$\mathcal{H}_2$-Optimal Linear Parametric Design[1]

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Abstract

We consider discrete-time linear time-invariant parameter dependent systems $\tilde{H}(z,p) := \sum_{i=1}^{n_p} p_i H_i(z)$. The parameter p is a design variable. $\mathcal{H}_2$-optimal parametric design problem is defined as $\min_p ||\tilde{H}(z,p)||_2^2$. The solution depends critically on the function $\tilde{H}(z,\cdot)$. Affine function $\tilde{H}(z,\cdot)$ corresponds to an open-loop design and leads to a convex quadratic optimization problem. An analytic solution is given in this case. Affine-fractional function $\tilde{H}(z,\cdot)$ corresponds to a closed-loop design and leads to a non-convex optimization problem. This case remains an open problem.

$\mathcal{H}_2$-Optimal Linear Parametric Design

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Abstract

We consider discrete-time linear time-invariant parameter dependent systems $\tilde{H}(z,p) := \sum_{i=1}^{n_p} p_i H_i(z)$. The parameter p is a design variable. $\mathcal{H}_2$-optimal parametric design problem is defined as $\min_p ||\tilde{H}(z,p)||_2^2$. The solution depends critically on the function $\tilde{H}(z,\cdot)$. Affine function $\tilde{H}(z,\cdot)$ corresponds to an open-loop design and leads to a convex quadratic optimization problem. An analytic solution is given in this case. Affine-fractional function $\tilde{H}(z,\cdot)$ corresponds to a closed-loop design and leads to a non-convex optimization problem. This case remains an open problem.

Keywords: parametric design, parameter optimization, multiple models.

1 Introduction

In the parametric approach, the designed system is described by a model which depends on parameters. The parameters are either uncontrollable, in which case they are called *parametric disturbances*, or they can be selected by the designer, in which case they are called *design parameters*.

A *parametric uncertainty* is considered in the design problem if a priori bounds are known or assumed for the parametric disturbances. Analysis and synthesis of linear time-invariant systems described with parametric uncertainty is a well developed subject [Ber87].

In the *parametric design*, one aims to select the control parameters so that certain specifications for the designed system are achieved. This approach is related to the root locus and parameter optimization methods [Che75], [Åst70, Ch.5].

There is no general theory for parametric design. The problem is approached numerically in [HC72, TM87]. Viewed as an optimization problem, the parametric design is non-convex, so that there is no guarantee that a global solution will be found.

Nevertheless parametric design is a useful problem in practice. It imposes a *structural constraint on the solution*. Common examples of control problems with such constraints are PID control and fixed order control. In general, it is more likely for a practical design to impose structural constraints and to restrict the freedom to the assignment of some free parameters than to deal with a completely unspecified structure.

A natural extension of the parametric design is *robust parametric design*, where both parametric disturbances and parametric control are treated. Such framework is symmetric with respect to both structured disturbance and structured control.

In the paper we consider the parametric design problem in the linear time-invariant case. The plant is a linear time-invariant system and the design parameters are fixed. The problem is defined as an optimal selection of the control parameters according to a given performance criterion. Two fundamentally different cases are the open-loop parametric design, e.g. approximation and estimation problems, and the closed-loop parametric design, e.g. regulation problem. The former is solved exactly while the later is a difficult problem that remains open.

We define the parametric design problem in Section 2. In Section 3, we distinguishes open-loop and closed-loop parametric design and give as examples the approximation, estimation, and control problems. In Section 4, we state the assumptions under which results are derived. In Section 5, we make a connection between the open-loop and the closed-loop linear parametric design, from one hand, and affine and affine-fractional in the parameters transfer functions, from the other hand. In Section 5, we solve the open-loop linear parametric design problem. The result is relevant for system approximation and estimation. In Section 7, we treat the closed-loop linear parametric design. In this case the problem is non-convex and the solution is an open question. The conclusions are given in Section 8.

2 Parametric design

Consider the following objects

$\mathcal{M}$	—	set of models
P	—	plant
$S(\cdot)$	—	solution structure
$C(\cdot,\cdot)$	—	connection specification
$m(\cdot)$	—	performance measure

$\mathcal{M}$ is a set of models which under sum and scalar multiplication, defined respectively as parallel connection and scaling of the outputs, is a linear vector space over $\mathbb{R}$. $P \in \mathcal{M}$ is a model of the plant. The solution is a system $S(p) \in \mathcal{M}$, which has the structure $S(\cdot)$, $S : \mathbb{R}^{n_p} \to \mathcal{M}$. $S(\cdot)$ is called a *parameter dependent system*. The argument of $S(\cdot)$ is called a *parameter*. The plant and the design solution are connected in a way specified by $C(\cdot,\cdot)$, $C : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. The resulting system $E(p) := C\big(P, S(p)\big)$ is the *error system*.

The design objectives are expressed by the minimization of the *performance index* $J(p) := m(E(p))$. The parametric design is the problem $\inf_p J(p)$.

$$\begin{array}{ccc} P & \xrightarrow{\arg\inf_p J(p)} & p_{\min} \\ \uparrow & \Uparrow & \uparrow \\ \text{plant} & \text{specification} & \text{solution} \\ & S(\cdot),\ C(\cdot,\cdot),\ m(\cdot) & \end{array}$$

Of special interest is the "linear in the parameters" design. A parametric design in which, $S(p) = \sum_{i=1}^{n_p} p_i S_i$, where $S_i \in \mathcal{M}$ are given, is called a *linear parametric design*. Informally, an linear parametric design can be viewed as an attempt to improve given designs by searching for a better solution over all linear combinations of the given ones. Such a view makes a connection with the multiple model approach [Lai76], [AM79, Ch10.1].

3 Open-loop and closed-loop parametric design

A parametric design is called an *open-loop parametric design* if

$$E(p) = C\big(P, S(p)\big) = H_{\mathrm{p}} + H_{\mathrm{l}} S(p) H_{\mathrm{r}}$$

for some systems $H_{\mathrm{p}} \in \mathcal{M}, H_{\mathrm{l}} \in \mathcal{M}$ and $H_{\mathrm{r}} \in \mathcal{M}$, independent of p.

Example 1 (Approximation problem) A parametric design in which $E(p) = -P + S(p)$ is called an approximation problem (see Figure 1).

In the approximation problem, the plant P is a reference model to be approximated by the design solution $S(p)$. The approximation problem is an open-loop parametric design ($H_{\mathrm{r}} = H_{\mathrm{l}} = I$ and $H_{\mathrm{p}} = -P$).

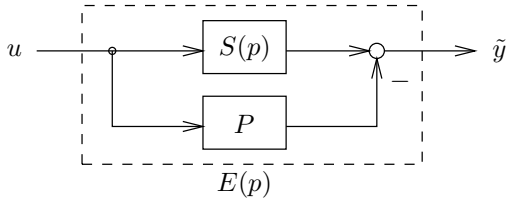


Figure 1: Approximation problem scheme

Example 2 (Estimation problem) A parametric design in which

$$E(p) = -P_{xw} + S(p) \begin{bmatrix} P_{yw} \\ I \end{bmatrix}$$

is called an estimation problem (see Figure 2).

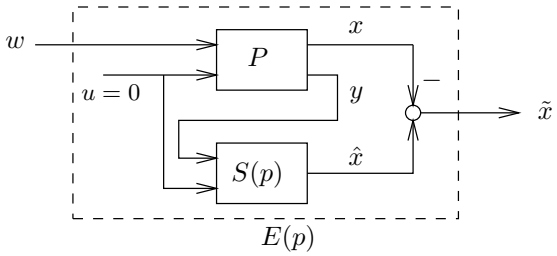


Figure 2: Estimation problem scheme

In the estimation problem, the plant P is a 2×2 port system $\left[\begin{smallmatrix} P_{xw} & P_{xu} \\ P_{yw} & P_{yu} \end{smallmatrix}\right]$. The input is partitioned into, unmeasurable input (or disturbance) w, and measurable input u. The output is partitioned into, unmeasurable output x, to be estimated, and measurable output y.

$S(p)$ is the observer which is a 1×2 port system with input $[y^T\ u^T]^T$, and output, the estimate $\hat{x}$. The design objective is minimization of the effect of w on the error of estimation $\tilde{x}$. $E(p)$ is defined accordingly as the channel from w to $\tilde{x}$. The estimation problem is an open-loop parametric design.

Note 1 (Scaling) Left and right scaling factors $F_{\mathrm{l}} \in \mathcal{M}$ and $F_{\mathrm{r}} \in \mathcal{M}$ with shaping filters can be used to incorporate spectral information of the disturbances and frequency emphasis in the criterion. The error function becomes $F_{\mathrm{l}} E(p) F_{\mathrm{r}}$. Note that, the scaled problem can be reformulated as another standard problem.

A parametric design is called a *closed-loop parametric design* if

$$E(p) = \big(I + H_{\mathrm{D,l}} \bar{S}(p) H_{\mathrm{D,r}}\big)^{-1} \big(H_{\mathrm{N,p}} + H_{\mathrm{N,l}} S(p) H_{\mathrm{N,r}}\big)$$

for some systems $H_{\mathrm{N,p}} \in \mathcal{M}$, $H_{\mathrm{N,l}} \in \mathcal{M}$, $H_{\mathrm{N,r}} \in \mathcal{M}$, $H_{\mathrm{D,l}} \in \mathcal{M}$, and $H_{\mathrm{D,r}} \in \mathcal{M}$, independent of p. For an l output ports error system, $\bar{S}(p) = \mathrm{diag}\{\underbrace{S(p) \cdots S(p)}_{l}\}$.

Example 3 (Control problem) A parametric design in which

$$E(p) = \left(\begin{bmatrix} I & \\ & I \end{bmatrix} + \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} S(p) & \\ & S(p) \end{bmatrix} \begin{bmatrix} I & \\ & P \end{bmatrix} \right)^{-1} \times \left(\begin{bmatrix} P & I \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} S(p) \begin{bmatrix} P & I \end{bmatrix} \right) \quad (1)$$

is called a control problem (see Figure 3).

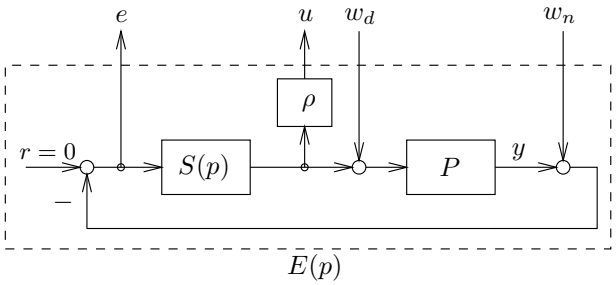


Figure 3: Control problem scheme

In the control problem, the design solution $S(p)$ is the controller. The design objective is minimization of the effect of the disturbance w_d and the noise w_n on the error of regulation e with admissible control signal u. Accordingly $E(p)$ has input $[w_d^T \ w_n^T]^T$ and output $[e^T \ \rho u^T]^T$, where ρ is a design parameter. A trail and error is needed, to find a ρ which gives admissible amplitude of u.

To see the connection between (1) and Figure 3 note that

$$E(p) = \begin{bmatrix} (I + PS(p))^{-1}P & (I + PS(p))^{-1} \\ (I + S(p)P)^{-1}S(p)P & (I + S(p)P)^{-1}S(p) \end{bmatrix} = \begin{bmatrix} I + PS(p) & \\ & I + S(p)P \end{bmatrix}^{-1} \begin{bmatrix} P & I \\ S(p)P & S(p) \end{bmatrix}.$$

The control problem is a closed-loop parametric design. Note 1 holds for the control problem also.

4 Assumptions

In the rest of the paper, the linear parametric design is considered for discrete-time systems with open-loop or closed-loop connection and $\mathcal{H}_2$- norm performance measure. With these assumptions, $\mathcal{M}$ is the set of all discrete-time linear time-invariant systems. The plant model is $P(z)$. The design solution is

$$S(z,p) = \sum_{i=1}^{n_p} p_i S_i(z)$$

for a given set of systems $\{S_i(z)\}_{i=1}^{n_p}$. The error system is

$$\tilde{H}(z,p) := C\big(P(z), S(z,p)\big) = H_{\mathrm{p}}(z) + H_{\mathrm{l}}(z)S(z,p)H_{\mathrm{r}}(z),$$

in the open-loop case, and

$$\tilde{H}(z,p) := C\big(P(z), S(z,p)\big) = \big(I + H_{\mathrm{D,l}}(z)S(z,p)H_{\mathrm{D,r}}(z)\big)^{-1} \times \big(H_{\mathrm{N,p}}(z) + H_{\mathrm{N,l}}(z)S(z,p)H_{\mathrm{N,r}}(z)\big),$$

in the closed-loop case. The performance index is

$$J(p) = ||\tilde{H}(z,p)||_2^2.$$

In the open-loop case,

$$\tilde{H}(z,p) = -H_0(z) + \sum_{i=1}^{n_p} p_i H_i(z), \quad (2)$$

where

$$H_0(z) := -P(z), \ H_i(z) := H_{\mathrm{l}}(z)S_i(z)H_{\mathrm{r}}(z).$$

The "affine in the parameters" $n_y \times n_u$ transfer function $\tilde{H}(z,p)$, given by (2), is written more compactly as

$$\tilde{H}(z,p) = \tilde{H}(z)(\tilde{p} \otimes I_{n_u}), \quad (3)$$

where

$$\tilde{H}(z) := \big[H_0(z) \ H_1(z) \cdots H_{n_p}(z)\big], \quad \tilde{p} := \big[-1 \ p^T\big]^T.$$

In the closed-loop case

$$\tilde{H}(z,p) = \big(I_{n_y} + H_D(z)(p \otimes I_{n_y})\big)^{-1} \times \big(H_N(z)(\tilde{p} \otimes I_{n_u})\big), \quad (4)$$

where

$$H_{\mathrm{D}}(z) := \big[H_{\mathrm{D,l}}(z)S_1(z)H_{\mathrm{D,r}}(z) \cdots H_{\mathrm{D,l}}(z)S_{n_p}(z)H_{\mathrm{D,r}}(z)\big],$$
$$H_{\mathrm{N}}(z) := \big[-H_{\mathrm{N},p}(z) \ H_{\mathrm{N,l}}(z)S_1(z)H_{\mathrm{N,r}} \cdots H_{\mathrm{N,l}}(z)S_{n_p}(z)H_{\mathrm{N,r}}\big].$$

With these assumptions and notation, the problem to be considered is

$$\inf_p ||\tilde{H}(z,p)||_2^2$$

with $\tilde{H}(z,p)$ given either by (3) or by (4).

5 Affine and affine-fractional transfer functions

Affine and affine-fractional parameter dependent matrix transfer functions are defined next.

Definition 1 (Affine transfer function) *An $n_y \times n_u$ parameter dependent transfer function $\tilde{H}(z,p)$ is affine in the parameter $p \in \mathbb{R}^{n_p}$ if there is an $n_y \times n_u(n_p+1)$ transfer function $\tilde{H}(z)$, independent of p, such that (3) holds for all $p \in \mathbb{R}^{n_p}$*

By definition, an "affine in p" error transfer function $\tilde{H}(z,p)$ is equivalent to open-loop linear parametric design. Next, the implication of the open-loop property of the linear parametric design (or equivalently of the "affine in p" property of the error transfer function) is shown on the state space realizations of the error transfer function.

Theorem 1 *An $n_y \times n_u$ parameter dependent transfer function $\tilde{H}(z,p)$ is affine in $p \in \mathbb{R}^{n_p}$ if and only if there are $\tilde{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbb{R}^{\tilde{n}\times n_u}$, independent of p, and $\tilde{C} : \mathbb{R}^{n_p} \to \mathbb{R}^{n_y\times\tilde{n}}$, $\tilde{D} : \mathbb{R}^{n_p} \to \mathbb{R}^{n_y\times n_u}$ affine in p such that $\big(\tilde{A},\tilde{B},\tilde{C}(p),\tilde{D}(p)\big)$ is a realization of $\tilde{H}(z,p)$.*

Proof: ($\Rightarrow$) Assume that $\tilde{H}(z,p)$ is affine in p. Then there is a $\tilde{H}(z)$, independent of p such that $\tilde{H}(z,p) = \tilde{H}(z)(\tilde{p}\otimes I_{n_u})$. Partition $\tilde{H}(z) = \big[H_0(z)\ H_1(z)\cdots H_{n_p}(z)\big]$, where $H_i(z)$ is $n_y \times n_u$ and let $\big(A_i,B_i,C_i,D_i\big)$ be a realization of $H_i(z)$ of order n_i. Then a realization of $\tilde{H}(z,p)$ is

$$\big(\tilde{A},\tilde{B},\tilde{C}(p),\tilde{D}(p)\big) := \Big(\begin{bmatrix} A_0 & & \\ & \ddots & \\ & & A_{n_p}\end{bmatrix}, \begin{bmatrix} B_0 \\ \vdots \\ B_{n_p}\end{bmatrix}, \big[-C_0\ p_1C_1\ \cdots p_{n_p}C_{n_p}\big],\ -D_0+\sum_{i=1}^{n_p} p_iD_i\Big). \quad (5)$$

$\tilde{C}(p)$ and $\tilde{D}(p)$ are affine in p,

$$\begin{aligned}\tilde{C}(p) &= (\tilde{p}^T\otimes I)\tilde{C},\ \tilde{C} := \mathrm{diag}\{C_0\ C_1\cdots C_{n_p}\},\\ \tilde{D}(p) &= (\tilde{p}^T\otimes I)\tilde{D},\ \tilde{D} := \big[D_0^T\ D_1^T\cdots D_{n_p}^T\big]^T.\end{aligned} \quad (6)$$

($\Leftarrow$) Assume that $\big(\tilde{A},\tilde{B},\tilde{C}(p),\tilde{D}(p)\big)$ is a realization of $\tilde{H}(z,p)$, $\tilde{A} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbb{R}^{\tilde{n}\times n_u}$ are independent of p, and $\tilde{C} : \mathbb{R} \to \mathbb{R}^{n_y\times\tilde{n}}$, $\tilde{D} : \mathbb{R} \to \mathbb{R}^{n_y\times n_u}$ are affine in p. Then $\tilde{C}(p) = \tilde{C}_0 + \sum_{i=1}^{n_p} p_i\tilde{C}_i$, and $\tilde{S}(p) = \tilde{D}_0 + \sum_{i=1}^{n_p} p_i\tilde{D}_i$ for some $\tilde{C}_i$, $\tilde{D}_i$ independent of p.

$$\begin{aligned}\tilde{H}(z,p) &= \tilde{C}(p)\big(zI-\tilde{A}\big)^{-1}\tilde{B}+\tilde{D}(p)\\ &= H_0+\sum_{i=1}^{n_p} p_iH_i(z),\end{aligned}$$

where $H_i(z) := C_i\big(zI-\tilde{A}\big)^{-1}\tilde{B}+D_i,\ i=0,1,\ldots,n_p$. ■

The fact that the parameters do not enter into the state matrix informally means that there is no dynamics involved with the variation of p. A change in p affects immediately the output.

Definition 2 (Affine-fractional transfer function) *An $n_y \times n_u$ parameter dependent transfer function $\tilde{H}(z,p)$ is affine-fractional in the parameter $p \in \mathbb{R}^{n_p}$ if there is an $n_y \times n_y$ invertible transfer function $H_{\mathrm{D}}(z,p)$ and an $n_y \times n_u$ transfer function $H_{\mathrm{N}}(z,p)$, both affine in p, such that (4) holds for all $p \in \mathbb{R}^{n_p}$*

By definition, an "affine-fractional in p" transfer function $\tilde{H}(z,p)$, is equivalent to closed-loop linear parametric design. Next, the implication of the closed-loop property of the linear parametric design (or equivalently the affine-fractional in p property of the error transfer function) is shown on the state space realizations of $\tilde{H}(z,p)$.

Theorem 2 *If a proper, parameter dependent transfer function $\tilde{H}(z,p)$ is affine-fractional in p and has the form*

$$\tilde{H}(z,p) = \big(I-H_{\mathrm{D}}(z,p)\big)^{-1}H_{\mathrm{N}}(z,p)$$

for some $n_y\times n_y$ $H_{\mathrm{D}}(z,p)$ and $n_y\times n_u$ $H_{\mathrm{N}}(z,p)$ strictly proper transfer functions, affine in p, then there are $A(p)$ and $C(p)$, affine in p, and B independent of p such that $\big(A(p),B,C(p)\big)$, is a realization of $H(z,p)$.

Proof: Let $\big(A_{\mathrm{N}},B_{\mathrm{N}},C_{\mathrm{N}}(p)\big)$ and $\big(A_{\mathrm{D}},B_{\mathrm{D}},C_{\mathrm{D}}(p)\big)$, where $C_{\mathrm{N}}(p)$ and $C_{\mathrm{D}}(p)$ are affine in p, are realizations of $H_{\mathrm{N}}(z,p)$ and $H_{\mathrm{D}}(z,p)$, respectively. Using Theorem 1, such realizations are possible, because of the strict properness assumptions ($D_{\mathrm{N}} = D_{\mathrm{D}} = 0$) and the affine property of $H_{\mathrm{N}}(z,p)$ and $H_{\mathrm{D}}(z,p)$. Then

$$\begin{aligned}\tilde{H}(z,p) = \Big(I-C_{\mathrm{D}}(p)\big(zI-A_{\mathrm{D}}\big)^{-1}B_{\mathrm{D}}\Big)^{-1}\\ \times C_{\mathrm{N}}(p)\big(zI-A_{\mathrm{N}}\big)^{-1}B_{\mathrm{N}}.\end{aligned}$$

Using the matrix inversion lemma, this is equivalent to

$$\begin{aligned}\tilde{H}(z,p) = \Big(I+C_{\mathrm{D}}(p)\big(zI-A_{\mathrm{D}}-B_{\mathrm{D}}C_{\mathrm{D}}(p)\big)^{-1}B_{\mathrm{D}}\Big)\\ \times\Big(C_{\mathrm{N}}(p)\big(zI-A_{\mathrm{N}}\big)^{-1}B_{\mathrm{N}}\Big).\end{aligned}$$

$\tilde{H}(z,p)$ is a series connection of the systems

$$\big(A_{\mathrm{D}}-B_{\mathrm{D}}C_{\mathrm{D}}(p),B_{\mathrm{D}},C_{\mathrm{D}}(p)\big)$$

and

$$\big(A_{\mathrm{N}},B_{\mathrm{N}},C_{\mathrm{N}}(p)\big).$$

Then $\tilde{H}(z,p)$ has a realization

$$\big(\tilde{A}(p),\tilde{B},\tilde{C}(p)\big) := \Big(\begin{bmatrix} A_{\rm N} & 0 \\ B_{\rm D}C_{\rm N}(p) & A_{\rm D}+B_{\rm D}C_{\rm D}(p)\end{bmatrix}, \begin{bmatrix} B_{\rm N} \\ 0\end{bmatrix}, \begin{bmatrix} 0 & C_{\rm D}(p)\end{bmatrix}\Big) \quad (7)$$

with $\tilde{A}(p)$ and $\tilde{C}(p)$ affine in p. ∎

Note that in this case there is dynamics involved with the parameter variation.

6 Open-loop linear parametric design

Consider $\inf_p ||\tilde{H}(z,p)||_2^2$ with $\tilde{H}(z,p) = \tilde{H}(z)(\tilde{p}\otimes I)$. Using the state space realization (5-6)

$$||\tilde{H}(z,p)||_2^2 = \mathrm{tr}\left((\tilde{p}^T\otimes I)\big(\tilde{C}\tilde{W}_c\tilde{C}^T+\tilde{D}\tilde{D}^T\big)(\tilde{p}\otimes I)\right),$$

where

$$\tilde{A}\tilde{W}_c\tilde{A}^T-\tilde{W}_c+\tilde{B}\tilde{B}^T=0.$$

Let $p_0 := -1$ and

$$W_{c,ij} := \tilde{W}_c(s_i : s_{i+1}-1, s_j : s_{j+1}-1),\ i,j=0,1,\ldots,n_p$$

where $s_0 = 1$ and

$$s_i = 1+\sum_{k=0}^{i-1} n_k, \quad i=1,\ldots,n_p.$$

Then

$$\begin{aligned}||\tilde{H}(z,p)||_2^2 &= \mathrm{tr}\left(\sum_{i=0}^{n_p}\sum_{j=0}^{n_p} p_ip_j\big(C_iW_{c,ij}C_j^T+D_iD_j^T\big)\right)\\ &= \sum_{i=0}^{n_p}\sum_{j=0}^{n_p} p_ip_j\underbrace{\mathrm{tr}\left(C_iW_{c,ij}C_j^T+D_iD_j^T\right)}_{F_{ij}}\\ &:= \tilde{p}^T\tilde{F}\tilde{p}.\end{aligned}$$

So that $||\tilde{H}(z,p)||_2^2$ is quadratic in p. From $\tilde{W}_c\geq 0$ and Lemma 1, follows that $\tilde{F}\geq 0$ thus p is optimal if and only if $Fp=g$, where

$$\tilde{F}=\begin{bmatrix} f & g^T\\ g & F\end{bmatrix}, \quad f\in\mathbb{R}.$$

Assuming that $F>0$,

$$p_{\min} = F^{-1}g$$

is the unique minimum and

$$\min_p ||\tilde{H}(z,p)||_2 = \sqrt{f-p_{\min}^TF^{-1}p_{\min}}$$

Lemma 1 *Let $n, n_p\in\mathcal{Z}^+$, $A=A^T\in\mathbb{R}^{n.n_p\times n.n_p}$ and define the block partitioning*

$$A := \begin{bmatrix} A_{11} & \cdots & A_{1n_p}\\ \vdots & & \vdots\\ A_{n_p1} & \cdots & A_{n_pn_p}\end{bmatrix}, \quad A_{ij}\in\mathbb{R}^{n\times n}.$$

Then $\big[\mathrm{tr}\,(A_{ij})\big]$ is positive definite (positive semidefinite) if A is positive definite (positive semidefinite).

Proof: Let $p\in\mathbb{R}^{n_p}$ and define

$$Z(p) := p\otimes I_n \in \mathbb{R}^{n.n_p\times n}.$$

Note that $\mathrm{rank}\,(Z(p)) = n$ for all $p\in\mathbb{R}^{n_p}$ and

$$\begin{aligned}Z(p)^TAZ(p) &= \big[p_1I\cdots p_{n_p}I\big]A\begin{bmatrix} p_1I\\ \vdots\\ p_{n_p}I\end{bmatrix}\\ &= \sum_{i,j} p_ip_jA_{ij}.\end{aligned} \quad (8)$$

Then

$$\begin{aligned}\mathrm{rank}\,(Z(p))=n,\ A>0 &\Rightarrow Z(p)^TAZ(p)>0\\ &\Rightarrow \mathrm{tr}\left(Z(p)^TAZ(p)\right)>0\end{aligned}$$

and from (8)

$$\mathrm{tr}\left(\sum_{i,j}p_ip_jA_{i,j}\right)>0 \Rightarrow \sum_{i,j}p_ip_j\mathrm{tr}\,(A_{i,j})>0$$

$$\Rightarrow p^T\big[\mathrm{tr}\,(A_{ij})\big]p>0$$

While p is arbitrary $\big[\mathrm{tr}\,(A_{ij})\big]>0$. The proof when $A\geq 0$ is analogous. ∎

7 Closed-loop linear parametric design

Consider $\inf_p ||\tilde{H}(z,p)||_2^2$ with $\tilde{H}(z,p)$ given by (4). Using the state space realization (7)

$$||\tilde{H}(z,p)||_2^2 = \mathrm{tr}\left((\tilde{p}^T\otimes I)\big(\tilde{C}^T\tilde{W}_c(p)\tilde{C}\big)(\tilde{p}\otimes I)\right),$$

where

$$\tilde{A}(p)\tilde{W}_c(p)\tilde{A}^T(p)-\tilde{W}_c(p)+\tilde{B}\tilde{B}^T=0.$$

Then

$$||\tilde{H}(z,p)||_2^2 = \tilde{p}^T\tilde{F}(p)\tilde{p},$$

where

$$\tilde{F}_{ij} := \mathrm{tr}\left(\tilde{C}_i^T\tilde{W}_c(p)\tilde{C}_j\right).$$

Consider the control problem defined in Example 2. In the context of the current assumptions this problem is

discrete-time $\mathcal{H}_2$-*optimal control problem.* Given are three systems $H_i(z)$, $i = 0, 1, 2$. Let $K_i(z)$ be the $\mathcal{H}_2$-optimal controller for $H_i(z)$. The two parameter linearly structured controller $K(z, p) = p_1K_1(z)+p_2K_2(z)$ is introduced and the structural design problem is: find p that minimizes $||\tilde{H}(z, p)||_2$.

Figure 4 shows the plot of the cost function $J(p)$. We compute the minimum by using general purpose optimization (MATLAB's `fminsearch`). The initial approximation is selected as the solution of the following approximation problem

$$K_0(z) = p_1K_1(z) + p_2K_2(z),$$

i.e., we approximate the optimal controller $K_0(z)$ by the parameterized controller $K(z, p)$. The approximation problem is easy to solve but is suboptimal in terms of the control problem objective function.

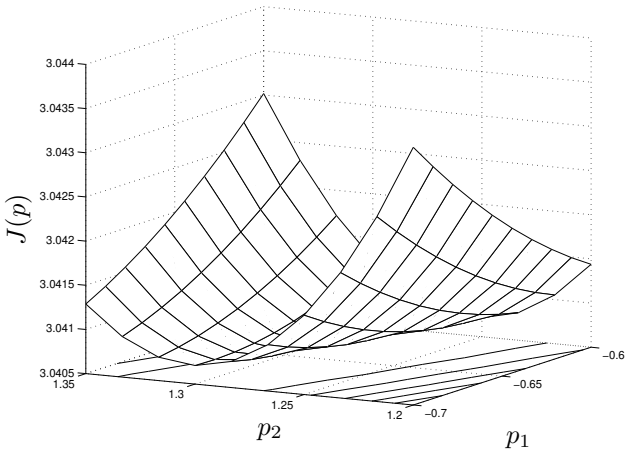


Figure 4: Control example, performance index

8 Conclusions

$\mathcal{H}_2$-optimal design using the freedom to assign control parameters is considered. The relevance of this problem and some connections with existing research areas were discussed in the introduction. A contribution of the paper is the observation that the usual notion of open-loop and closed-loop design in the parametric design is related respectively with affine and affine-fractional error transfer function. Furthermore, the open-loop and the closed-loop linear parametric design lead to special state space realizations. The open-loop linear parametric design is solved. This problem is convex quadratic and can be considered as an easy problem. The closed-loop linear parametric design problem leads to a non-convex optimization problem. An example of controller design subject to a structural constraint is shown. There is not general and efficient methodology to solve the problem.

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