{t\in\mathbb{N}}$ and the rescaled evolution set H is equal to the rescaled evolution set of $1/(1 - z\alpha(x))$.



(a)

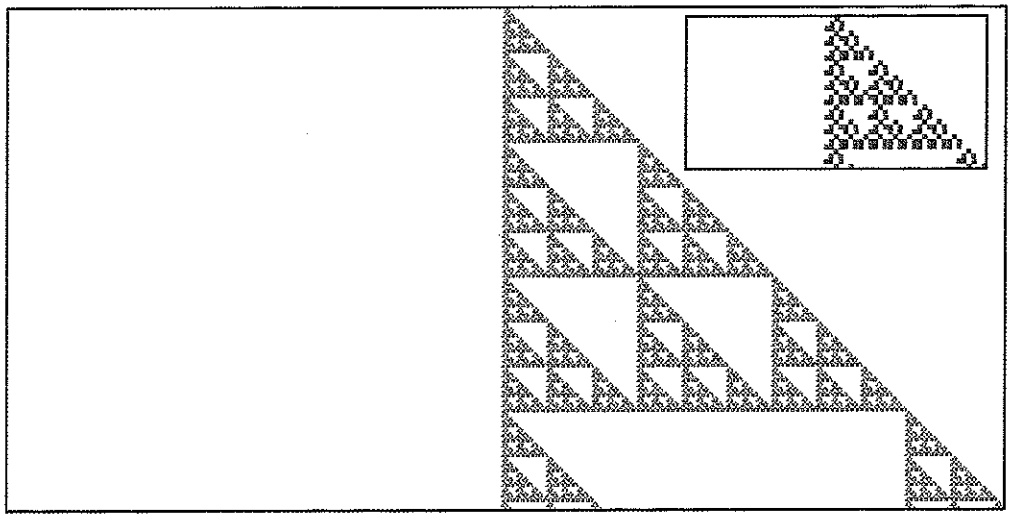


(b)

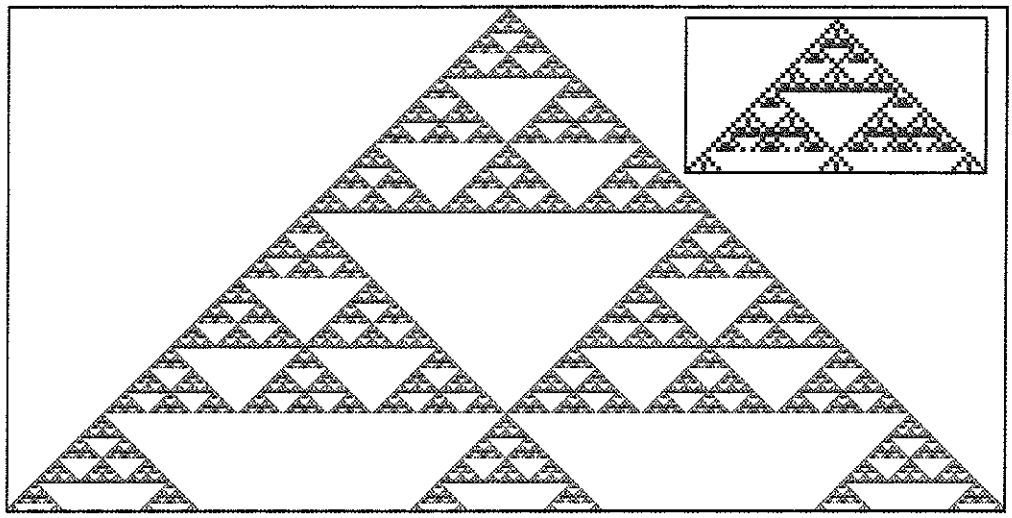
Fig. 11. (a) Orbit slice 0 of the CA on $\mathbb{Z} \oplus \mathbb{Z}_6$ with state values in $\mathbb{F}_3$ and local rule defined by the matrix $R = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$, with initial configuration $\rho_0 = (1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0)^T$. All other orbit slices are similar, although differing in detail, as shown in (b). Color code: white = 0, green = 1, red = 2.

Proof. All double sequences being rational functions with coefficients in $\mathbb{F}_q$ are p-automatic. Due to the weak p-Fermat property of $\alpha(x)$, we have that $(p^t)_{t\in\mathbb{N}}$ is a scaling sequence of $1/(1 - z\alpha(x))$. Then it is plain that $(p^t)_{t\in\mathbb{N}}$ is also a scaling sequence for $1/(1 - z^{p^\gamma}\alpha(x)^{p^\gamma})$ and both rescaled evolution sets are the same.

By Lemma 1, $h_0(x)/(1 - z^{p^\gamma}\alpha(x)^{p^\gamma})$ and $1/(1 - z\alpha(x))$ have the same rescaled evolution sets. We obviously have for all $t \in \mathbb{N}$ that



(a)



(b)

Fig. 12. Orbit slices 0 of the CA on $\mathbb{Z} \oplus \mathbb{Z}_6$ with state values in $\mathbb{F}_3$ and local rule as in Fig. 11, with (a) initial configuration $\rho_0 = (1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^T$, (b) initial configuration $\rho_0 = (1\ \ 0\ \ 0\ \ 2\ \ 0\ \ 0)^T$.

$$Y_{p^t}\left(\frac{h_0(x)}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}}\right) \subset Y_{p^t}(H(x,z))\,.$$

Let $j \in \{0, \ldots, p^\gamma - 1\}$ and let $t = t'p^\gamma + j$. Furthermore, let the ith coefficient of the Laurent polynomial $\alpha(x)^{t'p^\gamma}h_j(x)$ be different from zero. There exists d_j^- and d_j^+ such that $\operatorname{supp} h_j(x) \subset [d_j^-, d_j^+]$. The assumption on the ith coefficient of $\alpha(x)^{t'p^\gamma}h_j(x)$ implies that at least one of the coefficients $a_{i-d_j^+}$, $a_{i-d_j^+ + 1}, \ldots, a_{i-d_j^-}$ of $\alpha(x)^{t'p^\gamma}$ is different from zero.

This shows that

$$Y_{p^t}(H(x,z)) \subset O_{D+p^\gamma}\left(Y_{p^t}\left(\frac{1}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}}\right)\right)$$

where $O_\varepsilon(\ )$ denotes the ε-collar and $D = \max\{|d_j^-|, |d_j^+|\,|\, j = 0, \ldots, p^\gamma - 1\}$. Combining both inclusions we see that the rescaled evolution set of $H(x, y)$ is equal to that of $1/(1 - z\alpha(x))$. ■

As a consequence we note

Theorem 5. *Let the assumptions of Theorem 4 be satisfied. If the rational function of the orbit in slice j is different from zero, then $(p^t)_{t\in\mathbb{N}}$ is a scaling sequence.*

The proof is obvious.

We conclude with a discussion of the case $m = p^\gamma$. Corollary 1 then gives

$$\sum_{t=0}^{\infty} \mathbf{R}(x)^{tp^\gamma} z^{tp^\gamma} = \frac{1}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}} F_0 ,$$

where $\alpha(x) = r_0(x) + \cdots + r_{p^\gamma-1}(x)$ and $F_0 = \mathrm{Id}$. Therefore, see Corollary 2

$$\sum_{t=0}^{\infty} \mathbf{R}(x)^t z^t = \frac{1}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}} (\mathrm{Id} + z\mathbf{R}(x) + \cdots + z^{p^\gamma-1}\mathbf{R}(x)^{p^\gamma-1}) .$$

If the initial configuration $\rho_=(c_0, \ldots, c_{p^\gamma-1})^T$, then we have for the slice j the following representation

$$\frac{1}{1 - z^{p^\gamma}\alpha(x)^{p^\gamma}} (c_j + z h_{1,j}(x) + \cdots + z^{p^\gamma-1} h_{p^\gamma-1,j}(x)) \qquad (26)$$

with polynomials $h_{i,j}$.

Lemma 6. *If $m = p^\gamma$ and if $c(x)$ defines an initial configuration as in Theorem 4 and if the slice j does not represent the rational function zero, then the rescaled evolution set exists and equals*

$$\mathcal{A}(r_0 + \cdots + r_{p^\gamma-1}) .$$

Proof. The rational function in slice j is given by Eq. (26), which was supposed to be different from zero. Therefore Lemma 5 proves the assertion. ■

For the initial configuration $\rho_0 = (1, 1, \ldots, 1)^T$ we not only have equal rescaled limit sets but also equal orbits.

Lemma 7. *If the initial value $c(x) = \sum \rho_k x^k$ satisfies the requirements of Theorem 4 and if additionally $\rho_0 \equiv 1$, then all orbit slices are identical and are equal to the orbit of $(r_0(x) + r_1(x) + \cdots + r_{m-1}(x))$.*

Proof. The assertion follows from the fact that $\rho_0 = (1, 1, \ldots, 1)^T$ is an eigenvector of $\mathbf{R}(x)$ with eigenvalue $r_0(x) + \cdots + r_{m-1}(x)$. ■

5. The Case $\mathbb{Z} \oplus G$

In this section we consider the general case of an LCA defined on $\mathbb{F}_q^{\mathbb{Z}\oplus G}$, where $(G, +)$ is a finite Abelian group. As in the previous sections we distinguish two cases, namely the order of G, denoted as $|G|$, and p are relatively prime; and p divides $|G|$. From now on m denotes the order of G, i.e. $m = |G|$.

Theorem 6. *Let G be a finite Abelian group such that p and $m = |G|$ are relatively prime. If $\mathbf{R}(x)$ is as in Eq. (3), then there exist an m' and an $m \times m$-matrix W with coefficients in $F_q^{(m')}$ such that*

$$\mathbf{R}(x) = W \,\mathrm{diag}((\alpha_g(x))_{g\in G}) W^{-1}$$

with Laurent polynomials $\alpha_g(x) \in \mathbb{F}_q^{(m')}[x, x^{-1}]$, $g \in G$.

Proof. The classification Theorem for finite Abelian groups, e.g. [Gallian, 1994], gives $G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l}$ with m_i divides m_{i+1} for $i = 0, \ldots, l-1$, and $m = m_1 m_2 \cdots m_l$.

We shall show that the matrix $\mathbf{R}(x)$ is a circulant of level l, see [Davis, 1974, p. 188], and Theorem 5.8.4. in [Davis, 1974] guarantees the existence of the matrix W.

Since $\mathbf{R}(x) = (r_{g-h}(x))_{g,h\in G}$, where $r_g(x)$, $g \in G$, are Laurent polynomials we study the $m \times m$-matrix $\tilde{G} = (g-h)_{g,h\in G}$ with entries in G. We shall show that this matrix is circulant of level l, implying the assertion for $\mathbf{R}(x)$. To this end we write $G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{l-1}} \oplus \mathbb{Z}_{m_l} = H \oplus \mathbb{Z}_{m_l}$. This gives $G = H \cup (1+H) \cup (2+H) \cdots \cup (m_l - 1 + H)$ with mutually disjoint sets. Arranging the rows and columns of $\tilde{G}$ according to this partition we obtain

	H	$1+H$	$2+H$	$\cdots$	$(m_l-1)+H$
H	$\tilde{H}$	$(m_l-1)+\tilde{H}$	$(m_l-2)+\tilde{H}$	$\cdots$	$1+\tilde{H}$
$1+H$	$1+\tilde{H}$	$\tilde{H}$	$(m_l-1)+\tilde{H}$	$\cdots$	$2+\tilde{H}$
$\vdots$					
$(m_l-1)+H$	$(m_l-1)+\tilde{H}$	$(m_l-2)+\tilde{H}$	$\cdots$	$1+\tilde{H}$	$\tilde{H}$

,

where $\tilde{H} = (g-h)_{g,h\in H}$ is the corresponding matrix for the group H. We see that $\tilde{G}$ is a block circulant matrix [Davis, 1974, p. 176]. Since $H = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{l-1}}$, we see that $\tilde{H}$ is also a block circulant matrix. By induction we finally arrive at $\mathbb{Z}_{m_1}$ having a circulant matrix $\tilde{\mathbb{Z}}_{m_1}$, see Eq. (4). This shows that $\tilde{G}$ is a circulant of level l. By Theorem 5.8.4. in [Davis, 1974] and due to the fact that $\gcd(p,m) = 1$ it follows that there exists a matrix $W \in \mathbb{F}_q^{(m_l)}$ such that $W^{-1}\mathbf{R}(x)W$ is in diagonal form. ■

Remark 3. For G a finite Abelian group and $\tilde{G} = (g-h)_{g,h\in G}$ as above we can consider the matrices $A_g = (a_{g_1,g_2}(g)) \in \mathbb{F}_q^{G\times G}$ defined as $a_{g_1,g_2}(g) = 1$ if and only if $g_1 - g_2 = g$ and $a_{g_1,g_2} = 0$ otherwise. The map $g \mapsto A_g$ is a faithful representation of G, i.e. $A_{g+h} = A_g A_h$ and $g \mapsto A_g$ is injective, see e.g. [James & Liebeck, 1993].

Moreover, we can write the matrix $\mathbf{R}(x) = \sum_{g\in G} r_g(x) A_g$ where every matrix A_g is circulant of level l. In case $G = \mathbb{Z}_m$ this corresponds to Eq. (7) with $P^j = A_j$, $j \in \{0,\ldots,m-1\} = \mathbb{Z}_m$.

The diagonalization of $\mathbf{R}(x)$ allows us to describe the limit sets of slices.

Theorem 7. *(LCA on $\mathbb{Z}\oplus G$ with states in F_q, $q = p^n$, $|G|$ and p relatively prime) Let $c(x) = \sum_{k\in\mathbb{Z}} \rho_k x^k$ be such that $\rho_k \equiv 0$ for $k \neq 0$ and $\rho_0 \not\equiv 0$. Let the rational functions $H_{\alpha_g}(x,z) = 1/(1-z\alpha_g(x))$, $g \in G$, $\alpha_g(x)$ as in Theorem 6, be $\mathbb{F}_q^{(m_l)}$-linearly independent.*

If $g \in G$, then either $\mathcal{A}(\underline{c}, R)_g = \emptyset$ or there exists $\lambda_{g,h}$, $h \in G$, such that

$$\mathcal{A}(\underline{c}, R)_g = \bigcup_{\lambda_{g,h}\neq 0} \mathcal{A}\left(\frac{1}{1-z\alpha_h(x)}\right).$$

Proof. Since $\mathbf{R}(x)$ is diagonizable we have that slice g is a rational function of the form

$$\sum_{h\in G} \frac{\lambda_{g,h}}{1-z\alpha_h(x)},$$

with $\lambda_{g,h} \in \mathbb{F}_q^{(m_l)}$. Theorem 9 in the Appendix now induces the assertion. ■

Finally, we study the case where p divides m the order of G. Analogous to the case of a cyclic group we have a direct decomposition $G = G(p) \oplus G'$, where $G(p) = \{g \,|\, \text{there exists } s \in \mathbb{N}\colon\ p^s g = 0\}$ is the subgroup of all elements in G with order being a power of p. The order of the subgroup G' is equal to m' and m' and p are relatively prime. Note further that the order of $G(p)$ is equal to p^γ for a $\gamma \in \mathbb{N}$. As for the case $G = \mathbb{Z}_m$ we have.

Theorem 8. *Let G be a finite Abelian group such that p divides $m = |G|$. If $\mathbf{R}(x)$ is a local rule, then there exist m' and an $m\times m$-matrix W with coefficients in $\mathbb{F}_q^{(m')}$ and $\alpha_j(x) \in \mathbb{F}_q^{(m')}[x, x^{-1}]$, $j = 0,\ldots,m'-1$ such that*

$$\mathbf{R}(x)^{p^\gamma} = W \operatorname{diag}(\alpha_0(x),\ldots,\alpha_0(x),\ldots, \alpha_{m'-1}(x),\ldots,\alpha_{m'-1}(x))W^{-1},$$

where each entry is repeated p^γ-times.

Proof. We have $G = G(p)\oplus G'$ with $|G(p)| = p^\gamma$ and $|G'| = m'$ and p and m' are relatively prime. Now the same arguments as in the proof of Lemma 4 apply. ■

Corollary 3. *There exist $m\times m$-matrices F_j, $j = 0,\ldots,m'-1$, with coefficients in $\mathbb{F}_q^{(m')}$ such that*

$$\begin{aligned} F_{\mathbf{R}}(x,z) &= \sum_{t=0}^{\infty} \mathbf{R}(x)^t z^t \\ &= \sum_{j=0}^{m'-1} \frac{1}{1-z^{p^\gamma}} F_j(\mathrm{Id} + z\mathbf{R}(x) + z^2\mathbf{R}(x)^2 \\ &\quad + \cdots + z^{p^\gamma-1}\mathbf{R}(x)^{p^\gamma-1}). \end{aligned}$$

As a consequence we have that Theorem 4 also holds for the case where G is a finite Abelian group. The existence of nontrivial rescaled evolution sets in slice g can be deduced in the same way as we already did in case $G = \mathbb{Z}_m$.

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References

Allouche, J.-P. & Mendès France, M. [1995] "Automata and automatic sequences," *Beyond Quasicristals*, eds. Axel, F. & Gratias, D. (Springer/Les Éditions de Physique), pp. 293–365.

Allouche, J.-P, v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1996] "Linear cellular automata, finite automata and Pascal's triangle," *Discrete Appl. Math.* **66**, 1–22.

Allouche, J.-P., Courbage, M. & Skordev, G. [2001] "Notes on cellular automata," *Cubo*, Matemática Educational Univ. de la Frontera (Chile) **2**, 213–244.

Allouche, J.-P. & Skordev, G. [2001] "Remarks on permutive cellular automata," Inst. Dyn. Syst. Univ. Bremen, Report **460**.

Aso, H. & Honda, N. [1985] "Dynamical characteristics of linear cellular automata," *J. Comp. Syst. Sci.* **30**, 291–317.

Barbé, A. & v. Haeseler, F. [2003] "Limit sets of automatic sequences," *Adv. Math.* **175**(2), 169–196.

Davis, P. J. [1974] *Circulant Matrices* (John Wiley, NY).

Edgar, G. A. [1990] *Measure, Topology and Fractal Geometry* (Springer-Verlag, NY).

Gallian, J. A. [1994] *Contemporary Abstract Algebra* (D. C. Heath, Lexington).

Garzon, M. [1995] *Modeling of Massive Parallelism* (Springer-Verlag, NY).

James, G. & Liebeck, M. [1993] *Representation and Characters of Groups* (Cambridge University Press, Cambridge).

Kargapolov, M. I. & Merzljakov, J. I. [1979] *Fundamentals of the Theory of Groups* (Springer-Verlag, NY).

Lidl, R. & Niederreiter, H. [1984] *Finite Fields* (Addison-Wesley, Cambridge).

Martin, O., Odlyzko, A. & Wolfram, S. [1984] "Algebraic properties of cellular automata," *Commun. Math. Phys.* **93**, 219–258.

Petersen, A. [1997] *Automatic Sequences, Rational Functions and Geometry*, PhD Thesis, Univ. Bremen.

Salon, O. [1986] "Suites automatiques à multi-indices," *Séminaire de Théorie des Nombres de Bordeauxa, exposé* **4**, 4-01–4-27 with Appendix of Shallit, J., 4-29A–4-36A.

Salon, O. [1987] "Suites automatiques à multi-indices et algébricité," *C. R. Acad. Sci. Paris, Série I* **305**, 501–504.

Takahashi, S. [1992] "Self-similarity of linear cellular automata," *J. Comput. Syst. Sci.* **44**, 114–140.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1992a] "Linear cellular automata, substitutions, hierarchical iterated function systems," in *Fractal Geometry and Computer Graphics*, eds. Encarnacao, J. L., Peitgen, H.-O., Sakes, G. & Englert, G. (Springer-Verlag, NY), pp. 3–23.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1992b] "On the fractal structure of rescaled evolution sets of cellular automata and attractors of dynamical systems," Inst. Dyn. Syst., Univ. Bremen, Report **278**.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1993] "Cellular automata, matrix substitutions and fractals," *Ann. Math. Artif. Intell.* **8**, 345–362.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1995a] "Global analysis of self-similarity features of cellular automata: selected examples," *Physica* **D86**, 64–80.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [1995b] "Multifractal decomposition of rescaled evolution set of equivariant cellular automata," *Rand. Comput. Dyn.* **3**, 93–119.

v. Haeseler & Petersen, A. [1998] "Automaticity of rational functions," *Beiträge zur Algebra und Geometrie* **39**, 219–229.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [2000] "On the fractal structure of the rescaled evolution set of Carlitz sequences of polynomials," *Discr. Appl. Math.* **103**, 89–109.

v. Haeseler, F., Peitgen, H.-O. & Skordev, G. [2001] "Self-similar structure of rescaled evolution sets of cellular automata I, II," *Int. J. Bifurcation and Chaos* **11**(4), 927–941.

v. Haeseler, F. [2003] *Automatic Sequences* (W. De Gruyter, Berlin).

Willson, S. [1984] "Cellular automata can generate fractals," *Discr. Appl. Math.* **8**, 91–99.

Willson, S. [1992] "Calculating growth rates and moments for additive cellular automata," *Discr. Appl. Math.* **35**, 47–65.

Wolfram, S. [1994] *Cellular Automata and Complexity* (Addison-Wesley, Reading, MA).

Appendix

In this section we prove the result that the rescaled evolution set of a sum of orbits is, under certain conditions, equal to the union of the rescaled evolution sets of the orbits. The proof relies on several facts stated in [Petersen, 1997].

Theorem 9. *Let* $a_i(x) \in \mathbb{F}_q[x]$, $i = 1, \ldots, M$, *be polynomials such that the rational functions* $H_i(x, y) = 1/(1 - ya_i(x))$ *are* $\mathbb{F}_q$*-linearly independent, i.e.* $0 = \sum \lambda_i H_i(x, y)$ *with* $\lambda_i \in \mathbb{F}_q$, $i = 1, \ldots, M$, *implies* $\lambda_i = 0$ *for all* $i = 1, \ldots, M$.

If

$$S(x, y) = \sum_{i=1}^{M} \lambda_i H_i(x, y)$$

Now let $(i,j) \in \mathbb{N}^2$ such that $\varepsilon_{i,j} = 1$, then $\underline{t}_{i,j} \neq 0$ and there exists $(i', j') \in \mathbb{N}^2$ such that $\|(i, j) - (i', j')\|_\infty < L$ and $\chi_{i',j'} = 1$. This shows that

$$Y_{p^t}(\chi_{\tau_0}) \subset O_L(Y_{p^t}(\chi_{\mathbf{t}}))\,,$$

where $O_L(\ )$ denotes the L-collar. Together with the inclusion $Y_{p^t}(\chi_{\mathbf{t}}) \subset Y_{p^t}(\chi_{\tau_0})$ we obtain

$$\lim_{t\to\infty} \frac{\mathbf{1}}{p^t} Y_{p^t}(\hat{S}) = \mathcal{A}(T_1)\,.$$

Final arguments

We have seen that for every maximal element in Q_G^* the rescaled evolution set is given as $\mathcal{A}(T)$ where $T(x, y) = 1/(1 - ya_i(x))$. Moreover, these rescaled evolution sets are contained in the rescaled evolution set of S. By construction of the partial order on Q_G we have that

$$\bigcup_{i=1}^{m} \mathcal{A}(H_j(x, y)) \subset \bigcup_{\langle H\rangle \in Q_G^*} \mathcal{A}(\langle H\rangle) \subset \mathcal{A}(S)\,.$$

As we have already seen, the left-hand side of the above inclusion contains $\mathcal{A}(S)$, therefore both sets are equal.