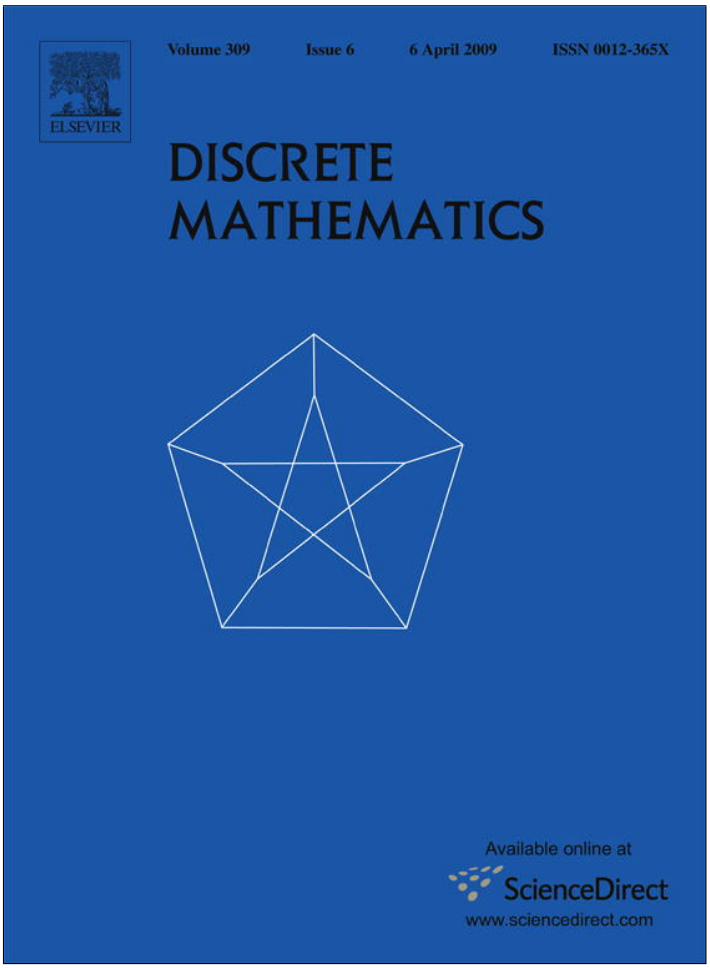
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Frame cellular automata: Configurations, generating sets and related matroids

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Abstract

We introduce a frame cellular automaton as a broad generalization of an earlier study on quasigroup-defined cellular automata. It consists of a triple $(\mathcal{F}, \mathcal{R}, E_{\mathcal{F}})$ where, for a given finite set X of cells, the frame $\mathcal{F}$ is a family of subsets of X (called elementary frames, denoted $S_i, i = 1, \ldots, n$), which is a cover of X. A matching configuration is a map $\underline{c} : X \to G$ which attributes to each cell a state in a finite set G under restriction of a set of local rules $\mathcal{R} = \{R_i \mid i = 1, \ldots n\}$, where R_i holds in the elementary frame S_i and is determined by an $(|S_i|\text{-}1)$-ary quasigroup over G. The frame associated map $E_{\mathcal{F}}$ models how a matching configuration can be grown iteratively from a certain initial cell-set. General properties of frames and related matroids are investigated. A generating set $S \subset X$ is a set of cells such that there is a bijection between the collection of matching configurations and G^S. It is shown that for certain frames, the algebraically defined generating sets are bases of a related geometric-combinatorially defined matroid.
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1. Introduction and overview

1.1. Background

Let us start by providing some background from which the subject of this paper has emerged. In [1], the observation that symmetric triangular state-configurations appear in the orbits of the cellular automaton (CA) which generates Pascal's triangle mod 2, led us to investigate the combinatorics and enumeration of these configurations. Paper [2] showed that similar symmetric configurations occur in a broader class of CA, namely those for which the evolution rule is determined by certain symmetric quasigroups. In [3], the triangular template of cells on which these symmetric configurations appear was considered as an isolated cellular array, and the conditions for symmetry inspired a random dynamic (recursive) feedback mechanism whose attractors correspond to the possible symmetric configurations. Crucial in this feedback scheme was the concept of generating sets (see further). The complete characterization of

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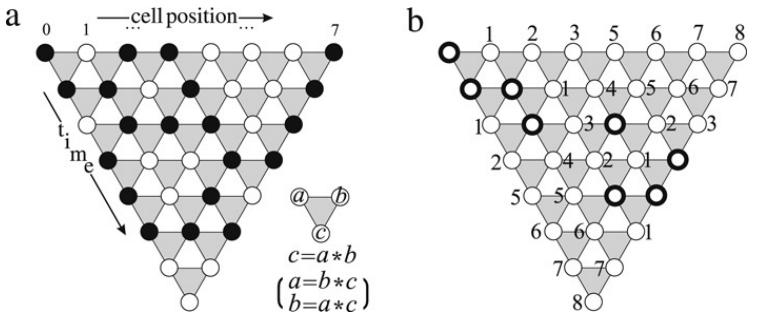


Fig. 1. (a) A triangular configuration in a cellular automaton orbit with local rule defined by the quasigroup $(G, \star) = (\{0, 1\},$ addition mod 2) (the states of the cells are indicated by the colors black = 1, white = 0). The local rule applies in the gray-colored *elementary* triangles. A skew time-axis is used for emphasizing the symmetry of the rule in each elementary triangle: the state of any two cells in such an elementary triangle determines the state of the third cell. (b) The fat cells form a nontrivial primitive generating set: when the state of these cells is known, the whole configuration can be obtained by applying the local evolution rule iteratively. Numbers at the cells indicate in which iteration the corresponding cell-state is determined. Cell-states themselves are not indicated, i.e., white cells in (b) do not indicate that these cells have state 0, in contrast to what is the case under (a).

generating sets for such triangular configurations was the subject in [4], where an unexpected link was uncovered between the algebraic characterization of generating sets and a geometric-combinatorially defined matroid.

The investigation in the present paper aims at finding the broadest structure under which the concepts and properties for the triangular structures studied in [4], are still valid, especially concerning this generating set/matroid relationship. It will become clear that cellular structures, called "frames" in this context, together with CA-type evolution rules determined by multary (n-ary) quasigroups, provide the proper generalization.

First, we briefly recall the simple case of the quasigroup-based triangular configurations. After that, we will show how to generalize the underlying concepts.

Let $(G, \star)$ be a quasigroup, i.e., a set G equipped with a binary operator $\star$ such that for all $a, b \in G$, the equations $a \star x = b$ and $y \star a = b$ have a unique solution x and y, respectively. Consider the bi-infinite one-dimensional cellular automaton, with overall state $\underline{g} = (g(k))_{k\in\mathbb{Z}} \in G^{\mathbb{Z}}$, $g(k)$ being the state of the cell at position k. Starting from the initial overall state $\underline{g}_0$, the state updates at discrete times $t \in \mathbb{N}$, according to the map $A : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$, defined by the *local evolution rule*

$$A(\underline{g})(k) = g(k) \star g(k + 1).$$

This produces, at time t, the CA-state $\underline{g}_t = A^t(\underline{g}_0)$. The two-dimensional sequence $(\underline{g}_t)_{t\in\mathbb{N}} = (A^t(\underline{g}_0))_{t\in\mathbb{N}} = (g_t(k))_{k\in\mathbb{Z},t\in\mathbb{N}}$ represents the *orbit*, i.e., the "state-time" evolution, of the CA.

If the initial state $\underline{g}_0$ is known only for cells at positions in the interval $[s, s + N]$, then $\underline{g}_t$ is only known at the positions in $[s, s + N - t]$. Fig. 1(a) illustrates that this corresponds to a triangular domain in the classical graphical representation of the orbit space.

As indicated in Fig. 1(a), the local evolution rule defined by the quasigroup applies to each gray-colored *elementary triangle*, i.e., it is local to these elementary triangles. In the construction of the CA's orbit, the rule applies from top to bottom, in the sense that the states of the upper two cells in an elementary triangle determines the state of the bottom cell. However, the rule being determined by a quasigroup implies that knowledge of the state of *any* two cells in such an elementary triangle (not necessarily the two upper cells) determines the state of the remaining third cell. Taking this aspect into account, such triangular configurations were considered in [2–4] as isolated objects with interesting properties of their own. In the context of these papers, a *matching configuration* is defined as a configuration for which the cell-states satisfy the local rule in each elementary triangle. It was observed that it is possible to construct such a matching configuration starting from a minimal set of cells with known state by applying the local rule iteratively, as illustrated in Fig. 1(b). Any such minimal set of cells from which it is possible to determine the complete configuration by applying the local rule iteratively was called a *primitive generating set*. Trivial primitive generating sets are formed by the cells on the respective sides of the triangular array. For nontrivial primitive generating sets, other than the one in Fig. 1(b), see [4]. Clearly, there is a bijection between the set of all possible configurations on a primitive generating set and the set of all possible matching configurations [4]. Actually, primitive generating sets form a subset of a larger

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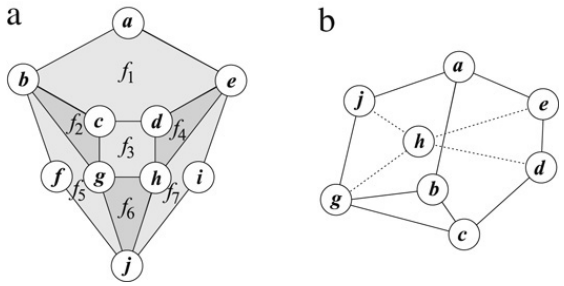


Fig. 2. (a) A "planar" frame, the gray-colored polygons f_1 to f_7 are the elementary frames. (b) A "polyhedral" frame, the faces are the elementary frames.

class of *generating sets*, i.e., sets for which a bijection exists between all configurations on such a set and all matching configurations.

The dynamics of generating the complete matching configuration from a primitive generating set, by applying the local rule in the elementary triangles iteratively, can be considered as a prototype of what we propose to call a *frame cellular automaton* (FCA). For the example at hand, such an FCA consists of the triple $(\mathcal{F}, R, E_{\mathcal{F}})$, where

$\mathcal{F}$: is a set of cells, organized in a so-called *frame*, which here is the collection of the elementary gray-colored triangles in Fig. 1.

R: is a local rule that applies in each elementary triangle: the addition modulo 2 of all cell-states in each triangle equals 0.

$E_{\mathcal{F}}$: is the extension map associated with the frame, occasionally just denoted E. Given any set S of cells in the frame whose states are known while the states of the other cells are unknown, this map determines which cells in the frame can be attributed a state by application of the local rule in each elementary triangle, whenever possible. The condition is that the state of two cells in an elementary triangle needs to be known in order to expand with the third cell in that triangle. As such, it does not depend on the particular rule. For example, in Fig. 1(b), let S coincide with the primitive generating set indicated by the fat cells, then $E(S) = S \cup \{1\}$, where $\{1\}$ symbolizes the set of cells labelled 1. In a similar way, $E(E(S)) = E(S) \cup \{2\}$, etc.

1.2. Generalization

The rest of the introduction presents, in an informal and summarizing way, some additional examples showing how the prototype triangular FCA introduced above, and related questions, can be generalized in a broad sense. The underlying mathematical theory will be developed in the remaining sections.

1.2.1. An intuitive introduction to frames

As a first illustrative example displaying rather general features, and which is neither too simple nor too complicated, consider the structures displayed in Fig. 2. Fig. 2(a) is a plane composition of three triangles, three tetragons, and one pentagon. Fig. 2(b) represents a polyhedron built from the same collection of polygons. Such structures are examples of *frames* (a name that we propose for reasons which are intuitively clear). Each of the composing polygons is called an elementary frame.

In general, a frame is a finite set of cells X (the set of vertices in the figures) together with a list of distinct nonempty subsets of X (the vertices of the composing polygons in the figure), whose union is X. The elements of the list are called *elementary frames*, and actually form a cover of X [9,11]. They generalize the elementary triangles in the prototype FCA. Fig. 3 provides additional examples.

A *configuration* of a frame is a map from X to a finite set of states G (sometimes referred to as colors in the examples). A *matching configuration* satisfies the restrictions imposed by a set $\mathcal{R}$ of *local rules* for the elementary frames. A local rule for an elementary frame with n cells is such that if the states of $(n-1)$ cells are known, then the state of the remaining n-th cell is uniquely determined. This is equivalent to saying that this rule is generated by an $(n-1)$-ary quasigroup with values in G [8], see also Section 5. The specification of a set of local rules (one rule for every elementary frame) provides the generalization for the uniform rule in the triangular prototype FCA.

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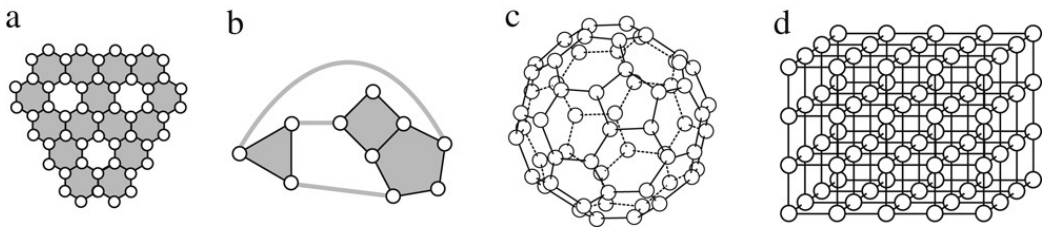


Fig. 3. Some additional "geometrically inspired" frames. Elementary frames are occasionally displayed in gray. For the buckyball frame (c), elementary frames may for example correspond to edges, or faces, or a combination of both. In the cubical point lattice (d), elementary frames may correspond to edges, faces, or even elementary cubes.

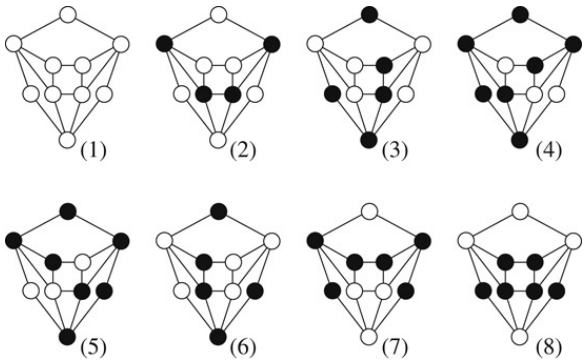


Fig. 4. All matching configurations for the frame in Fig. 2(a) and the "even blacks" rule.

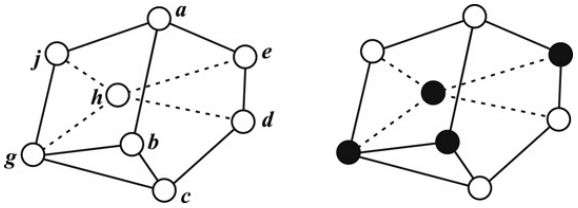


Fig. 5. All matching configurations for the frame in Fig. 2(b) and the "even blacks" rule.

As an example, consider the frames in Fig. 2. The state set is $G = \{\text{black, white}\}$, where the local rule for every elementary frame is:

"An elementary frame contains an even number of black cells".

We call this the "*even blacks*" rule. Note that this rule is indeed an $(n-1)$-ary quasigroup for an elementary frame containing n cells (if the number of black cells in $(n-1)$ of its cells is known, the state of the remaining cell is known). Of course, as elementary frames may have cells in common, colors (states) appearing in one elementary frame influence the possible colors in adjacent elementary frames. This restricts the possible distribution of colors over the cells of the complete frame. Figs. 4 and 5 show the matching configurations for the "even blacks" rule for the frames in Fig. 2(a), (b), respectively.

Observe that the "even blacks"-rule can be stated in algebraic terms: Let $\mathbb{F}_2 = \{0, 1\}$ be the field with two elements. If white stands for 0 and black for 1, then the local matching rules specify that, in any elementary frame f_j of the frame, it must hold that $\sum_{i=1}^{n_j} c_{j,i} = 0$, where the elementary frame f_j contains n_j cells, the i-th cell in it having color $c_{j,i} \in \mathbb{F}_2$. In this setting the question of the existence of matching configurations is merely a question on the existence of solutions of a set of linear equations in $\mathbb{F}_2$. However, as already illustrated in [4] for the prototype triangular FCA, not all local rules are expressible as linear equations.

1.2.2. Some questions

Some interesting questions that arose in [4] for the triangular prototype FCA also arise in the general case. They form the central theme of the present paper.

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First Question: what is the number of matching configurations?. The answer depends on the frame as well as on the collection of local rules. We explain a sensible method to find a solution with the help of the frame in Fig. 2(a) and the "even blacks" rule. Start with all cells in a noncolored state. Select in step (1) an elementary frame, say f_2, which contains three cells. Then attribute (black or white) colors to two of its cells, say c and g. Then the "even blacks" rule determines the color of the third cell b, and we have a matching configuration restricted to f_2. Select in step (2) an elementary frame which is adjacent to f_2, say f_5, which has four cells. The two cells b and g, which are common to f_2 and f_5 are already colored; by fixing the color of a third cell, say f, the color of the remaining fourth cell j is uniquely determined. In step (3), consider f_6 which already contains the two colored cells g and j, so the color of h is uniquely determined. This continues in subsequent steps with the elementary frames f_3, f_4, f_7 and f_1, where the local rule can be applied without fixing the colors of additional cells first. This illustrates how the whole configuration can be "grown" from knowledge of the color of the cells c, f, g by applying the local rule iteratively. This set is minimal, as knowing the states on a proper subset of $\{c, g, f\}$ will not determine the overall configuration. The conditions

(i) minimality

(ii) extensibility: the ability to grow a complete matching configuration by iterative application of the local rule, makes this set $\{c, g, f\}$ a so-called *primitive set*, similar to a primitive (generating) set for the triangular prototype. Any subset of cells in a frame satisfying these two conditions will be called a primitive set. Notice that, because of the generalizations considered further, the qualifications "generating" and "primitive" have been uncoupled, in contrast to what was the case for the triangular prototype (see the Second Question for considerations about "generating").

In the example at hand, any assignment of colors to a primitive set produces a matching configuration. That is because the underlying frame is *completely reducible*. A frame is reducible if there exists an elementary frame that contains a cell which is not contained in any other elementary frame. Reducing a frame removes all elementary frames of this type. The remaining frame may be reducible or irreducible. If a frame is such that all elementary frames can be removed by successive reductions, then it is called completely reducible. If a frame is irreducible, or not completely reducible, like the frame in Fig. 2(b), the "growth"-procedure described above may lead to conflicts as it may happen that an elementary frame is already completed at a certain step without having used its own local rule.

One of the main results in this paper is Theorem 5.3: If a frame is completely reducible and if P is a primitive set for it (satisfying the aforementioned conditions (i) and (ii)), then for every collection of local rules there exist $|G|^{|P|}$-different matching configurations. In other words, there exists a bijection between the set G^P and the set of all matching configurations. This also means that all primitive sets have the same cardinality. In Fig. 6, all sets marked with a P are primitive sets for the frame of Fig. 2(a).

The process which "grows" the complete matching configuration by extending a primitive generating set involves a generalized extension map $E_{\mathcal{F}}$. With this we have completed the triple {frame $\mathcal{F}$, set of rules R, extension map $E_{\mathcal{F}}$} for the present example, as a generalization of a the triple defining the triangular prototype FCA. It should be clear that, although this prototype was obtained by considering the orbit of a classical one-dimensional cellular automaton, the frame cellular automaton in the present example is no longer related to a classical cellular automaton.

In order to study the existence of matching configurations for frames which are not completely reducible, we introduce the concept of *lifts*. A lift is a completely reducible frame which in a certain sense covers a not completely reducible frame. For example, frame (a) in Fig. 2 is a lift of frame (b) in the same figure: if one identifies the cells a, f, and i in frame (a), one obtains frame (b). A matching configuration for (b) is obtained from a matching configuration for (a) for which the states (colors) on the cells a, f, and i are identical. The matching configurations (1) and (2) in Fig. 4 are the only ones for which this is the case. The corresponding matching configurations for the frame (b) are shown in Fig. 5.

Second Question: about existence and properties of generating sets. As already noted, if a frame is completely reducible and has a primitive set P, then there exists a bijection between G^P and the set of matching configurations. This raises the second question: Is there a broader class of sets $S \subset X$ such that there exists a bijection between G^S and all matching configurations? These sets are called *generating sets*. All generating sets have the same cardinality. It is clear that for completely reducible frames, primitive sets form a subset of the generating sets. This is illustrated in Fig. 6. But the question is also valid for frames which are not completely reducible. In that case, primitive sets are not necessarily generating.

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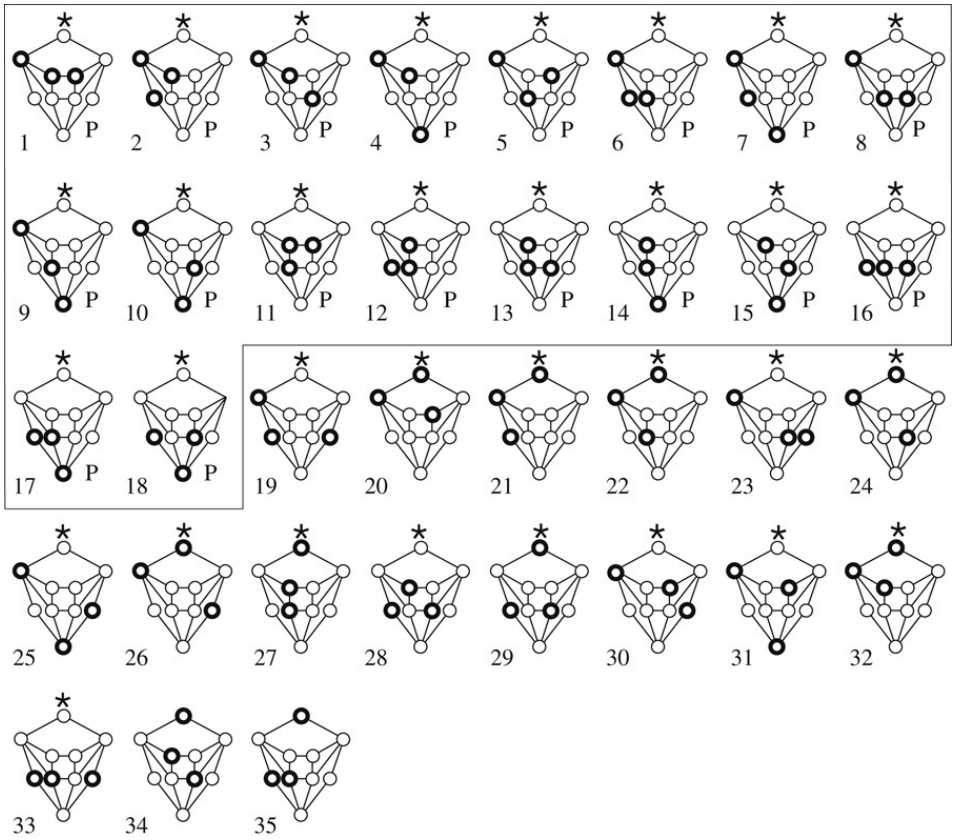


Fig. 6. From 1 to 18: all primitive sets (modulo vertical reflection) for the completely reducible frame of Fig. 2(a). From 1 to 35: all generating sets (modulo vertical reflection). There is a bijection between all possible configurations on a generating set and the matching configurations in Fig. 4. Sets marked with $*$ (all except the two last) are also generating for the 3-state rule, see also Fig. 9.

Contrarily to primitive sets (generating or not) which only depend on the frame, nonprimitive generating sets also depend on the rule. We illustrate this with a second rule for the frames in Fig. 2. This is the so called *3-state rule*. Let $\mathbb{F}_3 = \{0, 1, 2\}$ be the field with three elements, and let the local rules be given as follows:

- In every elementary frame except f_1 the sum of the values is 0 (in $\mathbb{F}_3$),
- In f_1 the values have to satisfy $2a + 2b + c + 2d + e = 0$.

Fig. 7 shows all matching configurations for the completely reducible frame of Fig. 2(a) for this 3-state rule. Fig. 8 does so for the irreducible frame Fig. 2(b).

As Fig. 6 shows, there are subsets of X (sets 34, 35) which are generating for the "even blacks" rule but not generating for the 3-state rule. On the other hand, there also exist generating sets for the 3-state rule which are not generating sets for the "even blacks" rule (all sets in Fig. 9(a)). This remark leads to the question about a necessary condition for a subset S of X to be a generating set.

The answer is given in Theorem 5.10 for arbitrary frames and in Theorem 5.11 for completely reducible frames. The latter says that the collection of possible(!) generating sets forms the set of bases of a geometric-combinatorially defined matroid (the sets in Fig. 6 together with the sets in Fig. 9(a)) This puts certain restrictions on the way S is distributed over the frame.

Can it happen that an FCA has no generating set? It will be shown that the answer is affirmative, in which case a weak form of generating set, called *minimal identifying set*, will be considered.

1.3. Summary and overview

The paper is organized as follows. In Section 2 we formalize the notion of frames and study their basic properties. We consider two means to convert irreducible frames into completely reducible ones that are "close" to it: by constructing a minimal lift, and a maximal subframe. Associated with these are "defect"-numbers (measuring the

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Fig. 7. All 27 matching configurations that satisfy the 3-state rule for the completely reducible frame of Fig. 2(a) (0 = white, 1 = gray, 2 = black).

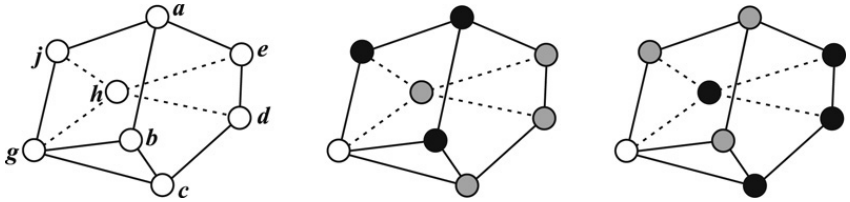


Fig. 8. The three matching configurations for the irreducible frame in Fig. 2(b) and the 3-state rule. Any cell, except g, constitutes a generating set as it is a minimal set for which a bijection exists between all its three possible states and the set of matching configurations.

degree of deficiency of a frame with respect to its closest completely reducible form), and the main Theorem 2.12 in this section states that both defect-numbers are equal.

In Section 3, we discuss the extension map $E_{\mathcal{F}}$ of a frame as a tool to model the above mentioned "growth-process". Primitive sets are introduced and their properties discussed. This section concludes with a formula for the cardinality of a primitive set for a given frame in terms of the number of cells, the number of elementary frames, and the defect number from Section 2.

Section 4 gives the "geometric-combinatorial" definition of $\mathcal{F}$-*independent* sets, where $\mathcal{F}$ is a frame. These $\mathcal{F}$-independent sets are a weak form of the independent sets in a matroid, but do form a matroid in several cases (e.g.,this is the case if the frame is completely reducible). We discuss sufficient conditions for a frame to have $\mathcal{F}$-independent sets which form a matroid. However, the full characterization of frames which yield a matroid of $\mathcal{F}$-independent sets remains an open problem.

In Section 5, we introduce local rules and matching configurations and define generating sets and their weaker form, the *identifying sets*. We demonstrate that generating sets, as well as minimal identifying sets, are $\mathcal{F}$-independent sets. This section concludes with some observations on linear rules, in particular on the relation between the $\mathcal{F}$-independent sets and the matroids defined by a linear rule.

2. Frames

2.1. Elementary definitions and properties

In this section, we study the basic properties of frames. For instance, the frame shown in Fig. 2 can be considered as consisting of a set of cells (vertices), i.e., $X = \{a, b, c, d, e, g, h, i, j\}$, and a list of elementary frames f_1, $f_2, \ldots, f_7$, where each f_j is a subset of X. The list of elementary frames can be considered as a map from an index-set $I = \{1, \ldots, 7\}$ into the subsets of X. Moreover one has that $X = f_1 \cup \cdots \cup f_7$.

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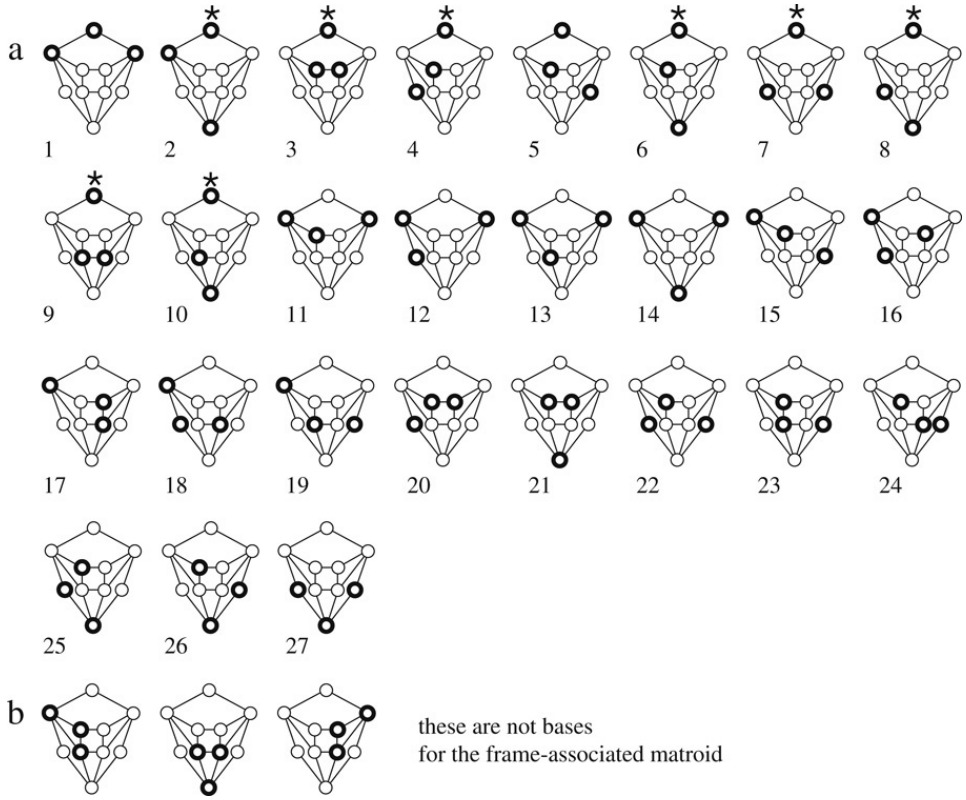


Fig. 9. (a) The sets of fat cells (modulo vertical reflection) together with the sets in Fig. 6 form the set of bases for a matroid associated with the frame. Sets marked $*$ in this figure and in Fig. 6 are the generating sets for the 3-state rule. (b) The only triples of cells in the frame that are not bases of the associated matroid: these cannot be generating sets (whatever matching rules are considered).

Definition 2.1. Let X and I be finite sets and let $\tau : I \to \mathcal{P}(X)$. The triple $\mathcal{F} = (X, I, \tau)$ is called a *frame* if

$$\tau(i) \neq \emptyset \quad \text{for all } i \in I, \quad \tau(i) \neq \tau(j) \quad \text{for } i \neq j, \quad \text{and} \quad X = \bigcup_{j \in I} \tau(j).$$

In agreement with the previous descriptions, elements of X will usually be named cells, those of I indices.

If $X = \emptyset$ and $I = \emptyset$, then no map τ exists. However, we consider the triple $(\emptyset, \emptyset,)$ as the empty frame. From now on all sets are considered to be nonempty, unless stated otherwise.

Let $\mathcal{F} = (X, I, \tau)$ be a frame and let $X' \subseteq X$, $I' \subseteq I$ be such that $X' = \bigcup_{i \in I'} \tau(i)$, then the frame $\mathcal{F}' = (X', I', \tau)$ is called a *subframe* of $\mathcal{F}$, denoted as $\mathcal{F}' \subseteq \mathcal{F}$. The subframes of the form $(\tau(i), \{i\}, \tau)$, $i \in I$ and denoted by $\tau(i)$ for short, are called *elementary frames*.

$T_{\mathcal{F}} = \{\tau(i) \mid i \in I\}$ denotes the set of elementary frames. $T_{\mathcal{F}}$ is actually a cover of X, see [7,9] for details. Note that $|T_{\mathcal{F}}| = |I|$.

A frame $\mathcal{F} = (X, I, \tau)$ is *connected* if there do not exist two nonempty subframes (X_1, I_1, τ), (X_2, I_2, τ) of $\mathcal{F}$ such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, and $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$.

The difference between the number of cells of a frame and the number of elementary frames, denoted

$$\text{ind}(\mathcal{F}) = |X| - |I|\,,$$

is called the *index* of the frame $\mathcal{F}$.

If $f : X \to Y$ is any map, then f induces a map from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. This map will also be denoted as f and it is defined by $f(S) = \{f(s) \mid s \in S\}$ for all $S \in \mathcal{P}(X)$. A morphism between frames is a map which preserves the elementary subframes. To be more precise:

Definition 2.2. Let $\mathcal{F} = (X, I, \tau)$ and $\mathcal{G} = (Y, J, \sigma)$ be frames. A *morphism* $\psi : \mathcal{G} \to \mathcal{F}$ is a pair $\psi = (\lambda, \rho)$ where $\lambda : Y \to X$ and $\rho : J \to I$ are maps such that $\lambda(\sigma(j)) = \tau(\rho(j))$ for all $j \in J$.

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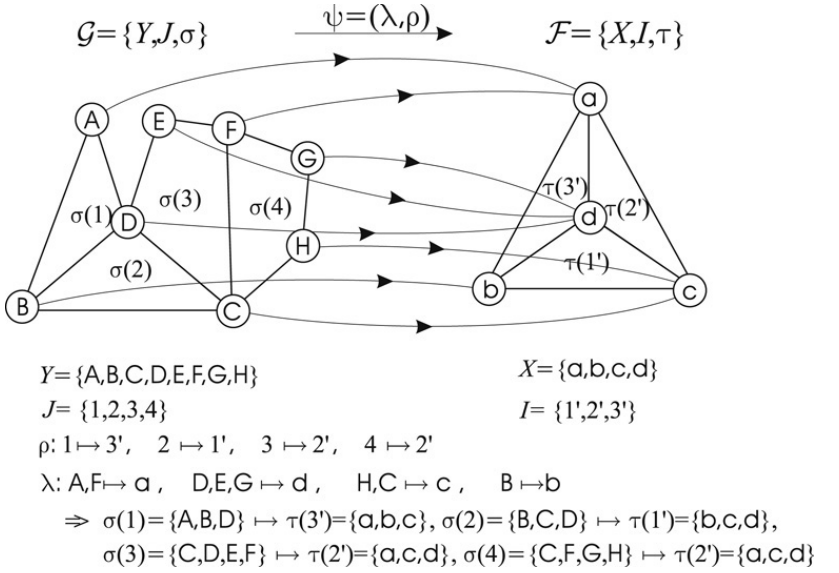


Fig. 10. Example of an epimorphism between the frames $\mathcal{G}$ and $\mathcal{F}$: the arrows depict the map $\lambda : Y \to X$.

As usual, if λ and ρ are both injective, the morphism ψ is called *monomorphism*. If λ and ρ are both surjective, then ψ is called an *epimorphism*, and if λ and ρ are both bijective, then ψ is called an *isomorphism*.

Since epimorphisms will recur frequently later, we illustrate this concept in Fig. 10.

The following definition introduces two important maps.

Definition 2.3. Let $\mathcal{F} = (X, I, \tau)$ be a frame. The *frame-map* $\mathbb{I}_{\mathcal{F}} : \mathcal{P}(X) \to \mathcal{P}(I)$ is defined as $\mathbb{I}_{\mathcal{F}}(S) = \{i \mid i \in I \text{ and } \tau(i) \subseteq S\}$.

The *support-map* $\mathbb{X}_{\mathcal{F}} : \mathcal{P}(I) \to \mathcal{P}(X)$ is defined as $\mathbb{X}_{\mathcal{F}}(I') = \bigcup_{i \in I'} \tau(i)$ for $I' \subset I$.

In other words, $\mathbb{I}_{\mathcal{F}}(S)$ indicates the elementary frames contained in a subset S, while $\mathbb{X}_{\mathcal{F}}(I')$ indicates the cells of X supported by the elementary frames indexed by $I' \subseteq I$. We agree to omit the subscript $\mathcal{F}$ if there is no risk of confusion. Note that the triple $(S, \mathbb{I}(S), \tau)$ is not necessarily a frame for $S \subset X$. On the other hand, for any $I' \subseteq I$ the triple $(\mathbb{X}(I'), I', \tau)$ is a subframe of $\mathcal{F}$. Moreover, for any $S \subseteq X$ the triple $\mathcal{F}(S) = (\mathbb{X}(\mathbb{I}(S)), \mathbb{I}(S), \tau)$ is a subframe of $\mathcal{F}$. It is called the *subframe induced by* S. Note that $\mathcal{F}(X) = \mathcal{F}$ and that $\mathcal{F}(\emptyset)$ is the empty frame.

Two special classes of frames, whose relevance will be clarified in the next section, are the classes of completely reducible and of not completely reducible frames.

Definition 2.4. The frame $\mathcal{F} = (X, I, \tau)$ is called *reducible* if there exists an $x \in X$ that belongs to a unique elementary frame $\tau(i)$. Then the subframe that remains after removing $\tau(i)$, i.e., $(\mathbb{X}_{\mathcal{F}}(I \setminus \{i\}), I \setminus \{i\}, \tau)$, is called a *reduction* of $\mathcal{F}$. If a frame is not reducible, it is called *irreducible*.

Fig. 11(a) shows a reducible frame and its reduction. A frame $\mathcal{F}$ is called *completely reducible* if there exists a sequence $(\mathcal{F}_k)_{k=1,\ldots,n}$ of frames, with $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{F}_n = (\emptyset, \emptyset,)$, and such that $\mathcal{F}_{k+1}$ is a reduction of $\mathcal{F}_k$ for all $k = 1, \ldots, n-1$. Fig. 11(b) shows a completely reducible frame, and the successive reductions.

The following theorem, whose proof is obvious, characterizes completely reducible frames in terms of their subframes.

Theorem 2.5. *A frame is completely reducible if and only if every subframe is completely reducible.*

Fig. 12 shows examples of irreducible frames with a positive as well as a negative index. However, as the next lemma shows, the index for completely reducible frames can not be negative.

Lemma 2.6. *If the frame $\mathcal{F}$ is completely reducible, then* $\mathrm{ind}(\mathcal{F}) \geq 0$.

Proof. The proof proceeds via induction on $|I|$. If $|I| = 1$, then completely reducible frames are of the form $(X, \{1\}, \tau)$ and $\tau(1) = X$. Therefore $\mathrm{ind}(X, \{1\}, \tau) = |X| - 1 \geq 0$.

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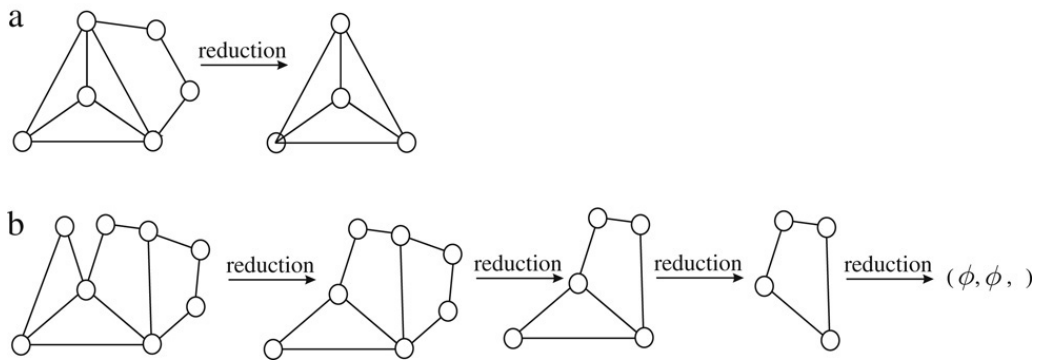


Fig. 11. (a) A reducible frame and its irreducible reduction. (b) A completely reducible frame and one of its reduction sequences.

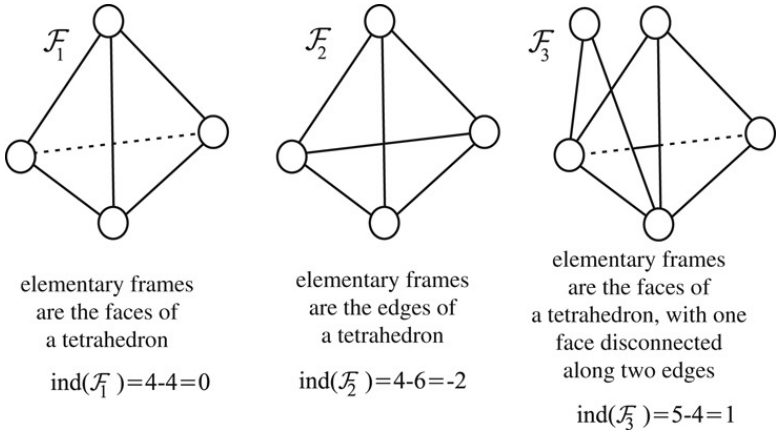


Fig. 12. Not completely reducible frames with 0, negative and positive index.

Now assume that the assertion is true for all completely reducible frames (Y, J, σ) with $|J| \leq N-1$. Now let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame with $|I| = N$. Then there exists a reduction $\mathcal{F}' = (X', I', \tau)$ of $\mathcal{F}$ that is completely reducible and satisfies $|I'| = N-1$. Due to the induction hypothesis, it follows $|X'|-|I'| = |X'|-(N-1) \geq 0$. Since $|X| \geq |X'|+1$ and $N = |I| = |I'|+1$ it follows that $|X|-|I| \geq |X'|+1-(|I'|+1) = |X'|-|I'| \geq 0$, which proves the assertion. □

Notice that the converse of Lemma 2.6 is not true, as can be seen in Fig. 12.

The next two subsections introduce two ways of constructing a completely reducible frame which is in some sense close to a given irreducible frame.

2.2. *Minimal completely reducible lifts*

Definition 2.7. Let $\psi : \mathcal{G} \to \mathcal{F}$ be an epimorphism between the frames $\mathcal{G}$ and $\mathcal{F}$. If $\mathcal{G}$ is completely reducible, then $\mathcal{G}$ is called a *lift* of $\mathcal{F}$ (with respect to the epimorphism ψ).

Example 2.8. Frame $\mathcal{G}$ in Fig. 10 is completely reducible, and as $\psi = (\lambda, \rho)$ establishes an epimorphism from $\mathcal{G}$ onto $\mathcal{F}$, it is a lift of $\mathcal{F}$.

The existence of lifts for any frame is guaranteed by the next theorem.

Theorem 2.9. *Every frame $\mathcal{F} = (X, I, \tau)$ has a lift.*

Proof. Given $\mathcal{F} = (X, I, \tau)$, consider the triple (X', I, τ'), where $X' = \bigcup_{i \in I} (\tau(i) \times \{i\})$ and $\tau' : I \to \mathcal{P}(X')$ is defined as $\tau'(i) = \tau(i) \times \{i\}$. Clearly this triple is a frame $\mathcal{G}$ which "detaches" the elementary frames of $\mathcal{F}$. Since $\tau'(i) \cap \tau'(j) = \emptyset$ whenever $i \neq j$, the frame $\mathcal{G}$ is completely reducible. It remains to construct an epimorphism

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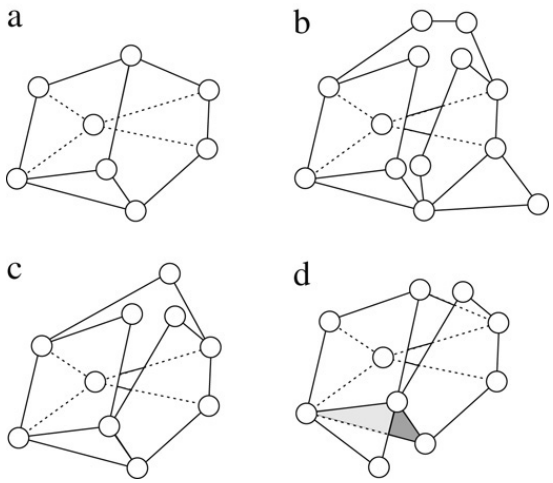


Fig. 13. Three frames (*b*), (*c*), and (*d*) which are lifts of the frame (*a*). The gray face in (*d*) does not represent an elementary frame. The frames (*c*) and (*d*) are minimal lifts and are non-isomorphic.

$\psi : \mathcal{G} \to \mathcal{F}$. To this end define $\lambda : X' \to X$ as $\lambda(x, i) = x$ and $\rho : I \to I$ as the identity. Then λ and ρ are surjective and, by construction $\lambda(\tau'(i)) = \tau(i)$ for all $i \in I$. This proves the assertion. □

Since every frame has a lift one can introduce the notion of minimal lift. A frame $\mathcal{G} = (Y, J, \sigma)$ is a *minimal lift* of the frame $\mathcal{F} = (X, I, \tau)$ if $\mathcal{G}$ has as few extra cells and extra elementary frames as possible as compared to $\mathcal{F}$, i.e., if $|Y| - |X| + |J| - |I|$ is as small as possible among all lifts of $\mathcal{F}$. We call this value the *lift-defect* and denote this value as $\delta^L(\mathcal{F})$, i.e.,

$$\delta^L(\mathcal{F}) = \min\{|Y| - |X| + |J| - |I| \mid (Y, J, \sigma) \text{ is a lift of } \mathcal{F} = (X, I, \tau)\}. \tag{1}$$

Fig. 13 shows three possible lifts of the irreducible frame displayed in Fig. 2(b) from the introduction: one is non-minimal, and two are minimal.

Lemma 2.10. *If $\mathcal{G} = (Y, J, \sigma)$ is a minimal lift of the frame $\mathcal{F} = (X, I, \tau)$, then $|J| = |I|$.*

Proof. Since $\mathcal{G}$ is a lift of $\mathcal{F}$, there exist surjective maps $\lambda : Y \to X$ and $\rho : J \to I$. This shows that $|J| \geq |I|$. Now assume that there exist $j_0, j_1 \in J$ such that $j_0 \neq j_1$ and $\rho(j_0) = \rho(j_1)$. Let $J' = J \setminus \{j_0\}$, $Y' = \bigcup_{j \in J'} \sigma(j)$ and let $\sigma' : J' \to \mathcal{P}(Y')$ be defined as $\sigma'(j) = \sigma(j)$, $j \in J'$. Then the triple (Y', J', σ') is a completely reducible subframe of $\mathcal{G}$. Moreover, due to its construction it is a lift of $\mathcal{F}$. This shows that $\mathcal{G}$ is not a minimal lift, as $|Y| > |Y'|$ and $|J'| = |J| - 1$ imply that $|Y| - |X| + |J| - |I| > |Y'| - |X| + |J'| - |I|$, which is a contradiction to the minimality of $\mathcal{G}$. In other words, the map $\rho : J \to I$ is injective. Since ρ is also surjective, it follows that $|J| = |I|$. □

Due to Lemma 2.10 we may always assume that a minimal lift of (X, I, τ) is of the form (Y, I, σ) and that the morphism ψ is given as $\psi = (\lambda, \mathrm{id})$, where id is the identity on the set I. As a consequence, the definition of $\delta^L(\mathcal{F})$ in Eq. (1) can be rewritten as

$$\delta^L(\mathcal{F}) = \min\{|Y| - |X| \mid (Y, I, \tau) \text{ is a lift of } \mathcal{F} = (X, I, \tau)\}. \tag{2}$$

2.3. Maximal completely reducible subframes

As illustrated in Fig. 13(c)–(d), a minimal lift of an irreducible frame $\mathcal{F} = (X, I, \tau)$ can be interpreted as separating some of $\mathcal{F}$'s elementary frames by duplicating certain of its vertices, without changing the number of elementary frames. However, there is an alternative way of transforming $\mathcal{F}$ into a related completely reducible frame. This is done by removing elementary frames in such a way that the remaining subframe still covers X while being completely reducible (this is illustrated in Fig. 14).

Notice that such procedure is always possible. In fact, if $\mathcal{F} = (X, I, \tau)$ is a frame, then the elementary frames of $\mathcal{F}$ are clearly completely reducible subframes of $\mathcal{F}$. It is therefore meaningful to look for completely reducible

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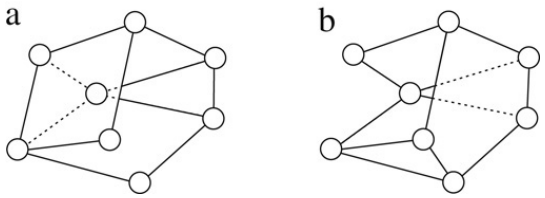


Fig. 14. Two completely reducible subframes of the irreducible frame in Fig. 13(a), both obtained by removing two elementary frames.

subframes of $\mathcal{F}$ which are as large as possible, in the sense that they contain a maximal number of elementary frames, i.e., we are looking for completely reducible subframes $(\mathbb{X}_{\mathcal{F}}(I'), I', \tau),\ I' \subseteq I$, such that $|I| - \left|I'\right|$ is minimal. These are called *maximal completely reducible subframes*. The minimal value of $|I| - \left|I'\right|$ will be called the *subframe-defect* and is denoted $\delta^S(\mathcal{F})$, i.e,

$$\delta^S(\mathcal{F}) = \min\{|I| - \left|I'\right| \mid (\mathbb{X}_{\mathcal{F}}(I'), I', \tau) \subseteq \mathcal{F} \text{ and completely reducible}\}. \tag{3}$$

Lemma 2.11. *If $\mathcal{F}' = (\mathbb{X}_{\mathcal{F}}(I'), I', \tau)$ is a maximal completely reducible subframe of $\mathcal{F} = (X, I, \tau)$, then $\mathbb{X}_{\mathcal{F}}(I') = X$.*

Proof. Suppose that $\mathbb{X}(I') \neq X$, then there exist an $x \in X \setminus \mathbb{X}_{\mathcal{F}}(I')$ and a $j \in I$ such that $x \in \tau(j)$ and such that the subframe $(\mathbb{X}_{\mathcal{F}}(I' \cup \{j\}), I' \cup \{j\}, \tau)$ is also completely reducible, since x only belongs to the elementary frame $\tau(j)$. Since $\left|I' \cup \{j\}\right| > \left|I'\right|$ this is a contradiction to the maximality of $\mathcal{F}'$. □

Hence, as in the case of the minimal lift, we can rewrite Eq. (3) as

$$\delta^S(\mathcal{F}) = \min\{|I| - \left|I'\right| \mid (X, I', \tau) \subseteq \mathcal{F} \text{ is completely reducible}\}. \tag{4}$$

2.4. Relation between subframe- and lift-defects

Theorem 2.12. *If $\mathcal{F}$ is a frame, then $\delta^S(\mathcal{F}) = \delta^L(\mathcal{F})$.*

Proof. If $\mathcal{F}$ is a completely reducible frame, then the result is trivial. Hence, we shall consider the case where $\mathcal{F}$ is not completely reducible, and show that (**a**): $\delta^L(\mathcal{F}) \leq \delta^S(\mathcal{F})$, and (**b**): $\delta^S(\mathcal{F}) \leq \delta^L(\mathcal{F})$.

(**a**) Let $\mathcal{F} = (X, I, \tau)$ be a frame. We consider a maximal completely reducible subframe (X, I', τ) of $\mathcal{F}$ and we construct a lift of $\mathcal{F}$ using this subframe. Let $U = I \setminus I'$ and let $g : U \to \mathbb{X}_{\mathcal{F}}(U)$ be a map such that $g(u) \in \tau(u)$. Now define $Y = X \cup U$ and the map $\sigma : I \to \mathcal{P}(Y)$ by

$$\sigma(i) = \begin{cases} \tau(i) & \text{if } i \notin U \\ (\tau(i) \setminus \{g(i)\}) \cup \{i\} & \text{if } i \in U. \end{cases}$$

Then the triple $\mathcal{G} = (Y, I, \sigma)$ is a completely reducible frame. Finally, we define $\lambda : Y \to X$ as

$$\lambda(y) = \begin{cases} y & \text{if } y \in X \\ g(y) & \text{if } y \in U. \end{cases}$$

By the construction, one easily sees that $\psi = (\lambda, \mathrm{id}) : \mathcal{G} \to \mathcal{F}$ is an epimorphism. Since $\mathcal{G}$ is completely reducible and a lift of $\mathcal{F}$, it follows that (cf. Eq. (2))

$$\delta^L(\mathcal{F}) \leq \delta^S(\mathcal{F}) = |U| = |Y| - |X|, \tag{5}$$

which proves (a).

(**b**) The inequality $\delta^S(\mathcal{F}) \leq \delta^L(\mathcal{F})$ is shown by induction on the cardinality of I.

If $\mathcal{F} = (X, \{1\}, \tau)$ is a frame, then it is completely reducible and one has $\delta^L(\mathcal{F}) = \delta^S(\mathcal{F}) = 0$. We now assume that $\delta^S(\mathcal{F}) \leq \delta^L(\mathcal{F})$ for all frames $\mathcal{F} = (X, I, \tau)$ with $|I| \leq N$, or, equivalently, taking Inequality (5) into account, that

$$\delta^S(\mathcal{F}) = \delta^L(\mathcal{F}) \text{ if } |I| \leq N. \tag{6}$$

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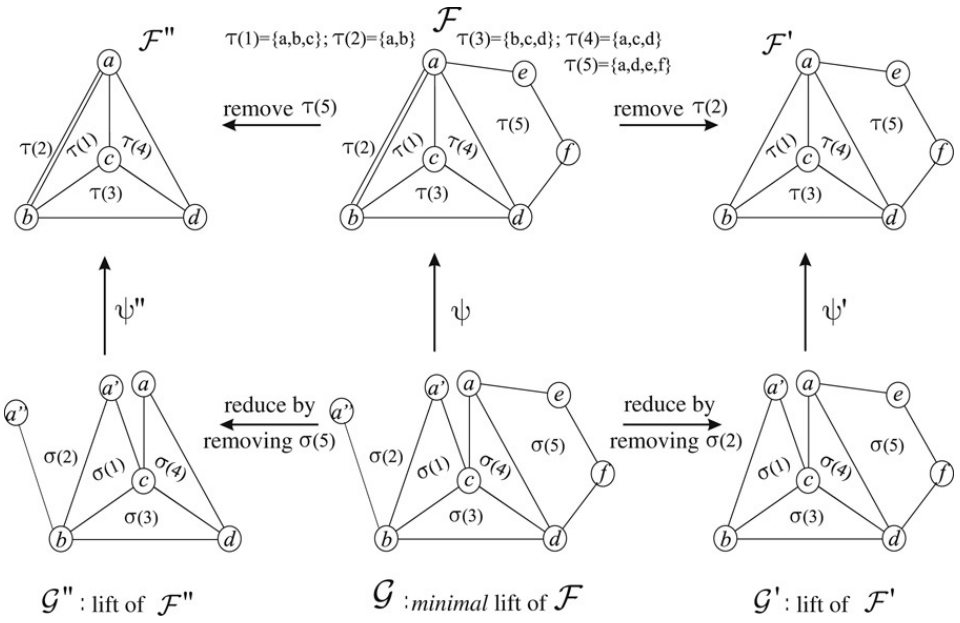


Fig. 15. Removing one elementary frame from a frame $\mathcal{F}$ and from a minimal lift $\mathcal{G}$ of $\mathcal{F}$ produces, to the right side, frame $\mathcal{F}'$ and an associated lift $\mathcal{G}'$; to the left side, frame $\mathcal{F}''$ and associated lift $\mathcal{G}''$. A part of Theorem 2.12 shows that $\mathcal{G}'$ (respectively $\mathcal{G}''$) is a minimal lift of $\mathcal{F}'$ (respectively $\mathcal{F}''$). Observe two distinct cases: for the removals to the right it holds that $|X| = |X'|$, for the removals to the left: $|X| > |X''|$.

Now let $\mathcal{F} = (X, I, \tau)$ with $|I| = N + 1$ and let $\mathcal{G} = (Y, I, \sigma)$ be a minimal lift with morphism $\psi = (\lambda, \mathrm{id})$. Since $\mathcal{G}$ is completely reducible there exists $j \in I$ such that $|Y'| < |Y|$, for $Y' = \mathbb{X}_{\mathcal{G}}(I \setminus \{j\})$. Since $\psi = (\lambda, \mathrm{id})$ and since $\mathcal{G}$ is a lift of $\mathcal{F}$, it follows that $\mathcal{G}' = (Y', I \setminus \{j\}, \sigma)$ is a lift of $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$, where $X' = \mathbb{X}_{\mathcal{F}}(I \setminus \{j\})$ and $\psi' : \mathcal{G}' \to \mathcal{F}'$ is given as $\psi' = (\lambda, \mathrm{id})$. This is illustrated in Fig. 15. Our goal is to show, by contradiction, that $\mathcal{G}'$ is a minimal lift of $\mathcal{F}'$, and then combine this with the induction hypothesis to find that $\delta^S(\mathcal{F}) \leq \delta^L(\mathcal{F})$. To demonstrate that $\mathcal{G}'$ is a minimal lift of $\mathcal{F}'$, we first prove four auxiliary results about the lift $\psi' : \mathcal{G}' \to \mathcal{F}'$.

A1. The map $\lambda : Y \setminus Y' \to X$ is injective.

Subproof. If $|Y \setminus Y'| = 1$, then the assertion is true. Now suppose that $|Y \setminus Y'| > 1$ and that there exist x, $y \in Y \setminus Y'$ such that $x \neq y$ and $\lambda(x) = \lambda(y)$. Then the triple $(Y \setminus \{y\}, I, \sigma^*)$, where $\sigma^* : I \to \mathcal{P}(Y \setminus \{y\})$ is defined as

$$\sigma^*(i) = \begin{cases} \sigma(i) & \text{if } i \neq j \\ \sigma(j) \setminus \{y\} & \text{if } i = j \end{cases}$$

is a completely reducible frame. By its very construction, it is also a lift of $\mathcal{F}$. Since $|Y \setminus \{y\}| = |Y| - 1$, this is a contradiction to the minimality of the lift $\mathcal{G}$, which concludes this subproof.

A2. If $x \in X \setminus X'$, then there exists a unique $y \in Y \setminus Y'$ such that $\lambda(y) = x$.

Subproof. If $x \in X \setminus X'$, then $x \in \tau(j)$ and $x \notin \tau(i)$ for $i \in I$ and $i \neq j$. Since $\lambda(\sigma(j)) = \tau(j)$ there exists a $y \in \sigma(j)$ such that $\lambda(y) = x$. It remains to show that $y \in Y \setminus Y'$. To this end, assume that there exists an $i \in I$, $i \neq j$, such that $y \in \sigma(i)$. Since $\lambda(\sigma(i)) = \tau(i)$ it follows that $\lambda(y) = x \in \tau(i)$, which is a contradiction with the fact that $x \notin \tau(i)$. The uniqueness of y follows from A1.

A3. If $|X \setminus X'| = 0$ and $|Y \setminus Y'| > 0$, then $|Y \setminus Y'| = 1$.

Subproof. Suppose that $|Y \setminus Y'| \geq 2$. Let $y \in Y \setminus Y'$, i.e., $y \notin \sigma(i)$ for all $i \in I$, $i \neq j$. Since $|X \setminus X'| = 0$ there exists an $i \in I$, $i \neq j$ such that $x = \lambda(y) \in \tau(i) \cap \tau(j)$. Therefore there exists an $y' \in \sigma(i)$ such that $\lambda(y') = x$ and $y \neq y'$. The triple $(Y \setminus \{y\}, I, \sigma^*)$, where σ^* is defined as

$$\sigma^*(k) = \begin{cases} \sigma(k) & \text{if } k \neq j \\ (\sigma(j) \setminus \{y\}) \cup \{y'\} & \text{if } k = j, \end{cases}$$

is a frame. Moreover, since $|Y \setminus Y'| \geq 2$, it follows that this frame is completely reducible. Due to its construction, it also follows that $\psi = (\lambda, \mathrm{id}) : (Y \setminus \{y\}, I, \sigma^*) \to \mathcal{F}$ is an epimorphism. This contradicts the minimality of (Y, I, σ).

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A4. If $\left|X \setminus X'\right| > 0$, then $\left|Y \setminus Y'\right| = \left|X \setminus X'\right| > 0$.

Subproof. By A1. and A2. it follows that $\left|X \setminus X'\right| \leq \left|Y \setminus Y'\right|$. It also follows from A1. and A2. that $\left|Y \setminus Y'\right| = 1$ implies $\left|X \setminus X'\right| = 1$. We therefore assume that $\left|Y \setminus Y'\right| \geq 2$. Now suppose that there exists an $y \in Y \setminus Y'$ such that $\lambda(y) \notin X \setminus X'$. This means that $\lambda(y) = x \in \tau(i) \cap \tau(j)$ for an $i \in I$, $i \neq j$, and that there exists a $y \notin Y \setminus Y'$, with $\lambda(y') = x$. Now the same construction as in A3. shows that (Y, I, σ) is not a minimal lift, which is a contradiction, and therefore one has $\left|X \setminus X'\right| = \left|Y \setminus Y'\right|$.

As the next step we will prove that the lift $\mathcal{G}' = (Y', I \setminus \{j\}, \sigma)$ is a minimal lift of $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$. To this end, we assume that $\mathcal{G}'$ is not minimal, i.e., there is a minimal lift $(Z, I \setminus \{j\}, \kappa)$ of $\mathcal{F}'$ with epimorphism $\phi = (\mu, \mathrm{id})$ and

$$|Z| < \left|Y'\right|. \tag{7}$$

For purely technical reasons (and without loss of generality) we shall assume that Z and X are disjoint.

Let $s : X' \to Z$ be any map such that $\mu(s(x)) = x$ for all $x \in X'$. Since $\mu(Z) = X'$ such a map exists. We distinguish two cases, namely, (1). $\left|X \setminus X'\right| = 0$ (from A3. above — see right part of Fig. 15 for an illustration where this is the case) and (2). $\left|X \setminus X'\right| > 0$ (from A4. above — see left part of Fig. 15).

(1) Let $\left|Y \setminus Y'\right| = 1$ and $\left|X \setminus X'\right| = 0$. Using the minimal lift $(Z, I \setminus \{j\}, \kappa)$ of $(X', I \setminus \{j\}, \tau)$ we construct a new lift of (X, I, τ). To this end, we choose an $x \in \tau(j)$ and consider $Z^* = Z \cup \{x\}$. The triple (Z^*, I, κ^*) with $\kappa^* : I \to \mathcal{P}(Z^*)$ defined as

$$\kappa^*(i) = \begin{cases} \kappa(i) & \text{if } i \neq j \\ (s(\tau(j)) \setminus \{s(x)\}) \cup \{x\} & \text{if } i = j, \end{cases}$$

is a frame. Since $(Z, I \setminus \{j\}, \kappa)$ is completely reducible and since $x \in \kappa^*(i)$ if and only if $i = j$ it follows that (Z^*, I, κ^*) is also completely reducible. Finally, the tuple $\psi^* = (\mu^*, \mathrm{id})$, where $\mu^* : Z^* \to X$ is defined as

$$\mu^*(z) = \begin{cases} \mu(z) & \text{if } z \neq s(x) \\ x & \text{if } z = s(x), \end{cases}$$

is an epimorphism from (Z^*, I, κ^*) onto (X, I, τ). Since $|Z^*| = |Z| + 1 < \left|Y'\right| + 1 = |Y|$ this is a contradiction to the minimality of the lift (Y, I, σ). Because $\left|Y'\right| = |Y| - 1$ and $\left|X'\right| = |X|$, it also follows that

$$\delta^L(X', I \setminus \{j\}, \sigma) = \delta^L(X, I, \tau) - 1. \tag{8}$$

(2) Now let $\left|Y \setminus Y'\right| = \left|X \setminus X'\right| > 0$. As above, we construct from the minimal lift $(Z, I \setminus \{j\}, \kappa)$ of $(X', I \setminus \{j\}, \tau)$ a new lift of (X, I, τ). Let $X \setminus X' = \{x_1, \ldots, x_n\}$ which we assume to be disjoint from the set Z. Let $Z^* = Z \cup \{x_1, \ldots, x_n\}$, then the triple (Z^*, I, κ^*), where $\kappa^* : I \to \mathcal{P}(Z^*)$ is defined as

$$\kappa^*(i) = \begin{cases} \kappa(i) & \text{if } i \neq j \\ s(\tau(j) \cap_{i \neq j} \tau(i)) \cup \{x_1, \ldots, x_k\} & \text{if } i = j, \end{cases}$$

is a completely reducible frame. If one defines $\mu^* : Z^* \to X$ as

$$\mu^*(z) = \begin{cases} \mu(z) & \text{if } z \notin \{x_1, \ldots, x_n\} \\ z & \text{if } z \in \{x_1, \ldots, x_n\}, \end{cases}$$

then, by its construction $\psi^* = (\mu^*, \mathrm{id})$ defines a lift of (X, I, τ). Since $|Z^*| = |Z| + |\{x_1, \ldots, x_n\}| < \left|Y'\right| + |\{x_1, \ldots, x_n\}| = |Y|$ this is a contradiction to the minimality of the lift (Y, I, σ). It also follows that

$$\begin{aligned} \delta^L(X', I \setminus \{j\}, \sigma) &= \left|Y'\right| - \left|X'\right| \\ &= \left|Y'\right| + \left|Y \setminus Y'\right| - \left|X'\right| - \left|X \setminus X'\right| \\ &= |Y| - |X| = \delta^L(X, I, \tau). \end{aligned} \tag{9}$$

This shows that $\mathcal{G}' = (Y', I \setminus \{j\}, \sigma)$ is a minimal lift of $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$. As a consequence,

$$\delta^L(X', I \setminus \{j\}, \tau) = \left|Y'\right| - \left|X'\right|. \tag{10}$$

As $Y' \subseteq Y$ and $X' \subseteq X$, we have that $|X| = \left|X \setminus X'\right| + \left|X'\right|$ and $|Y| = \left|Y \setminus Y'\right| + \left|Y'\right|$, yielding

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$\left|Y'\right| - \left|X'\right| = |Y| - |X| - \left|Y \setminus Y'\right| + \left|X \setminus X'\right| = \delta^L(\mathcal{F}) - \left|Y \setminus Y'\right| + \left|X \setminus X'\right|$, the last equality because $\mathcal{G}$ is a minimal lift of $\mathcal{F}$. Using the considerations A2. and A3. above, this leads to

$$\left|Y'\right| - \left|X'\right| = \begin{cases} \delta^L(X, I, \tau) - 1 & \text{if } \left|X'\right| = |X| \\ \delta^L(X, I, \tau) & \text{if } \left|X'\right| <| X | . \end{cases} \tag{11}$$

Now, since $|I \setminus \{j\}|$ has cardinality N, the induction hypothesis (6) implies that

$$\delta^S(X', I \setminus \{j\}, \tau) = \delta^L(X', I \setminus \{j\}, \tau) = \left|Y'\right| - \left|X'\right|, \tag{12}$$

where the last equality follows from Eq. (10). Thus, $\delta^S(\mathcal{F}')$ elementary frames can be removed from $\mathcal{F}'$ such that a completely reducible subframe of $\mathcal{F}'$ remains. This means

- In case $X = X'$, there exists a subset U of $I \setminus \{j\}$ with $|U| = \delta^L(|F|) - 1$, such that $(X, I \setminus (U \cup \{j\}), \tau)$ is a completely reducible subframe of $\mathcal{F}' = (X, I \setminus \{j\}, \tau)$. This implies that $(X, I \setminus (U \cup \{j\}), \tau)$ is a completely reducible subframe of $\mathcal{F} = (X, I, \tau)$, though not necessarily maximal. As a consequence, it follows that $\delta^S(\mathcal{F}) \leq |U \cup \{j\}| = |U| + 1 = \delta^L(\mathcal{F})$.
- In case $\left|X'\right| < |X|$ (or $X' \subsetneq X$), there exists a $(X, I \setminus (U \cup \{j\}), \tau)$ such that $|U| = \delta^L(\mathcal{F})$ and such that $(X', (I \setminus \{j\}) \setminus U, \tau)$ is a completely reducible subframe of $(X', I \setminus \{j\}, \tau)$. Since $\tau(j)$ is not completely contained in X' and since $X' \cup \tau(j) = X$ it follows that $(X, I \setminus U, \tau)$ is a completely reducible subframe of (X, I, τ). This shows that $\delta^S(\mathcal{F}) \leq |U| = \delta^L(\mathcal{F})$.

Thus we have shown that $\delta^L(\mathcal{F}) \leq \delta^S(\mathcal{F})$ and $\delta^S(\mathcal{F}) \leq \delta^L(\mathcal{F})$ and therefore $\delta^L(\mathcal{F}) = \delta^S(\mathcal{F})$. □

Due to the above theorem, the following definition is justified.

Definition 2.13. Let $\mathcal{F}$ be a frame, the *defect* of $\mathcal{F}$ is denoted as $\delta(\mathcal{F})$ and is equal to $\delta^L(\mathcal{F}) = \delta^S(\mathcal{F})$.

The name defect seems proper as it gives a kind of distance between an irreducible frame and "closest" completely reducible frames.

Corollary 2.14. *Every frame $\mathcal{F} = (X, I, \tau)$ has a minimal lift $\mathcal{G} = (X \cup U, I, \sigma)$ with $|U| = \delta(\mathcal{F})$ and with epimorphism $\psi = (\lambda, \mathrm{id})$ such that the maps $\lambda_{|\sigma(i)} : \sigma(i) \to \tau(i)$ are bijective for all $i \in I$ and such that $\lambda(x) = x$ for all $x \in X \subset X \cup U$.*

Proof. There exists a subset U of I such that $|U| = \delta(\mathcal{F})$. The first part of the proof of Theorem 2.12 shows how to construct a minimal lift with the required property. □

As a further consequence we note

Corollary 2.15. *Let $\mathcal{F} = (X, I, \tau)$ be a frame.*

(1) *If $\mathcal{F}' \subseteq \mathcal{F}$, then $\delta(\mathcal{F}') \leq \delta(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is reducible, then $\delta(\mathcal{F}) = \delta(\mathcal{F}')$ for the reduction $\mathcal{F}'$ of $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is completely reducible, then $\delta(\mathcal{F}) = 0$.*
(4) $|X| - |I| + \delta(\mathcal{F}) \geq 0$.

Proof. (1) is obvious; (2) follows from the proof of Theorem 2.12; (3) is obvious. (4) By Corollary 2.14, there exists a minimal lift $\mathcal{G} = (X \cup U, I, \sigma)$ of $\mathcal{F}$ with $|U| = \delta(\mathcal{F})$. Lemma 2.6 implies that $\mathrm{ind}(\mathcal{G}) = |X \cup U| - |I| \geq 0$. This shows that $|X| + \delta(\mathcal{F}) - |I| \geq 0$. □

3. The extension map of a frame, and primitive generating sets

In this section we move a step further in the direction of determining matching configurations on frames. To study the primitive sets as mentioned in the introduction we introduce a map E which models the fact that the configuration on an elementary frame can be completed if it is known on all but one cells of the elementary frame.

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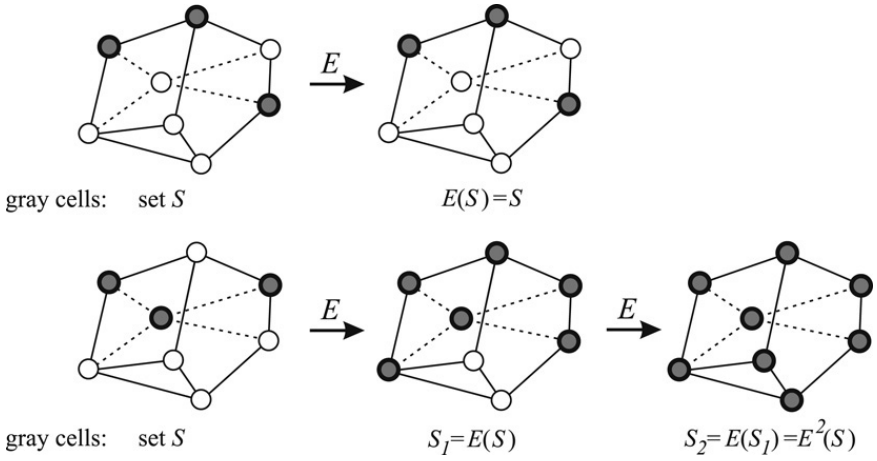


Fig. 16. The extension map applied to some sets S (given by the gray cells) in the frame from Fig. 2(b).

Definition 3.1. Let $\mathcal{F} = (X, I, \tau)$ be a frame and S a subset of X. The map $E = E_{\mathcal{F}} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as

$$E(S) = S \cup \bigcup_{\substack{i \in I \text{ and} \\ |\tau(i) \cap S| = |\tau(i)| - 1}} \tau(i)$$

is called the *extension map* (with respect to $\mathcal{F}$).

Fig. 16 illustrates this map for some sets S.

If (X, I, τ) is a frame such that some of its elementary frames contains only one element, then one has $E(\emptyset) \neq \emptyset$. On the other hand, if $E(\emptyset) \neq \emptyset$, one of the elementary frames contains only one element. This motivates the next definition.

Definition 3.2. The frame $\mathcal{F}$ is called *regular* if $E_{\mathcal{F}}(\emptyset) = \emptyset$, i.e., if each elementary frame contains at least two cells.

In order to avoid unnecessary technical details, we assume **from now on** that **all frames are regular**. Note that the empty frame is considered to be regular.

Since $S \subseteq E(S)$, it follows that $E^n(S) \subseteq E^{n+1}(S)$ for all $n \in \mathbb{N}$. Due to the finiteness of X, there exists an $n_0 = n_0(S) \in \mathbb{N}$ such that $E^n(S) = E^{n_0}(S)$ for all $n \geq n_0$.

Definition 3.3. Let $\mathcal{F} = (X, I, \tau)$ be a frame and $E : \mathcal{P}(X) \to \mathcal{P}(X)$ the corresponding extension map. The map $\Omega = \Omega_{\mathcal{F}} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined as

$$\Omega(S) = \bigcup_{n \in \mathbb{N}} E^n(S)$$

is called the *limit map* (of the frame $\mathcal{F}$).

Note that $\Omega(S) = E^{n_0}(S)$, where n_0 was defined above. In other words, $\Omega(S)$ is a fixed point of the extension map. The next lemma collects some elementary properties of the extension and limit maps.

Lemma 3.4. *Let E be the extension map of the frame $\mathcal{F} = (X, I, \tau)$, and let Ω be the associated limit map. For $S_1, S_2 \subseteq X$ the following assertions are true.*

(1) *If $S_1 \subseteq S_2$, then $E(S_1) \subseteq E(S_2)$.*
(2) *If $S_1 \subseteq S_2$, then $\Omega(S_1) \subseteq \Omega(S_2)$.*
(3) $\Omega(\Omega(S_1)) = \Omega(S_1)$.
(4) $\Omega(\Omega(S_1) \cup \Omega(S_2)) = \Omega(S_1 \cup S_2)$.

Proof. The assertions (1), (2), and (3) are obvious. To prove (4), one notes that $S_1 \cup S_2 \subseteq \Omega(S_1) \cup \Omega(S_2) \subseteq \Omega(S_1 \cup S_2)$, and an application of Ω and (2) yields $\Omega(S_1 \cup S_2) \subseteq \Omega(\Omega(S_1) \cup \Omega(S_2)) \subseteq \Omega^2(S_1 \cup S_2)$. Since $\Omega(S) = \Omega^2(S)$ for all $S \subseteq X$, the assertion follows. □

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Note that Ω is a closure operator (see [5] for a definition and an overview of related subjects). In the context of closure operators, the sets S with the property $\Omega(S) = S$ are called closed sets. These closed sets are of importance for the considerations in Section 4.

The following lemma will be of considerable help in the following sections. It provides a relation between the new cells x generated by an application of the extension map, i.e., $x \in E(S) \setminus S$, and the corresponding new elementary frames. The lemma basically says that every new cell x requires its own elementary frame. i.e., $x \in \tau(j) \subset E(S)$ and $\tau(j) \not\subset S$.

Lemma 3.5. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. If $S \subseteq X$, then there exists an injective map $\zeta : E(S) \setminus S \to \mathbb{I}_{\mathcal{F}}(E(S)) \setminus \mathbb{I}_{\mathcal{F}}(S)$, such that $x \in \tau(\zeta(x))$ for all $x \in E(S) \setminus S$.*

Proof. Due to the definition of E, for $x \in E(S) \setminus S$ there exists an elementary frame $\tau(j)$ such that $|S \cap \tau(j)| = |\tau(j)| - 1$ and such that $x \in \tau(j)$. If $y \in E(S) \setminus S$ and $y \neq x$, then y does not belong to $\tau(j)$. This proves the assertion. □

Iterating the above argument one obtains

Corollary 3.6. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. If $S \subseteq X$, then there exists an injective map $\xi : \Omega(S) \setminus S \to \mathbb{I}_{\mathcal{F}}(\Omega(S)) \setminus \mathbb{I}_{\mathcal{F}}(S)$ such that $x \in \tau(\xi(x))$ for all $x \in \Omega(S) \setminus S$.*

Definition 3.7. Let $\mathcal{F}$ be a frame with limit map Ω. Let $S \subseteq X$, then

- S is called *Ω-generating* if $\Omega(S) = X$.
- S is called *Ω-minimal* if S is Ω-generating and no proper subset $S' \subset S$ is Ω-generating.
- $S \subseteq X$ is called *Ω-primitive* if S is Ω-minimal and $|S| = \min\{|S'| \mid S' \text{ is } \Omega\text{-minimal}\}$.

Since X itself is Ω-generating, it follows that Ω-primitive sets exist. Moreover, note that every Ω-generating set S contains an Ω-minimal set. However, it is not true that every Ω-minimal set contains an Ω-primitive set (an example of this, related to the triangular frame of the type in Fig. 1 from the introduction, can be found in Fig. 13 in [4]).

Note that, if $\mathcal{F} = (X, I, \tau)$ is a frame such that there exist $i, j \in I$, $i \neq j$, with $\tau(i) = \tau(j)$, then $\mathcal{F}' = (X, I \setminus \{j\}, \tau)$ is a subframe of $\mathcal{F}$ such that $E_{\mathcal{F}} = E_{\mathcal{F}'}$. Consequently, $\Omega_{\mathcal{F}} = \Omega_{\mathcal{F}'}$. Keeping this in mind and combining it with Corollary 3.6 one obtains

Theorem 3.8. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. If $S \subseteq X$ is Ω-generating, then*

$$|S| \geq |X| - |I|\,.$$

Proof. Let S be Ω-generating, i.e., $\Omega(S) = X$. Due to the above remark, we may assume that $\tau : I \to \mathcal{P}(X)$ is injective. If not, consider the maximal subframe (with respect to the cardinality of I') $\mathcal{F}' = (X, I', \tau)$ such that $\tau : I' \to \mathcal{P}(X)$ is injective. The $\Omega_{\mathcal{F}}$-generating sets are the same as the $\Omega_{\mathcal{F}'}$-generating sets.

By Corollary 3.6 there exists an injective map $\xi : \Omega(S) \setminus S \to \mathbb{I}_{\mathcal{F}}(\Omega(S)) \setminus \mathbb{I}_{\mathcal{F}}(S)$. This shows that $|X \setminus S| = |X| - |S| \leq |I \setminus \mathbb{I}_{\mathcal{F}}(S)| \leq |I|$, and the assertion follows. □

Therefore one sees that $|X| - |I|$ is a lower bound for the cardinality of the Ω-generating sets of the frame (X, I, τ). The next theorem shows that this lower bound can be achieved for completely reducible frames.

Theorem 3.9. *Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame, then there exists an Ω-primitive set $S \subset X$ such that*

$$|S| = |X| - |I|\,.$$

Proof. The proof is by induction on $|I|$. If $|I| = 1$, then the assertion is obviously true.

Now assume that the assertion is true for all completely reducible frames (Y, J, σ) with $|J| \leq N$. If $\mathcal{F} = (X, I, \tau)$ is a completely reducible frame with $|I| = N + 1$, then there exists an $x \in X$ and a unique $j \in I$ such that $x \in \tau(j)$. Let $X' = \mathbb{X}_{\mathcal{F}}(I \setminus \{j\})$, then $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$ is a completely reducible frame (Theorem 2.5) with $|I \setminus \{j\}| = N$. Due to the induction hypothesis there exists an $\Omega_{\mathcal{F}'}$-primitive set $S' \subset X'$ such that $|S'| =$

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$\left|X'\right| - |I \setminus \{j\}| = \left|X'\right| - N$. Then the set $S = S' \cup \left(\tau(j) \setminus (\{x\} \cup (\tau(j) \cap X'))\right)$ is clearly $\Omega_{\mathcal{F}}$-generating and the cardinality of S is $|S| = \left|S'\right| + \left|\tau(j) \setminus (\{x\} \cup (\tau(j) \cap X'))\right|$. Using the fact that $\left|S'\right| = \left|X'\right| - N$ and that $\left|\tau(j) \setminus (\{x\} \cup (\tau(j) \cap X'))\right| = |X| - \left|X'\right| - 1$, this becomes

$$|S| = \left|X'\right| - N + \left|\tau(j) \setminus (\{x\} \cup (\tau(j) \cap X'))\right| = |X| - (N + 1).$$

This shows that S' is an $\Omega_{\mathcal{F}}$-primitive set. □

Combining the previous theorem with the fact that, by definition, all Ω-primitive set have the same (minimal) cardinality, we have

Corollary 3.10. *Every Ω-primitive set of a completely reducible frame $\mathcal{F} = (X, I, \tau)$ has cardinality* $\operatorname{ind}(\mathcal{F}) = |X| - |I|$.

The following lemma will be helpful for later considerations.

Lemma 3.11. *Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame and let $P \subset X$ be an $\Omega_{\mathcal{F}}$-primitive set. For $n \in \mathbb{N}$ define the subframes $\mathcal{F}_n = (X_n, I_n, \tau)$ where $I_n = \mathbb{I}_{\mathcal{F}}(E^n(P))$, $X_n = \mathbb{X}_{\mathcal{F}}(I_n)$, and $E = E_{\mathcal{F}}$ denotes the associated extension map. Then for every $n \in \mathbb{N}$ it holds that $|P| = |X_n \cup P| - |I_n|$.*

Proof. Let $P_n = X_n \cap P$, then P_n is $\Omega_{\mathcal{F}_n}$-primitive. Since $\mathcal{F}_n$ is a completely reducible frame (Theorem 2.5), it follows by Theorem 3.9 that $|P_n| = |X_n| - |I_n|$ for all $n \in \mathbb{N}$, implying $|X_n \setminus P| = |I_n|$. Since $|P \cup X_n| = |X_n| + |P \setminus P_n|$ and $|P_n| = |P| - |P \setminus P_n|$, it follows that $|P| = |P \cup X_n| - |I_n|$ for all $n \in \mathbb{N}$. □

For completely reducible frames there exists a second characterization of Ω-primitive sets. To this end, we introduce two new notions.

Definition 3.12. Let $\mathcal{F} = (X, I, \tau)$ be a frame and $E : \mathcal{P}(X) \to \mathcal{P}(X)$ the associated extension map. For $S \in \mathcal{P}(X)$ the function $f_S : X \to \mathbb{N} \cup \{\infty\}$ defined as

$$f_S(x) = \begin{cases} \min\{n \in \mathbb{N} \mid x \in E^n(S)\} & \text{if } x \in \Omega(S) \\ \infty & \text{otherwise} \end{cases}$$

is called the *evolution function* of S (with respect to $\mathcal{F}$).

This function gives, for any element of X, the minimal number of applications of the extension map which are necessary to cover that element, starting from a given set S. We also define

Definition 3.13. Let $\mathcal{F} = (X, I, \tau)$ be a frame. A map $f : X \to \mathbb{N}$ is called *$\mathcal{F}$-map* if it satisfies

(1) For every elementary frame $\tau(i)$ there exists a unique $x \in \tau(i)$ such that $f(x) > 0$ and $f(x) \geq f(y)$ for all $y \in \tau(i)$.
(2) For every $x \in X$ such that $f(x) > 0$ there exists a unique elementary frame $\tau(i)$ such that $x \in \tau(i)$ and $f(x) \geq f(y)$ for all $y \in \tau(i)$.

The first condition ensures that an $\mathcal{F}$-map is not constant if restricted to an elementary frame, while the second condition ensures that every $x \in X$ with $f(x) > 0$ is the local maximum in precisely one elementary frame.

Theorem 3.14. *A frame $\mathcal{F}$ has an $\mathcal{F}$-map, if and only if $\mathcal{F}$ is completely reducible.*

Proof. (a) $\mathcal{F}$ having an $\mathcal{F}$-map implies complete reducibility.

The proof is by induction on $|I|$. If $|I| = 1$, then $\mathcal{F} = (X, \{1\}, \tau)$ with $\tau(1) = X$ is the only frame that exists. Clearly, there exists an $\mathcal{F}$-map and $\mathcal{F}$ is completely reducible.

Now assume that every frame $\mathcal{G} = (Y, J, \sigma)$ with $|J| = N$ and with a $\mathcal{G}$-map is completely reducible. If $\mathcal{F} = (X, I, \tau)$ is a frame such that $|I| = N + 1$ and with an $\mathcal{F}$-map $f : X \to \mathbb{N}$, then there exists an $x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$. Since f is non-constant, $f(x) > 0$ and there exists a unique $j \in I$ such that $f(x) \geq f(y)$ for all $y \in \tau(j)$. Since x is a global maximum of f, there exists no $i \in I$, $i \neq j$, such that $x \in \tau(i)$. This shows that $\mathcal{F}$ is reducible. Let $X' = \mathbb{X}(I \setminus \{j\})$. Since f restricted to X' is an $\mathcal{F}'$-map for the frame

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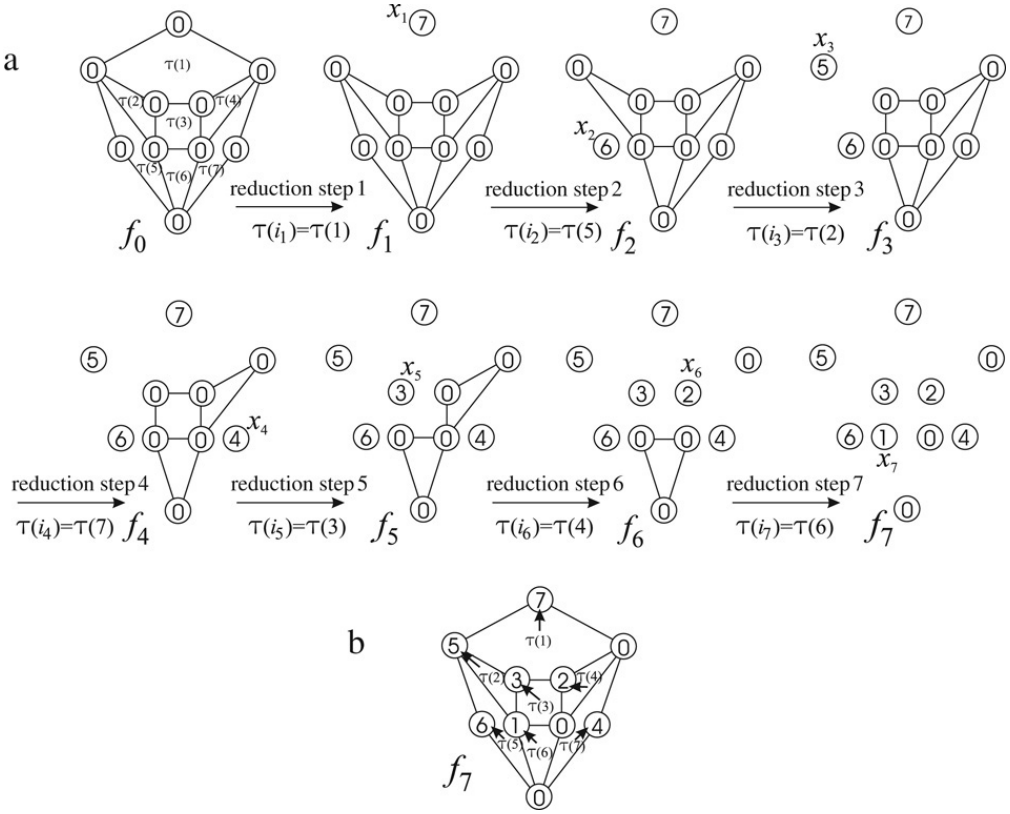


Fig. 17. (a) Construction of an $\mathcal{F}$-map for a completely reducible frame $\mathcal{F}$, as described in part (b) of the proof of Theorem 3.14, by successively removing elementary frames. The $\mathcal{F}$-map corresponds to the final map f_7. (b) The arrows indicate the local maximum of the $\mathcal{F}$-map for each elementary frame.

$\mathcal{F}' = (X', I \setminus \{j\}, \tau)$, it follows from the induction hypothesis that $\mathcal{F}'$ is completely reducible. This shows that $\mathcal{F}$ is completely reducible.

(b) If $\mathcal{F}$ is completely reducible, then it has an associated $\mathcal{F}$-map.

The proof is constructive, and is illustrated in Fig. 17. Let $\mathcal{F} = (X, I, \tau)$ be completely reducible. Then there is a sequence of nested subframes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{|I|} = (\emptyset, \emptyset,)$ such that removal of one elementary frame, say $\tau(i_j)$, leads from $\mathcal{F}_j$ to $\mathcal{F}_{j+1}$, for all $j = 0, 1, 2, \ldots, |I| - 1$. For every $\tau(i_j)$ we fix one cell $x_j \in \tau(i_j)$ which does not belong to another elementary frame of $\mathcal{F}_j$. Due to the complete reducibility this is possible.

Define the maps $f_j : X \to \mathbb{N}$, $j = 0, 1, 2, \ldots, |I|$, recursively as follows:

$$f_0(x) = 0 \quad \text{for all } x \in X$$

and for $j = 1, \ldots, |I|$ set

$$f_j(x) = \begin{cases} f_{j-1}(x) & \text{if } x \neq x_j \\ |I| - j + 1 & \text{if } x = x_j. \end{cases}$$

Then $f_{|I|}$ clearly satisfies the conditions for being an $\mathcal{F}$-map. □

Remark that the 0-labeled vertices in Fig. 17(b) constitute an Ω-primitive set for the underlying frame. This is no coincidence, as the next theorem shows.

Theorem 3.15. *Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame. $P \subset X$ is an $\Omega_{\mathcal{F}}$-primitive set if and only if there exists an $\mathcal{F}$-map $f : X \to \mathbb{N}$ such that*

$$P = f^{-1}(0) = \{x \in X \mid f(x) = 0\}.$$

Proof. The proof splits into two parts. In the first part, one shows that $f^{-1}(0)$ is an Ω-primitive set. In the second part, one shows that every Ω-primitive set defines an $\mathcal{F}$-map.

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The first part is proved by induction on $|I|$. If $|I| = 1$, then it is obvious that a $(X, \{1\}, \tau)$-map has $|\tau(1)| - 1$ zeros and these zeros form an Ω-primitive set of $(X, \{1\}, \tau)$.

Now assume that for every completely reducible frame $\mathcal{G} = (Y, J, \sigma)$ with $|J| \leq N$ and for every $\mathcal{G}$-map $g : X \to \mathbb{N}$ the set $g^{-1}(0)$ is $\Omega_{\mathcal{G}}$-primitive.

Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame with $|I| = N+1$ and let $f : X' \to \mathbb{N}$ be an $\mathcal{F}$-map. Obviously, there exists an $x \in X$ such that $f(x) \geq f(y)$ for all $y \in X$. Due to the properties of the $\mathcal{F}$-map f, there exists a unique $j \in I$ such that $x \in \tau(j)$. Then $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$, where $X' = \mathbb{X}_{\mathcal{F}}(I \setminus \{j\})$, is a completely reducible subframe of $\mathcal{F}$. Moreover, f restricted to the subframe $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{F}'$-map. Therefore, due to the induction hypothesis, it follows that $f^{-1}(0) \cap X'$ is an $\Omega_{\mathcal{F}'}$-primitive set. Since $f : X \to \mathbb{N}$ is an $\mathcal{F}$-map, it follows that $f(y) = 0$ whenever $y \in \tau(j) \setminus (\{x\} \cup X')$. This shows that the set $f^{-1}(0)$ is given as $\left(f^{-1}(0) \cap X'\right) \cup \{y \mid y \in \tau(j) \setminus (\{x\} \cup X')\}$, and it is $\Omega_{\mathcal{F}}$-generating. Moreover, one computes

$$\begin{aligned}\left|f^{-1}(0)\right| &= \left|f^{-1}(0) \cap X'\right| + \left|f^{-1}(0) \cap \left(\tau(j) \setminus X'\right)\right| \\ &= \left|X'\right| - |I \setminus \{j\}| + (\left|\tau(j) \setminus X'\right| - 1).\end{aligned}$$

This implies that $\left|f^{-1}(0)\right| = |X| - |I|$. Using the fact that $f^{-1}(0) \cap X'$ is $\Omega_{\mathcal{F}}$-primitive and Corollary 3.10, it follows that $f^{-1}(0)$ is $\Omega_{\mathcal{F}}$-primitive. This proves the first assertion.

For the second part of the proof consider a frame $\mathcal{F} = (X, I, \tau)$ and a subset P of X that is $\Omega_{\mathcal{F}}$-primitive together with its evolution function $f_P : X \to \mathbb{N}$. To complete the proof it suffices to show that f_P is an $\mathcal{F}$-map. Using the notation of Lemma 3.11 one has $|P| = |X_n \cup P| - |I_n|$ for all $n \in \mathbb{N}$. Taking into account that $|X_n \setminus P| = |I_n|$ (see proof Lemma 3.11), this formula can be rewritten for $n+1$, namely $|P| = |X_n \cup P| + |X_{n+1} \setminus X_n| - (|I_n| + |I_{n+1} \setminus I_n|)$ and one obtains $|P| = |X_n \cup P| - |I_n| + (|X_{n+1} \setminus X_n| - |I_{n+1} \setminus I_n|)$. Since $|X_n \cup P| - |I_n| = |P|$ it follows that $0 = |X_{n+1} \setminus X_n| - |I_{n+1} \setminus I_n|$ for all $n \in \mathbb{N}$. In other words, every cell $x \in X$ which is generated in the $(n+1)$th step, i.e., $f_P(x) = n+1$ requires an additional unique $i \in I_{n+1} \setminus I_n$ such that $x \in \tau(i)$. This establishes the existence of a bijective map $g : X \setminus P \to I$ such that $x \in \tau(g(x))$ for all $x \in X \setminus P$. Using the map g one concludes that f_P is an $\mathcal{F}$-map, as follows:

(1) Let $i \in I$, then there exists a minimal $n > 0$ such that $\tau(i) \subset E^n(P)$ and $x = g^{-1}(\tau(i))$ is such that $x \in \tau(i)$, $f_P(x) = n > 0$ and $f_P(x) \geq f_P(y)$ for all $y \in \tau(i)$.
(2) Let $x \in X$ and $f_P(x) = n > 0$, i.e., $x \notin P$, then $\tau(g(x))$ satisfies $\tau(g(x)) \subset E^n(P)$, $x \in \tau(g(x))$ and $f(x) \geq f(y)$ for all $y \in \tau(g(x))$.

This shows that f_P is an $\mathcal{F}$-map. □

As a consequence of the second part in the above proof we note

Corollary 3.16. *Let $\mathcal{F}$ be a completely reducible frame. If $P \subseteq X$ is $\Omega_{\mathcal{F}}$-primitive, then the evolution map $f_P : X \to \mathbb{N}$ is an $\mathcal{F}$-map.*

As a consequence of the already mentioned Theorems 3.8 and 3.9, $|X| - |T|$ provides a lower bound for the cardinality of a primitive set for every frame. In analogy to Theorem 3.9 we present a sharp lower bound for the cardinality of a primitive set of an arbitrary frame.

Theorem 3.17. *Let $\mathcal{F} = (X, I, \tau)$ be a frame and $\delta(\mathcal{F})$ its defect. If $S \subseteq X$ is $\Omega_{\mathcal{F}}$-generating then*

$$|S| \geq |X| - |I| + \delta(\mathcal{F}).$$

Proof. The proof is by induction on the value of $|I| + \delta(\mathcal{F})$. If $|I| + \delta(\mathcal{F}) = 1$, then $(X, \{1\}, \tau)$ with $\tau(1) = X$ is the only frame with $|I| + \delta(\mathcal{F}) = 1$. For this frame the assertion is true.

We now assume that the assertion is true for all frames $\mathcal{G} = (Y, J, \sigma)$ with $|J| + \delta(\mathcal{G}) \leq N$.

Let $\mathcal{F} = (X, I, \tau)$ be a frame such that $|I| + \delta(\mathcal{F}) = N+1$ and suppose there exists an $S \subset X$ such that S is $\Omega_{\mathcal{F}}$-generating and $|S| < |X| - |I| + \delta(\mathcal{F})$. It will be shown that this leads to a contradiction. With $f_S : X \to \mathbb{N}$ we denote the evolution function, see Definition 3.12, of the set S. The elementary inequality $\delta(\mathcal{F}) \leq |I|$ in connection with the assumption on S implies that $|S| < |X|$. In other words, there exists an $x \in X$ such that $f_S(x) > 0$ and

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$f_S(x) \geq f_S(y)$ for all $y \in X$. Moreover, there exists a $j \in I$ such that $x \in \tau(j)$ and $f_S(x) > f_S(y)$ for all $y \in \tau(j)$ with $x \neq y$.

We now distinguish two cases:

(1) $x \notin \tau(i)$ for all $i \in I, i \neq j$, i.e., x is contained in only one elementary frame of $\mathcal{F}$.
(2) x is contained in at least two elementary frames of $\mathcal{F}$.

These cases are treated as follows.

(1) Let $\tau(j)$ be the only frame containing x. The frame $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$, where $X' = \mathbb{X}_{\mathcal{F}}(I \setminus \{j\})$, is a reduction of $\mathcal{F}$, see Definition 2.4. Due to 2 of Corollary 2.15, it follows that $\delta(\mathcal{F}') = \delta(\mathcal{F})$. Together with the above mentioned assumption $\delta(\mathcal{F}) = N - |I| + 1$, the reduction then satisfies

$$\delta(\mathcal{F}') + |I \setminus \{j\}| = N.$$

Since S is $\Omega_{\mathcal{F}}$-generating and due to the choice of x, it follows that $S' = S \cap X'$ is $\Omega_{\mathcal{F}'}$-generating. Then the induction hypothesis implies that

$$\left|S'\right| \geq \left|X'\right| - |I \setminus \{j\}| + \delta(\mathcal{F}').$$

Now we have $\left|S'\right| = |S| - \left|\tau(j) \setminus X'\right| + 1$ and $\left|X'\right| = |X| - \left|\tau(j) \setminus X'\right|$. Using these equalities the above inequality transforms into $|S| \geq |X| - |I| + \delta(\mathcal{F})$. This is a contradiction to the assumption on S.

(2) If $x \in \tau(j) \cap \tau(i), i \neq j$, then we consider the frame $\mathcal{F}^* = (X \cup \{j\}, I, \tau^*)$, where τ^* is defined as

$$\tau^*(k) = \begin{cases} \tau(k) & \text{if } k \neq j \\ (\tau(j) \setminus \{x\}) \cap \{j\} & \text{if } k = j. \end{cases}$$

If $\lambda : X \cup \{j\} \to X$ is defined as

$$\lambda(y) = \begin{cases} y & \text{if } y \neq j \\ x & \text{if } y = j, \end{cases}$$

then $\psi = (\lambda, \mathrm{id}) : \mathcal{F}^* \to \mathcal{F}$ is an epimorphism. By the construction of $\psi : \mathcal{F}^* \to \mathcal{F}$, one has either $\delta(\mathcal{F}^*) = \delta(\mathcal{F})$ or $\delta(\mathcal{F}^*) = \delta(\mathcal{F}) - 1$.

Since $j \in \tau^*(k)$ if and only if $k = j$, it follows that $\mathcal{F}^*$ is reducible. Let $\mathcal{F}' = (X', I \setminus \{j\}, \tau^*)$, where $X' = \mathbb{X}_{\mathcal{F}^*}(I \setminus \{j\})$, be a reduction of $\mathcal{F}^*$. Then $S' = S \cap X'$ is a $\Omega_{\mathcal{F}'}$-generating set. Since $\delta(\mathcal{F}') = \delta(\mathcal{F}^*) \leq \delta(\mathcal{F})$, it follows that $\delta(\mathcal{F}') + |I \setminus \{j\}| \leq N$. Due to the induction hypothesis, if follows that $\left|S'\right| \geq \left|X'\right| - |I| + 1 + \delta(\mathcal{F}^*)$. Using the facts that $\left|S'\right| = |S| - \left|\tau^* \setminus X'\right| + 1$ and $\left|X'\right| = |X| + 1 - \left|\tau^* \setminus X'\right|$, the last inequality becomes $|S| \geq |X| - |I| + 1 + \delta(\mathcal{F}')$. On the other hand, we supposed that $|S| < |X| - |I| + \delta(\mathcal{F})$. This transforms the previous inequality into $\delta(\mathcal{F}) > 1 + \delta(\mathcal{F}')$. Since we have seen that $\delta(\mathcal{F}) - \delta(\mathcal{F}') \in \{0, 1\}$, this is a contradiction.

This shows that the assumption $|S| < |X| - |I| + \delta(\mathcal{F})$ leads to a contradiction, and completes the proof. □

In analogy to Theorem 3.9 we also note

Theorem 3.18. *If $\mathcal{F} = (X, I, \tau)$ is a frame, then there exists an $\Omega_{\mathcal{F}}$-primitive set $S \subseteq X$ such that*

$$|S| = |X| - |I| + \delta(\mathcal{F}).$$

Proof. It is known from Section 2 that $\mathcal{F} = (X, I, \tau)$ has a maximal completely reducible subframe $\mathcal{F}' = (X, I', \tau)$, $I' \subset I$, for which $\left|I \setminus I'\right| = |I| - \left|I'\right| = \delta(\mathcal{F})$. Due to Theorem 3.9, there exists an $\Omega_{\mathcal{F}'}$-primitive set S of cardinality $|X| - \left|I'\right|$. Since $\left|I'\right| = |I| - \delta(\mathcal{F})$ it follows that $|S| = |X| - |I| + \delta(\mathcal{F})$. Obviously S is also $\Omega_{\mathcal{F}}$-generating. Since $|S|$ is equal to the lower bound in Theorem 3.17, it follows that S is $\Omega_{\mathcal{F}}$-primitive. □

Due to its importance for later considerations we denote the cardinality of an Ω-primitive set of the frame $\mathcal{F}$ as the *primitive index*

$$\mu(\mathcal{F}) = |X| - |I| + \delta(\mathcal{F}) = \mathrm{ind}(\mathcal{F}) + \delta(\mathcal{F}). \tag{13}$$

We list some elementary properties of the primitive index μ.

Lemma 3.19. *Let $\mathcal{F} = (X, I, \tau)$ be a frame and let $j \in I$. The following two assertions are true*

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(1) *If* $\mathcal{F}' = (X, I \setminus \{j\}, \tau)$ *is a subframe of* $\mathcal{F}$*, then* $\mu(\mathcal{F}) \leq \mu(\mathcal{F}') \leq \mu(\mathcal{F}) + 1$.
(2) *If* $\mathcal{F}' = (X', I \setminus \{j\}, \tau)$ *is a subframe of* $\mathcal{F}$ *such that* $|X| - |X'| = \kappa > 0$*, then* $\mu(\mathcal{F}') = \mu(\mathcal{F}) + 1 - \kappa$.

Proof. (1) The assertion follows from the fact that $\delta(\mathcal{F}) - 1 \leq \delta(\mathcal{F}') \leq \delta(\mathcal{F})$.

(2). As $\kappa > 0$, $\mathcal{F}$ is reducible, and by 2. of Corollary 2.15, one has that $\delta(\mathcal{F}) = \delta(\mathcal{F}')$ and the assertion follows immediately. □

Corollary 3.20. *Let* $\mathcal{F} = (X, I, \tau)$ *be a frame and let* I' *be a subset of* I*. Then one has*
(1) $\mu\left(\mathbb{X}_{\mathcal{F}}(I'), I', \tau\right) \geq \mu\left(\mathbb{X}_{\mathcal{F}}(I'), \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I')), \tau\right)$
(2) $\mu\left(\mathbb{X}_{\mathcal{F}}(I'), \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I')), \tau\right) \geq \mu\left(E(\mathbb{X}_{\mathcal{F}}(I')), \mathbb{I}_{\mathcal{F}}(E(\mathbb{X}_{\mathcal{F}}(I'))), \tau\right)$*, where* E *is the extension map of* $\mathcal{F}$.

Proof. (1) follows from the fact that $I' \subseteq \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I'))$ and from repeated applications of Lemma 3.19.(1).

(2) is obvious if $E(\mathbb{X}_{\mathcal{F}}(I')) = \mathbb{X}_{\mathcal{F}}(I')$. If this is not the case, then for every $x \in E(\mathbb{X}_{\mathcal{F}}(I')) \setminus \mathbb{X}_{\mathcal{F}}(I')$ there exists a $j = j(x) \in I$ such that $\left|\tau(j) \cap \mathbb{X}_{\mathcal{F}}(I')\right| = |\tau(j)| - 1$. By repeated application of Lemma 3.19.(2)., it then follows that

$$\mu\left(\mathbb{X}_{\mathcal{F}}(I'), \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I')), \tau\right) = \mu\left(E(\mathbb{X}_{\mathcal{F}}(I')), \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I')) \cup \bigcup_{x \in E(\mathbb{X}_{\mathcal{F}}(I')) \setminus \mathbb{X}_{\mathcal{F}}(I')} j(x), \tau\right).$$

An application of (1) gives the desired estimate. □

4. Frames and matroids

In [4], it was shown that a particular matroid is associated with the frame displayed in Fig. 1. The bases of this matroid are possible generating sets for certain local rules on this frame. This section explores the existence of a similar matroid for general frames.

For convenience, we recall one of the many possible definitions of a matroid. A matroid is a pair $\mathfrak{M} = (X, \mathcal{I})$, where X is a (finite) set and $\mathcal{I}$ is a subset of $\mathcal{P}(X)$ such that the following conditions hold

(1) $\emptyset \in \mathcal{I}$.
(2) If $S \in \mathcal{I}$ and $S' \subseteq S$, then $S' \in \mathcal{I}$.
(3) If $S_1, S_2 \in \mathcal{I}$ such that $|S_2| = |S_1| + 1$, then there exist an $x \in S_2 \setminus S_1$ such that $S_1 \cup \{x\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are usually referred to as *independent sets* of the matroid $\mathfrak{M}$. Due to the third condition all maximal elements (with respect to inclusion) of $\mathcal{I}$ have the same cardinality. Any maximal element is called a *base* of the matroid. If B_1 and B_2 are bases of $\mathfrak{M}$, then they satisfy the exchange property, i.e.,

For every $x \in B_1 \setminus B_2$ there exists an $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a base of $\mathfrak{M}$.

On the other hand, if $\mathcal{B}$ is a collection of subsets of X such that all its elements have the same cardinality and the exchange property is satisfied, then $\mathcal{B}$ defines a matroid. For more details on matroids, see, e.g., [10,11].

As already noted in the previous section, the limit map Ω associated with a frame $\mathcal{F} = (X, I, \tau)$ is a closure operator. Since (special) closure operators are linked to matroids, it is natural to ask whether the closure operator Ω is related to a matroid. It is a well known fact, [10,11], that if Ω satisfies the exchange property, i.e., if

$$x, y \notin \Omega(S) \text{ then } x \in \Omega(S \cup \{y\}) \text{ implies } y \in \Omega(S \cup \{x\}),$$

then Ω is the closure operator of a matroid. Furthermore, in this case, the Ω-primitive subsets $P \subset X$ are bases of the matroid associated with Ω.

Unfortunately, in many cases Ω does not satisfy the exchange property. However, we will show that for certain frames it is possible to define a matroid $\mathcal{M}_{\mathcal{F}}$ even though Ω does not satisfy the exchange property.

To this end, we begin with a definition of $\mathcal{F}$-independent sets. These $\mathcal{F}$-independent sets serve as candidates for independent sets of a matroid related to the frame $\mathcal{F}$. We then characterize $\mathcal{F}$-independent sets in terms of the fixed points of Ω. At the end of this section we present two classes of frames for which the $\mathcal{F}$-independent sets form the collection of independent sets of a matroid.

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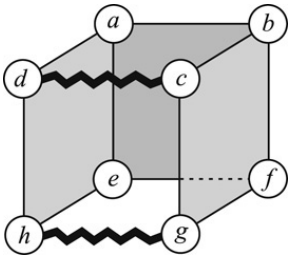


Fig. 18. A frame consisting of 5 elementary frames represented by the gray squares $\tau(1) = \{a, d, e, h\}$, $\tau(2) = \{a, b, e, f\}$, $\tau(3) = \{b, c, f, g\}$ and the zigzag-lines $\tau(4) = \{c, d\}$, $\tau(5) = \{g, h\}$.

Definition 4.1. Let $\mathcal{F} = (X, I, \tau)$ be a frame. A subset $S \subseteq X$ is called *$\mathcal{F}$-independent* if the following holds: If $S' \subseteq X$ is such that $\Omega(S) \subseteq \Omega(S')$, then $|S| \leq |S'|$. The collection of $\mathcal{F}$-independent sets is denoted as $\mathcal{I}_{\mathcal{F}}$.

Or, in other words: knowledge of S provides us with knowledge of $\Omega_{\mathcal{F}}(S)$, and S is $\mathcal{F}$-independent if one cannot obtain the same or more information, i.e., $\Omega_{\mathcal{F}}(S') \supseteq \Omega_{\mathcal{F}}(S)$, with less knowledge, i.e., $|S'| < |S|$.

Since $\Omega(\emptyset) \subseteq \Omega(S)$ for all $S \subseteq X$, it follows that $\emptyset$ is always an element of $\mathcal{I}_{\mathcal{F}}$. Furthermore, due to the definition of an Ω-primitive set, one has that Ω-primitivity of P implies that $P \in \mathcal{I}_{\mathcal{F}}$ and P is maximal (with respect to inclusion) in $\mathcal{I}_{\mathcal{F}}$.

Lemma 4.2. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. If $S \subseteq X$ is $\mathcal{F}$-independent, then $S' \subseteq S$ is also $\mathcal{F}$-independent.*

Proof. Suppose that S' is not $\mathcal{F}$-independent, then there exists an $S^* \subseteq X$ such that $\Omega(S') \subseteq \Omega(S^*)$ and $|S^*| < |S'|$. Since $\Omega(S) = \Omega(S' \cup (S \setminus S')) = \Omega(\Omega(S') \cup \Omega(S \setminus S'))$ and since $\Omega(S') \subseteq \Omega(S^*)$ it follows that $\Omega(S) \subseteq \Omega(\Omega(S^*) \cup \Omega(S \setminus S')) = \Omega(S^* \cup (S \setminus S'))$. Since $|S^* \cup (S \setminus S')| < |S|$ this is a contradiction to the fact that S is $\mathcal{F}$-independent. □

The collection of fixed points of $E_{\mathcal{F}}$ is denoted as

$$\mathcal{C}_{\mathcal{F}} = \{S \mid S \subseteq X \text{ and } S = E_{\mathcal{F}}(S)\}.$$

Another characterization of the elements of $\mathcal{C}_{\mathcal{F}}$ is the following: $S \in \mathcal{C}_{\mathcal{F}}$ if and only if there exists an $S' \subseteq X$ such that $S = \Omega(S')$.

Furthermore, it is well known, see e.g. [5], that $\mathcal{C}_{\mathcal{F}}$ together with the binary operations $C_1 \cap C_2$ (the meet) and $C_1 \vee C_2 = \Omega_{\mathcal{F}}(C_1 \cup C_2)$ (the joint) forms a complete lattice of subsets, denoted as $\mathcal{L}_{\mathcal{F}} = (\mathcal{C}_{\mathcal{F}}, \cap, \vee)$.

Let $\mathcal{F} = (X, I, \tau)$ be a frame and let μ be the primitive index defined in Eq. (13). If $C \in \mathcal{C}_{\mathcal{F}}$ is a fixed point of $\Omega_{\mathcal{F}}$, then it can be written as the disjoint union

$$C = C_0 \cup C_1,$$

where $C_0 = \mathbb{X}_{\mathcal{F}}(\mathbb{I}_{\mathcal{F}}(C))$, i.e., the cells of C which are contained in an elementary frame $\tau(i)$ such that $\tau(i) \subset C$, and the set $C_1 = C \setminus C_0$. Since $(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau)$ is a frame one can compute $\mu(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau)$. The *modified index*, denoted $\overline{\mu}$, is then defined as

$$\overline{\mu}(C) = |C_1| + \mu(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau), \tag{14}$$

which is the same as $\overline{\mu}(C) = |C| - |\mathbb{I}_{\mathcal{F}}(C)| + \delta(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau)$. Note that if $\mathbb{I}_{\mathcal{F}}(C) = 0$, then $\overline{\mu}(C) = |C|$. Moreover, note that $\overline{\mu}(C) = \min\{|S| \,|\, S \subseteq X, \Omega_{\mathcal{F}}(S) = C\}$.

Also note that $\overline{\mu}(X) = \mu(\mathcal{F})$ for every frame, and for a regular frame $\mathcal{F}$ one has $\emptyset \in \mathcal{C}_{\mathcal{F}}$ and $\overline{\mu}(\emptyset) = 0$.

Example 4.3. Consider the frame represented in Fig. 18. $C = \{a, b, g, h\}$ is a fixed point of the extension map. $\mathbb{I}_{\mathcal{F}}(C) = \{5\}$, $C_0 = \mathbb{X}_{\mathcal{F}}(\mathbb{I}_{\mathcal{F}}(C)) = \{g, h\}$, $\mu(C_0, \mathbb{I}_{\mathcal{F}}, \tau) = |C_0| - |\mathbb{I}_{\mathcal{F}}(C)| + \delta(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau) = 2 - 1 + 0 = 1$, $C_1 = C \setminus C_0 = \{a, b\}$, and it follows that $\overline{\mu}(C) = 2 + 1 = 3$.

Using the modified index $\overline{\mu}$ it is possible to give a characterization of $\mathcal{F}$-independent sets.

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Lemma 4.4. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. $S \subseteq X$ is $\mathcal{F}$-independent if and only if*

$$|S \cap C| \leq \overline{\mu}(C)$$

for all $C \in \mathcal{C}_{\mathcal{F}}$.

Proof. Let $S \in \mathcal{I}_{\mathcal{F}}$ be an $\mathcal{F}$-independent set. If $C \in \mathcal{C}_{\mathcal{F}}$, then $(S \cap C) \in \mathcal{I}_{\mathcal{F}}$, by Lemma 4.2. Due to the definition of $\overline{\mu}$, there exists a subset $P \in \mathcal{P}(X)$ such that $C = \Omega(P)$ and $|P| = \overline{\mu}(C)$. One therefore has $\Omega(S \cap C) \subseteq \Omega(C) = C = \Omega(P)$. The $\mathcal{F}$-independence of $S \cap C$ implies that $|S \cap C| \leq |P| = \overline{\mu}(C)$.

Now let $S \subseteq X$ be such that $|S \cap C| \leq \overline{\mu}(C)$ for all $C \in \mathcal{C}_{\mathcal{F}}$. Suppose S is not $\mathcal{F}$-independent. Then there exists an $S' \subseteq X$ such that $\Omega(S) \subseteq \Omega(S')$ and $\left|S'\right| < |S|$. Since $\Omega(S') \in \mathcal{C}_{\mathcal{F}}$ and due to the assumption on S one also has $|S| = \left|S \cap \Omega(S')\right| \leq \overline{\mu}(\Omega(S'))$. Since $\overline{\mu}(\Omega(S')) \leq \left|S'\right|$, one obtains $|S| \leq \left|S'\right|$. This is a contradiction to $\left|S'\right| < |S|$. In other words, S is $\mathcal{F}$-independent. □

The characterization of $\mathcal{F}$-independent sets in Lemma 4.4, namely,

$$\mathcal{I}_{\mathcal{F}} = \{S \mid S \subseteq X \text{ and } |S \cap C| \leq \overline{\mu}(C) \text{ for all } C \in \mathcal{C}_{\mathcal{F}}\},$$

is reminiscent of the construction of a matroid using a submodular function on a lattice of subsets, see, e.g., Proposition 12.1.11 in [10], which we present as

Theorem 4.5. *Let $\mathcal{F} = (X, I, \tau)$ be a frame and let $\mathcal{L}_{\mathcal{F}} = (\mathcal{C}_{\mathcal{F}}, \cap, \vee)$ be the complete lattice of fixed points of $E_{\mathcal{F}}$. If $\overline{\mu} : \mathcal{C}_{\mathcal{F}} \to \mathbb{N}$ is a submodular function, i.e., if*

$$\overline{\mu}(C_1 \cap C_2) + \overline{\mu}(C_1 \vee C_2) \leq \overline{\mu}(C_1) + \overline{\mu}(C_2)$$

for all $C_1, C_2 \in \mathcal{C}_{\mathcal{F}}$, then $\mathfrak{M}_{\mathcal{F}} = (X, \mathcal{I}_{\mathcal{F}})$ is a matroid.

Thus, for certain 'nice' frames, i.e., frames for which $\overline{\mu}$ is submodular, there exists a natural way to define a matroid related to the frame. Moreover, the $\mathcal{F}$-independent sets form the collection of independent sets of this matroid. In particular, the rank of the matroid $\mathfrak{M}_{\mathcal{F}}$ is equal to $\mu(\mathcal{F}) = \overline{\mu}(X)$ and the Ω-primitive sets are bases of the matroid.

Example 4.6. Fig. 19 presents a simplified lattice-representation of the fixed points of $\Omega_{\mathcal{F}}$ for the frame $\mathcal{F}$ that was used in Fig. 2(b). The $\overline{\mu}$-values associated with the different fixed points are indicated, and it can easily be checked that $\overline{\mu}$ is submodular. Lemma 4.4 allows us to identify the $\mathcal{F}$- independent sets as all singletons, all tuples, and all triples except those that form one of the triangular faces. Since $\overline{\mu}$ is submodular, Theorem 4.5 implies that these are also the independent sets of a matroid.

In order to determine the $\mathcal{F}$-independent sets of a frame $\mathcal{F}$ using Theorem 4.5 one has to compute first all fixed points of the limit map Ω. This task may be laborious. The next lemma shows that it is sufficient to work with the subframes of $\mathcal{F}$.

Theorem 4.7. *Let $\mathcal{F} = (X, I, \tau)$ be a frame. The set $S \subset X$ is $\mathcal{F}$-independent if and only if $\left|S \cap X'\right| \leq \mu(\mathcal{F}')$ holds for all subframes $\mathcal{F}' = (X', I', \tau)$ of $\mathcal{F}$.*

Proof. Suppose first that $\left|S \cap X'\right| \leq \mu(\mathcal{F}')$ holds for all subframes $\mathcal{F}'$ of $\mathcal{F}$. Let $C \in \mathcal{C}_{\mathcal{F}}$ be a fixed point of Ω, then we write $C = C_0 \cup C_1$, where $C_0 = \mathbb{X}_{\mathcal{F}}(\mathbb{I}_{\mathcal{F}}(C))$ and $C_1 = C \setminus C_0$. Then one has

$$|S \cap C| = |S \cap C_1| + |S \cap C_0|\,.$$

Since $(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau)$ is a subframe of $\mathcal{F}$ and since $|S \cap C_1| \leq |C_1|$, we have

$$|S \cap C| = |S \cap C_1| + |S \cap C_0| \leq |C_1| + \mu(C_0, \mathbb{I}_{\mathcal{F}}(C), \tau).$$

The right hand side of the above equation is the definition of $\overline{\mu}$, see Eq. (14). The whole equation then corresponds to the condition for $\mathcal{F}$-independency of S as given in Lemma 4.4.

Now assume that $S \subseteq X$ is independent, i.e., $|S \cap C| \leq \overline{\mu}(C)$ for all $C \in \mathcal{C}_{\mathcal{F}}$. For a given $I' \subseteq I$ we have to show that

$$\left|S \cap \mathbb{X}_{\mathcal{F}}(I')\right| \leq \mu(\mathbb{X}_{\mathcal{F}}(I'), I', \tau). \tag{15}$$

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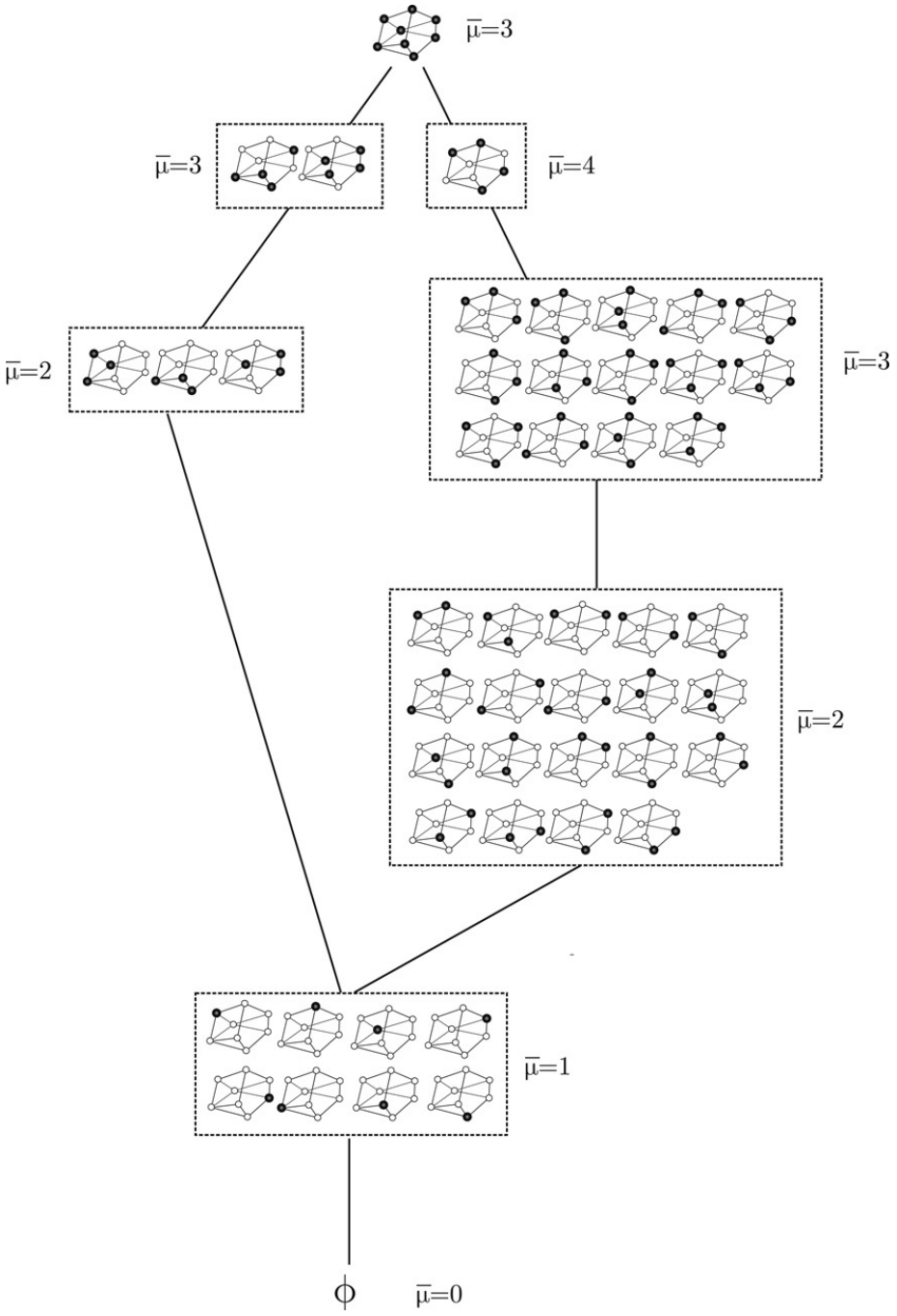


Fig. 19. The fixed points of $E_{\mathcal{F}}$ (sets formed by the gray cells) corresponding to the frame on top, and the corresponding subset-lattice. The lattice is represented in a simplified way: A connecting line between two collections of fixed points (enclosed in rectangles) indicates that there is a subset relationship between a fixed point in the lower collection and a fixed point in the higher collection. The values of $\overline{\mu}$ are indicated. Clearly, $\overline{\mu}$ satisfies the submodularity condition.

By 1 of Corollary 3.20 one has $\mu(\mathbb{X}_{\mathcal{F}}(I'), \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I')), \tau) \leq \mu(\mathbb{X}_{\mathcal{F}}(I'), I', \tau)$, and it is therefore sufficient to prove inequality (15) for those I' for which $I' = \mathbb{I}_{\mathcal{F}}(\mathbb{X}_{\mathcal{F}}(I'))$, which we do from now on.

If $\mathbb{X}_{\mathcal{F}}(I')$ is a fixed point of E, then one has

$$\left|S \cap \mathbb{X}_{\mathcal{F}}(I')\right| \leq \overline{\mu}(\mathbb{X}_{\mathcal{F}}(I')) = \mu(\mathbb{X}_{\mathcal{F}}(I'), I', \tau). \tag{16}$$

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If $\mathbb{X}_{\mathcal{F}}(I')$ is not a fixed point of E, then 2 of Corollary 3.20 implies that $\mu\left(E^{n+1}(\mathbb{X}_{\mathcal{F}}(I')), \mathbb{I}_{\mathcal{F}}(E^{n+1}(\mathbb{X}_{\mathcal{F}}(I'))), \tau\right) \leq \mu\left(E^{n}(\mathbb{X}_{\mathcal{F}}(I')), \mathbb{I}_{\mathcal{F}}(E^{n}(\mathbb{X}_{\mathcal{F}}(I'))), \tau\right)$ for all $n \in \mathbb{N}$. In particular,

$$\overline{\mu}(\Omega(\mathbb{X}_{\mathcal{F}}(I'))) = \mu\left(\Omega(\mathbb{X}_{\mathcal{F}}(I')), \mathbb{I}_{\mathcal{F}}(\Omega(\mathbb{X}_{\mathcal{F}}(I'))), \tau\right) \leq \mu\left(\mathbb{X}_{\mathcal{F}}(I'), I', \tau\right). \tag{17}$$

Since $\left|S \cap \mathbb{X}_{\mathcal{F}}(I')\right| \leq \left|S \cap \Omega(\mathbb{X}_{\mathcal{F}}(I'))\right|$, the Inequalities (16) and (17) imply that

$$\left|S \cap \mathbb{X}_{\mathcal{F}}(I')\right| \leq \left|S \cap \Omega(\mathbb{X}_{\mathcal{F}}(I'))\right| \leq \overline{\mu}(\Omega(\mathbb{X}_{\mathcal{F}}(I'))) \leq \mu(\mathbb{X}_{\mathcal{F}}(I'), I', \tau).$$

This proves the theorem. □

We now present two classes of frames for which $\overline{\mu}$ is submodular, and hence the $\mathcal{F}$-independent sets form a matroid. The first result concerns the class of completely reducible frames and generalizes the results concerning the frames in Fig. 1, see [4].

Theorem 4.8. *If $\mathcal{F} = (X, I, \tau)$ is a completely reducible frame, then $(X, \mathcal{I}_{\mathcal{F}})$ forms a matroid.*

Proof. According to Theorem 4.5, it is sufficient to show that the map $\overline{\mu}_{\mathcal{F}} : \mathcal{C}_{\mathcal{F}} \to \mathbb{N}$ is submodular on the lattice $\mathcal{L}_{\mathcal{F}} = (\mathcal{C}_{\mathcal{F}}, \cap, \vee)$.

By definition Eq. (14), one has

$$\overline{\mu}(C) = |C| - |\mathbb{I}_{\mathcal{F}}(C)| + \delta\left(\mathbb{X}_{\mathcal{F}}(\mathbb{I}_{\mathcal{F}}(C)), \mathbb{I}_{\mathcal{F}}(C), \tau\right)$$

for $C \in \mathcal{C}_{\mathcal{F}}$. Since $\mathcal{F}$ is completely reducible, one has $\delta(\mathcal{F}') = 0$ for every subframe $\mathcal{F}'$ of $\mathcal{F}$. This reduces $\overline{\mu}$ to $\overline{\mu}(C) = |C| - |\mathbb{I}_{\mathcal{F}}(C)|$ for $C \in \mathcal{C}$. It remains to show that

$$\overline{\mu}(C_1 \cap C_2) + \overline{\mu}(C_1 \vee C_2) \leq \overline{\mu}(C_1) + \overline{\mu}(C_2) \tag{18}$$

for all $C_1, C_2 \in \mathcal{C}$ and with $C_1 \vee C_2 = \Omega(C_1 \cup C_2)$. By definition, and because of complete reducibility, we have $\overline{\mu}(C_1 \cap C_2) = |C_1 \cap C_2| - |\mathbb{I}_{\mathcal{F}}(C_1 \cap C_2)|$ and $\overline{\mu}(C_1 \vee C_2) = |C_1 \cup C_2| + |(C_1 \vee C_2) \setminus (C_1 \cup C_2)| - |\mathbb{I}_{\mathcal{F}}(C_1 \vee C_2)|$.

Using these equalities and the fact that $|C_1 \cap C_2| + |C_1 \cup C_2| = |C_1| + |C_2|$, Inequality (18) transforms into

$$|(C_1 \vee C_2) \setminus (C_1 \cup C_2)| \leq -|\mathbb{I}_{\mathcal{F}}(C_1)| - |\mathbb{I}_{\mathcal{F}}(C_2)| + |\mathbb{I}_{\mathcal{F}}(C_1 \vee C_2)| + |\mathbb{I}_{\mathcal{F}}(C_1 \cap C_2)|.$$

The expression on the right hand side of the inequality is equal to the number of elementary frames contained in $C_1 \vee C_2$ but not in $C_1 \cup C_2$, while the left hand side is the number of 'new' cells obtained by an application of $\Omega_{\mathcal{F}}$ to $C_1 \cup C_2$. Thus Corollary 3.6 applies, i.e., the right hand side is indeed larger than or equal to the left hand side. □

The above theorem shows that for a completely reducible frame $\mathcal{F}$ the pair $\mathfrak{M}_{\mathcal{F}} = (X, \mathcal{I}_{\mathcal{F}})$ is a matroid $\mathcal{M}_{\mathcal{F}}$. The following theorem provides a closer description of this matroid: A subset B of X is a base if and only if there is a bijection between its complement $X \setminus B$ and the set of elementary frames.

Theorem 4.9. *Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame and $\mathcal{M}_{\mathcal{F}}$ the associated matroid. A subset $B \subseteq X$ is a base of $\mathcal{M}_{\mathcal{F}}$ if and only if there exists a bijective map $g_B : X \setminus B \to I$ such that $x \in \tau(g_B(x))$ for all $x \in X \setminus B$.*

Proof. Let $B \subseteq X$ such that there exists a bijective map $g_B : X \setminus B \to I$. For a given subset I' of I one has $\left|(X \setminus B) \cap \bigcup_{j \in I'} \tau(j)\right| \geq \{g_B^{-1}(j) \mid j \in I'\} = |I'|$. Using $\left|\bigcup_{j \in I'} \tau(j)\right| = \left|X \cap \bigcup_{j \in I'} \tau(j)\right| = \left|B \cap \bigcup_{j \in I'} \tau(j)\right| + \left|(X \setminus B) \cap \bigcup_{j \in I'} \tau(j)\right|$, this gives

$$\left|B \cap \bigcup_{j \in I'} \tau(j)\right| \leq \left|\bigcup_{j \in I'} \tau(j)\right| - |I'|.$$

By Theorem 4.7, this means that B is an independent set of $\mathcal{M}_{\mathcal{F}}$. Since the bijectivity of g_B implies that $|B| = |X| - |I|$, which equals the rank of $\mathcal{F}$, it follows that B is a base of $\mathcal{M}_{\mathcal{F}}$.

Now let B be a base in $\mathcal{M}_{\mathcal{F}}$. Since $\mathcal{F}$ is a regular frame and since B is a base it follows that $|\tau(j) \setminus B| \geq 1$ for all $j \in I$. In order to prove the assertion it suffices to establish the existence of a bijective map $g_B : X \setminus B \to \{\tau(j) \setminus B \mid j \in I\}$ such that $x \in g_B(x)$ for all $x \in X \setminus B$. In other words, the collection of subsets $(\tau(j) \setminus B : j \in I)$ possesses a transversal, e.g. [11], p.97. By Hall's marriage Theorem, [11], p.98, the existence of a transversal is

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guaranteed if for every subset I' of I the inequality $\left|\bigcup_{j\in I'}(\tau(j)\setminus B)\right| \geq |I'|$ is true. This is indeed the case since B being a base, one has, using Theorem 4.7, that

$$\left|\bigcup_{j\in I'}(\tau(j)\setminus B)\right| = \left|\left(\bigcup_{j\in I'}\tau(j)\right)\setminus B\right| = \left|\bigcup_{j\in I'}\tau(j)\right| - \left|\left(B\cap\bigcup_{j\in I'}\tau(j)\right)\right| \geq |I'|.$$

This shows that there exists a map g_B as required. □

A second class of frames for which a matroid exists consists of graphs, as shown by the next theorem. A graph does indeed induce a frame: the cells of the frame are the vertices, and the elementary frames are the couples of vertices connected by an edge. Furthermore, an arbitrary indexing of the elementary frames provides a proper index set.

Theorem 4.10. *Let $\mathcal{F} = (X, I, \tau)$ be a frame such that each elementary frame contains exactly two cells, then $(X, \mathcal{I}_\mathcal{F})$ forms a matroid.*

Proof. By Theorem 4.5, it is sufficient to demonstrate that the map $\overline{\mu} : \mathcal{C}_\mathcal{F} \to \mathbb{N}$ is submodular.

Since $|\tau(i)| = 2$ for all $i \in I$, the frame $\mathcal{F}$ can be considered as a graph. The vertices correspond to X and the edges correspond to I. Then it is clear that the connected components of this graph are fixed points of Ω. This shows that $C_1 \vee C_2 = \Omega(C_1 \cup C_2) = C_1 \cup C_2$ and $\overline{\mu}(C) =$ number of connected components of C. Therefore we have $\overline{\mu}(C_1 \cup C_2) + \overline{\mu}(C_1 \cap C_2) = \overline{\mu}(C_1) + \overline{\mu}(C_2)$ for all $C_1, C_2 \in \mathcal{C}_\mathcal{F}$. This proves the assertion. □

Remark 4.11. For graph-induced frames described by the conditions of Theorem 4.10, a subset $S \subset X$ is $\mathcal{F}$-independent if and only if any two distinct cells in S belong to different connected components of the graph.

We conclude this section with an example of a frame $\mathcal{F}$ such that $(X, \mathcal{I}_\mathcal{F})$ does not form a matroid.

Example 4.12. Let $\mathcal{F} = (\{a, b, c, d, e, f\}, I, \tau)$ be the frame depicted in the upper left corner of Fig. 21. By applying Theorem 4.7, one easily sees that $S_1 = \{a, d\}$ and $S_2 = \{b, d, f\}$ are maximal (with respect to inclusion) $\mathcal{F}$-independent sets. As the cardinalities of these maximal sets are different, it follows that $(X, \mathcal{I}_\mathcal{F})$ does not form a matroid.

One also notes that $\overline{\mu}$ is not submodular (cf. Theorem 4.5) since $\overline{\mu}(\{a, b, c, d\} \vee \{a, d, e, f\}) + \overline{\mu}(\{a, b, c, d\} \cap \{a, d, e, f\}) = 3 + 2$ which is greater than $\overline{\mu}(\{a, b, c, d\}) + \overline{\mu}(\{a, d, e, f\}) = 2 + 2$. However, by coincidence, the collection of all $\mathcal{F}$-independent sets of cardinality 3 forms the collection of bases of a matroid. This collection is given in Fig. 21(C), and one easily checks that this collection satisfies the exchange property.

5. Local rules, matching configurations, generating sets

As mentioned in the introduction, we are interested in attributing values to the cells in a frame in such a way that they match certain prescribed local rules in each elementary frame. An assignment of values which satisfies all local rules was called a matching configuration in the introduction. In the present, more formal context, it will be called $(\mathcal{F}, \mathcal{R})$-configuration, where $\mathcal{F}$ is a frame and $\mathcal{R}$ is a collection of local rules.

An $(\mathcal{F}, \mathcal{R})$-configuration is a map $\underline{c} : X \to G$, where G is a finite nonempty set such that $\underline{c}_{|\tau(i)}$ is of a specified type (the local rule).

Definition 5.1. Let $\mathcal{F} = (X, I, \tau)$ be a frame. A *rule* (with values in G) for $\mathcal{F}$ is a collection $\mathcal{R} = (R_i)_{i\in I}$ of subsets $R_i \subseteq G^{\tau(i)}$, $i \in I$ such that the following holds for all $i \in I$: For every $x \in \tau(i)$ and for every $f \in G^{\tau(i)\setminus\{x\}}$ there exists a unique $c \in R_i$ such that $c_{|\tau(i)\setminus\{x\}} = f$.

The sets R_i are referred to as *local rules*, as being local to an elementary frame. Each local rule R_i specifies the possible assignments of values in G to the elements of the elementary frame $\tau(i)$. The condition on the set R_i expressed in the above definition can also be stated as follows: If one knows $|\tau(i)| - 1$ values of a map $c : \tau(i) \to G$, then the map c is uniquely determined.

As a consequence of the definition we also note that the number of possible configurations on an elementary frame $\tau(i)$ is given as $|R_i| = |G|^{|\tau(i)|-1}$.

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If $|\tau(i)| = 2$, then the local rule R_i is given by a bijective map from G to G. If $|\tau(i)| = n \geq 3$ then a local rule R_i can be considered as a $(n-1)$-ary quasigroup (see e.g. [8], p. 426) i.e., there exists an $(n-1)$-ary operation represented by the map $F : G^{n-1} \to G$ such that for all $i \in \{1, \ldots, n-1\}$ and for all $x_j \in G$, $j \in \{0, \ldots, n-1\} \setminus \{i\}$, there exists a unique $x \in G$ such that $F(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{n-1}) = x_0$. In particular, for $|\tau(i)| = 3$ the local rule is given by an ordinary, i.e. binary, quasigroup, see e.g., [6].

We now formalize the concept of matching-configurations from the introduction by defining $(\mathcal{F}, \mathcal{R})$ matching configurations ($(\mathcal{F}, \mathcal{R})$-configurations for short).

Definition 5.2. Let $\mathcal{F} = (X, I, \tau)$ be a frame and let $\mathcal{R} = (R_i)_{i \in I}$ be a rule (with values in G) for $\mathcal{F}$. A map $\underline{c} : X \to G$ is called an $(\mathcal{F}, \mathcal{R})$ *(matching) configuration* if $\underline{c}_{|\tau(i)} \in R_i$ for all $i \in I$. The set of all $(\mathcal{F}, \mathcal{R})$-configurations is denoted $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$.

Given a frame $\mathcal{F}$ and a rule $\mathcal{R}$, then it is by no means clear whether $(\mathcal{F}, \mathcal{R})$-configurations exist. Thus, when we speak about $(\mathcal{F}, \mathcal{R})$-configurations we tacitly assume that they exist.

If $\mathcal{F}' = (X', I', \tau)$ is a subframe of $\mathcal{F} = (X, I, \tau)$ and if $\mathcal{R} = (R_i)_{i \in I}$ is a rule for $\mathcal{F}$, then $\mathcal{R}' = (R_i)_{i \in I'}$ is a rule for $\mathcal{F}'$. Moreover, if $\underline{c} \in \mathfrak{C}_{\mathcal{R}}(\mathcal{F})$, then $\underline{c}_{|X'} \in \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$. The converse is not true.

The already mentioned question on the existence of matching configurations has a complete answer for completely reducible frames.

Theorem 5.3. *Let $\mathcal{F} = (X, I, \tau)$ be a completely reducible frame. If $\mathcal{R} = (R_i)_{i \in I}$ is a rule (with values in G) for $\mathcal{F}$, then the number of $(\mathcal{F}, \mathcal{R})$ matching configurations is given by*

$$|\mathfrak{C}_{\mathcal{R}}(\mathcal{F})| = |G|^{\mu(\mathcal{F})}.$$

(Recall that, $\mathcal{F}$ being completely reducible, it holds that $\mu(\mathcal{F}) = |X| - |I|$).

Proof. By Theorem 3.9 there exists an $\Omega_{\mathcal{F}}$-primitive set $P \subseteq X$ such that $|P| = \mu(\mathcal{F})$. If $f_P : X \to \mathbb{N}$ denotes the evolution function (see Definition 3.12) of P, then by Corollary 3.16, f_P is an $\mathcal{F}$-map (Definition 3.13).

Since f_P is an $\mathcal{F}$-map, there exists for every $x \in X$ with $0 < f_P(x) \leq \max\{f_P(x) \mid x \in X\}$ a unique $\tau(i(x))$ such that $x \in \tau(i(x))$ and $f_p(y) < f_p(x)$ for all $y \in \tau(i(x))$, $y \neq x$.

Using this property, we extend any map $\zeta : P \to G$ to an $(\mathcal{F}, \mathcal{R})$-configuration. This is done recursively. For $n = 0$ we consider the map

$$\begin{aligned} c_0 : Y_0 &\to G \\ x &\mapsto \zeta(x), \end{aligned}$$

where $Y_0 = P = \{x \mid f_P(x) \leq 0\}$. If $c_n : Y_n \to G$, where $Y_n = \{x \mid f_P(x) \leq n\}$ is given, then $c_{n+1} : Y_{n+1} \to G$ is defined as

$$c_{n+1}(x) = \begin{cases} c_n(x) & \text{if } f_P(x) \leq n \\ r_{i(x)}(x) & \text{if } f_P(x) = n+1, \end{cases}$$

where $r_{i(x)}$ is the unique element of $R_{i(x)}$ which is determined by the values of c_n on $\tau(i(x)) \setminus \{x\}$. There exists an n_0 such that $c_{n_0} = c_{n_0+1}$ and this c_{n_0} is, by construction, an $(\mathcal{F}, \mathcal{R})$-configuration. It is obvious that every $(\mathcal{F}, \mathcal{R})$-configuration is obtained in this way. Thus there exists a bijection between G^P and $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$. This proves the assertion. □

Theorem 5.3 thus answers the "First Question" from Section 1.2.2 for completely reducible frames. It also induces an upper bound on the cardinality of $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$ for any frame, as follows.

Lemma 5.4. *If $\mathcal{F} = (X, I, \tau)$ is a frame with rule $\mathcal{R} = (R_i)_{i \in I}$, then*

$$|\mathfrak{C}_{\mathcal{R}}(\mathcal{F})| \leq |G|^{\mu(\mathcal{F})}.$$

Proof. Let $\mathcal{G} = (Y, I, \sigma)$ be a minimal lift with epimorphism $\Psi = (\lambda, \mathrm{id})$ as in Corollary 2.14. Then $\mathcal{R}$ can be lifted to a rule $\overline{\mathcal{R}} = (\overline{R}_i)_{i \in I}$ for $\mathcal{G}$ with values in G. To this end, one sets

$$\overline{R}_i = \{h : \sigma(i) \to G \mid h = f \circ \lambda,\ f \in R_i\}. \tag{19}$$

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If $\underline{c} \in \mathfrak{C}_{\mathcal{R}}(\mathcal{F})$, then the map $\underline{c}^+ : Y \to G$ defined as $\underline{c}^+(y) := \underline{c}(\lambda(y))$ is clearly an element of $\mathfrak{C}_{\mathcal{R}}(\mathcal{G})$. By Theorem 5.3 there are at most $|G|^{\mu(\mathcal{G})}$ different $(\mathcal{G}, \mathcal{R})$-configurations. Since every $(\mathcal{F}, \mathcal{R})$-configuration gives rise to a $(\mathcal{G}, \mathcal{R})$-configuration and since $\mu(\mathcal{G}) = \mu(\mathcal{F})$, the assertion follows. □

Remark 5.5. The above lemma also provides a way to obtain all $(\mathcal{F}, \mathcal{R})$-configurations with values in G. Let $\mathcal{G} = (X \cup U, I, \sigma)$ be a minimal lift of the frame $\mathcal{F}$, as in Corollary 2.14. If $P \subset X \cup U$ is $\Omega_{\mathcal{G}}$-primitive, then every assignment $\zeta : P \to G$ extends to a unique $(\mathcal{G}, \overline{\mathcal{R}})$-configuration $\underline{c} : X \cup U \to G$, where $\overline{\mathcal{R}}$ is defined as in Eq. (19). This configuration induces an $(\mathcal{F}, \mathcal{R})$-configuration if and only if $\lambda(x) = \lambda(y)$ implies $\underline{c}(x) = \underline{c}(y)$. In other words, the $(\mathcal{F}, \mathcal{R})$-configurations are those $(\mathcal{G}, \overline{\mathcal{R}})$-configurations which satisfy the above condition. This was already illustrated in the introduction by the relationship between the $(\mathcal{F}, \mathcal{R})$-configurations in Fig. 8 and Fig. 7.

Another method to compute $(\mathcal{F}, \mathcal{R})$-configurations is provided by the next lemma whose proof is obvious.

Lemma 5.6. *Let $\mathcal{F} = (X, I, \tau)$ be a frame with rule $\mathcal{R} = (R_i)_{i\in I}$ and values in G. If $I = I_1 \cup I_2$, with $I_1 \cap I_2 = \emptyset$ and if $\mathcal{F}_j = (X_j, I_j, \tau)$, where $X_j = \mathbb{X}_{\mathcal{F}}(I_j)$, and $\mathcal{R}_j = (R_i)_{i\in I_j}$, $j = 1, 2$, are frames and corresponding rules, respectively, then $\mathfrak{C}_{\mathcal{R}}(\mathcal{F}) = \{\underline{c} : X \to G \mid \underline{c}_{|X_j} = \underline{c}_j \in \mathfrak{C}_{\mathcal{R}_j}(\mathcal{F}_j)$ for $j = 1, 2$, and $\underline{c}_1(x) = \underline{c}_2(x)$ for all $x \in X_1 \cap X_2\}$.*

Remark 5.7. As already mentioned, the above lemma provides another method to find all $(\mathcal{F}, \mathcal{R})$-configurations. If $\mathcal{F} = (X, I, \tau)$ is a frame, then there exists a maximal completely reducible subframe $\mathcal{F}_1 = (X, I \setminus U, \tau)$, with $|U| = \delta(\mathcal{F})$. For an $\Omega_{\mathcal{F}_1}$-primitive set one easily computes the $(\mathcal{F}_1, \mathcal{R}_1)$-configurations, where $\mathcal{R}_1 = (R_i)_{i\in I\setminus U}$. An application of the above lemma for $I_1 = I \setminus U$ and $I_2 = U$ and keeping in mind that $X_2 \subset X_1 = X$, the $(\mathcal{F}, \mathcal{R})$-configurations are given as $\mathfrak{C}_{\mathcal{R}}(\mathcal{F}) = \{\underline{c} \in \mathfrak{C}_{\mathcal{R}_1}(\mathcal{F}_1) \mid \underline{c}_{|X_2} \in \mathfrak{C}_{\mathcal{R}_2}(\mathcal{F}_2)\}$.

Since, for completely reducible frames, there exists a bijection from $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$ with G^P, it is therefore natural to introduce the following

Definition 5.8. Let $\mathcal{F} = (X, I, \tau)$ be a frame and let $\mathcal{R} = (R_i)_{i\in I}$ a rule with values in G. A subset S of X is called a *generating set* of the $(\mathcal{F}, \mathcal{R})$-configurations if the map $\pi_S : \mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \to G^S$, $\underline{c} \mapsto \underline{c}_{|S}$ is bijective.

Lemma 5.9. *Let $\mathcal{F}$ be a completely reducible frame. Then its $\Omega_{\mathcal{F}}$-primitive sets are also generating sets.*

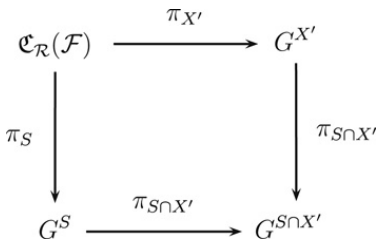
Proof. Let P be an $\Omega_{\mathcal{F}}$-primitive set. It was established in the proof of Theorem 5.3 that there exists a bijection between G^P and $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$. Thus P satisfies Definition 5.8 of a generating set. □

The relationship between generating sets and $\mathcal{F}$-independent sets is stated in

Theorem 5.10. *Let $\mathcal{F} = (X, I, \tau)$ be a frame with rule $\mathcal{R} = (R_i)_{i\in I}$. If $S \subseteq X$ is a generating set of the $(\mathcal{F}, \mathcal{R})$-configurations, then S is $\mathcal{F}$-independent.*

Proof. For a given $S \subset X$ we consider the restriction maps $\pi_S : G^X \to G^S$ defined as $\pi_s(f) = f_{|S}$.

We show that S is $\mathcal{F}$-independent with the help of Theorem 4.7. This says that S is $\mathcal{F}$-independent if and only if $\left|S \cap X'\right| \leq \mu(\mathcal{F}')$ for every subframe $\mathcal{F}' = (X', I', \tau)$ of $\mathcal{F}$. Let $\mathcal{F}' = (X', I', \tau)$ be a subframe. Consider the following commutative diagram



Since $\pi_{X'}(\mathfrak{C}_{\mathcal{R}}(\mathcal{F})) \subseteq \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$, Lemma 5.4 implies $|\pi_{X'}(\mathfrak{C}_{\mathcal{R}}(\mathcal{F}))| \leq |G|^{\mu(\mathcal{F}')}$. Since $\pi_{S\cap X'} \circ \pi_S$ is surjective, the commutativity of the diagram implies that $\pi_{S\cap X'} \circ \pi_{X'}$ is surjective, too. We therefore conclude that $|\pi_{X'}(\mathfrak{C}_{\mathcal{R}}(\mathcal{F}))| \geq |G|^{|S\cap X'|}$. Combining both inequalities one obtains $\left|S \cap X'\right| \leq \mu(\mathcal{F}')$. □

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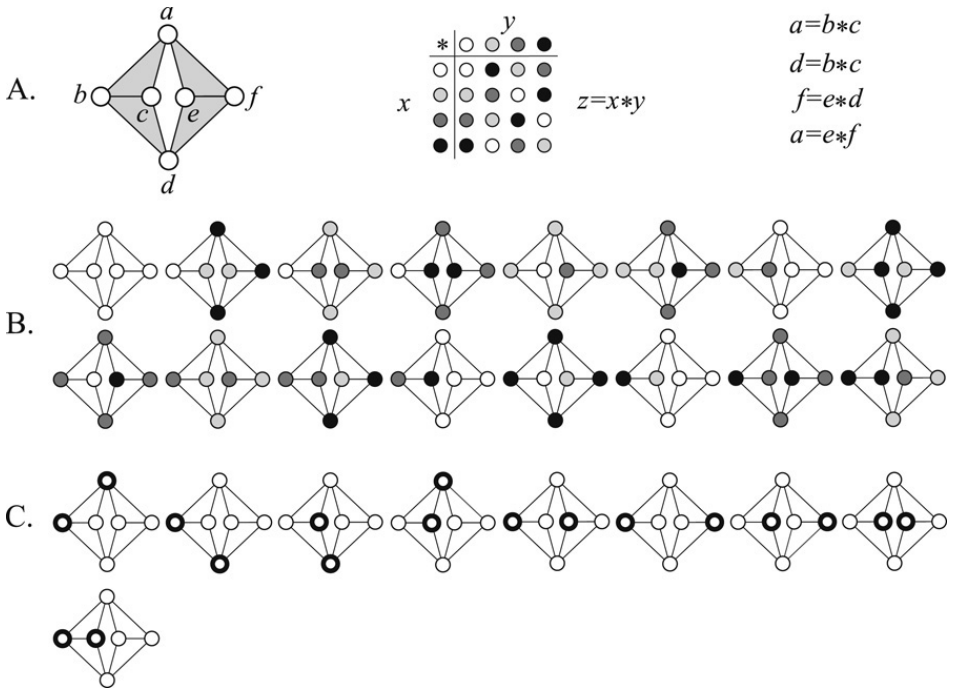


Fig. 20. A. An irreducible frame $\mathcal{F}$ and rule $\mathcal{R}$; B. All $(\mathcal{F}, \mathcal{R})$-configurations; C. All corresponding generating sets (indicated by the fat cells): they are $\mathcal{F}$-independent, but they are not bases of a matroid.

As a generalization of a result in [4], we note

Theorem 5.11. *Let $\mathcal{F}$ be a completely reducible frame. If $S \subseteq X$ is a generating set for the rule $\mathcal{R}$, then S is a base of the matroid $\mathfrak{M}_{\mathcal{F}} = (X, \mathcal{I}_{\mathcal{F}})$.*

Proof. According to Theorem 4.8, $(X, \mathcal{I}_{\mathcal{F}})$ is a matroid. If P is a $\Omega_{\mathcal{F}}$-primitive set, then P is a base of $\mathfrak{M}_{\mathcal{F}}$ and P is a generating set for $\mathcal{R}$. This implies that every generating set S has the same cardinality as P. Since $S \in \mathcal{I}_{\mathcal{F}}$ and $|S| = |P|$, it follows that S is a base of $\mathfrak{M}_{\mathcal{F}}$. □

Theorem 5.10 gives an answer to the "Second Question" in Section 1.2.2 concerning necessary conditions for a set to be generating: it must certainly be $\mathcal{F}$-independent. Via Theorem 4.7, this condition puts a limit on the number of cells in the intersection between a generating set and the subframes of $\mathcal{F}$. For completely reducible frames, Theorem 5.11 states that generating sets are bases of a matroid, and a condition on how the elements of a base can be distributed over the frame is now provided by Theorem 4.9.

Remark 5.12. In general, it is not true that any base in the matroid $\mathfrak{M}_{\mathcal{F}}$ of a completely reducible frame is a generating set for a given rule $\mathcal{R}$ (recall the examples in Figs. 6 and 9). However, like in [4], we conjecture that for every base there is a set of (nonlinear) quasigroups inducing a rule for which this base is generating.

The existence of generating sets depends on the frame as well on the rule. This was already illustrated by Fig. 8 in the introduction. Here follow some additional examples.

Example 5.13. (1) Consider the frame, the quasigroup, and the (nonlinear) local rules as shown in the top line of Fig. 20. Using the remark following Lemma 5.4 one finds that there exist 16 $(\mathcal{F}, \mathcal{R})$-configurations, which are shown in Fig. 20(B). All generating sets are displayed in part (C) of the figure.

(2) Consider the frame, the quasigroup, and the local rules as shown in the top line of Fig. 21. Using the remark following Lemma 5.4 one finds that there exist 32 $(\mathcal{F}, \mathcal{R})$-configurations, which are shown in Fig. 21(B). Knowledge of a configuration restricted to one of the displayed sets in Fig. 21(C) uniquely determines the whole configuration. However, in contrast to a generating set the values on these sets cannot take arbitrary values from the quasigroup.

The second example motivates the next definition.

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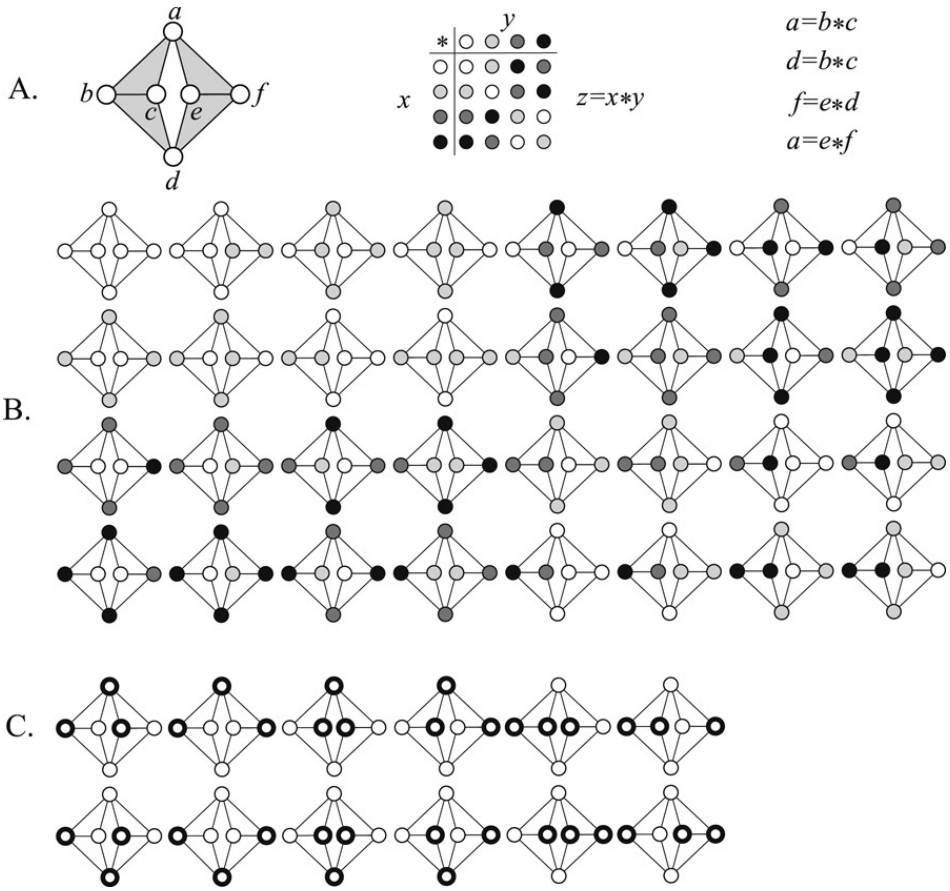


Fig. 21. A. An irreducible frame $\mathcal{F}$ and rule $\mathcal{R}$; B. All $(\mathcal{F}, \mathcal{R})$-configurations ; C. All corresponding minimal identifying sets (indicated by the fat cells): they are $\mathcal{F}$-independent and form the bases of a matroid. These sets happen also to be the primitive sets for this frame and for the maximal completely reducible subframe obtained by deleting the elementary frame $\{a, b, c\}$ (cf. Lemma 5.15).

Definition 5.14. Let $\mathcal{F} = (X, I, \tau)$ be a frame with rules $\mathcal{R}$ having values in G. A subset S of X is called an *identifying set* of the $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$-configurations if the restriction map $\pi_S : \mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \to G^S$ is injective.

Obviously, X is an identifying set. However, we are interested in minimal (with respect to cardinality) identifying sets. In case generating sets exist, they are also minimal identifying sets, as bijectivity (in the definition of generating sets) includes injectivity (in the definition of identifying sets). However, the qualification "identifying" will mainly be used when there are no generating sets (or when the possibility of this is not excluded). The sets shown in Fig. 21(C) present all minimal identifying sets for the frame and the rule considered in this figure. Note that all identifying sets are $\mathcal{F}$-independent sets. That this is not incidental will be established in what follows. In order to do so we begin with a lemma that guarantees the existence of certain $\Omega_{\mathcal{F}}$-primitive sets that are also identifying.

Lemma 5.15. *Let $\mathcal{F} = (X, I, \tau)$ be a frame and $\mathcal{R} = (R_i)_{i\in I}$ a rule. There exists an $\Omega_{\mathcal{F}}$-primitive set P such that P is identifying (for the $(\mathcal{F}, \mathcal{R})$-configurations).*

Proof. Let $\mathcal{F}' = (X, I \setminus U, \tau)$ (where $|U| = \delta_{\mathcal{F}}$, cf. Remark 5.7) be a maximal completely reducible subframe with rule $\mathcal{R}' = (R_i)_{i\in I\setminus U}$. Since $\mu(\mathcal{F}) = \mu(\mathcal{F}')$, it follows that every $\Omega_{\mathcal{F}'}$-primitive set is also $\Omega_{\mathcal{F}}$-primitive. Let P be $\Omega_{\mathcal{F}'}$-primitive. According to Lemma 5.9, since $\mathcal{F}'$ is completely reducible, P is generating for the $(\mathcal{F}', \mathcal{R}')$-configurations. Since $\mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \subseteq \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$, it follows that $\pi_P : \mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \to \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$ is injective. This shows that P is identifying for $\mathfrak{C}_{\mathcal{R}}(\mathcal{F})$-configurations. □

We are now prepared to prove

Theorem 5.16. *Let $\mathcal{F} = (X, I, \tau)$ be a frame and $\mathcal{R} = (R_i)_{i\in I}$ be a rule with values in G. If S is a minimal (with respect to cardinality) identifying set for $(\mathcal{F}, \mathcal{R})$-configurations, then S is $\mathcal{F}$-independent.*

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Proof. Assume that S is minimal identifying and not $\mathcal{F}$-independent, then, by Theorem 4.7, there exists a subframe $\mathcal{F}' = (X', I', \tau)$ of $\mathcal{F}$ such that $\left|S \cap X'\right| > \mu(\mathcal{F}')$.

With $\mathcal{R}' = (R_i)_{i\in I'}$ we denote the rule for $\mathcal{F}'$. Since $\mathcal{F}'$ is a subframe of $\mathcal{F}$ and due to the definition of $\mathcal{R}'$ one has $\mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \subseteq \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$, i.e., $\pi_{X'}(\mathfrak{C}_{\mathcal{R}}(\mathcal{F})) \subseteq \mathfrak{C}_{\mathcal{R}'}(\mathcal{F}')$. Lemma 5.15 applied to the frame $\mathcal{F}'$ with rule $\mathcal{R}'$ guarantees the existence of an $\Omega_{\mathcal{F}'}$-primitive set P' that is identifying for the $(\mathcal{R}', \mathcal{F}')$-configurations.

Let $S^* = (S \setminus X') \cup P$. Since $\left|S \cap X'\right| > \mu(\mathcal{F}') = \left|P'\right|$, it follows that $|S^*| < |S|$. We shall prove that $\pi_{S^*} : \mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \to G^{S^*}$ is injective.

If $\underline{c}_1, \underline{c}_2 \in \mathfrak{C}_{\mathcal{R}}(\mathcal{F})$ are different $(\mathcal{F}, \mathcal{R})$-configurations, then there exists $x \in S$, such that $\pi_S(\underline{c}_1)(x) \neq \pi_S(\underline{c}_2)(x)$, because S is identifying. If $x \in (S \setminus X')$, then $\pi_{S^*}(\underline{c}_1)(x) \neq \pi_{S^*}(\underline{c}_2)(x)$. Now suppose that $\underline{c}_1(s) = \underline{c}_2(s)$ for all $s \in S \setminus X'$, then there exists $x \in S \cap X'$ such that $\underline{c}_1(x) \neq \underline{c}_2(x)$. This means that $\pi_{X'}(\underline{c}_1)$ and $\pi_{X'}(\underline{c}_2)$ are different $(\mathcal{F}', \mathcal{R}')$-configurations and therefore, $\pi_{P'}(\pi_{X'}(\underline{c}_1)) \neq \pi_{P'}(\pi_{X'}(\underline{c}_2))$, since P' is identifying for $(\mathcal{F}', \mathcal{R}')$-configurations. This shows that $\pi_{S^*} : \mathfrak{C}_{\mathcal{R}}(\mathcal{F}) \to G^{S^*}$ is injective. In other words S^* is identifying for $(\mathcal{F}, \mathcal{R})$-configurations. Since $|S^*| < |S|$, it follows that S is not minimal. This is a contradiction. □

We conclude with some remarks on linear rules. To this end we assume that G is a finite field, denoted as $\mathbb{F}$. If $\mathcal{F} = (X, I, \tau)$ is a frame, then a linear rule $\mathcal{R} = (R_i)_{i\in I}$ is given as

$$R_i = \left\{ \zeta \in \mathbb{F}^{\tau(i)} \mid \sum_{x\in\tau(i)} a_{i,x}\zeta(x) = 0 \right\},$$

where $a_{i,x} \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$. In other words, the rule $\mathcal{R}$ can be considered as a $|I| \times |X|$-matrix $A = (a_{i,x})_{i\in I, x\in X}$ with entries in $\mathbb{F}$. It has the property that $a_{i,x} \neq 0$ if and only if $x \in \tau(i)$. Any matrix satisfying this condition will be called an $\mathcal{F}$-matrix. Obviously, every $\mathcal{F}$-matrix A defines a rule $\mathcal{R} = \mathcal{R}(A)$ for $\mathcal{F}$, and one has

$$\mathfrak{C}_{\mathcal{R}}(\mathcal{F}) = \ker(A) = \left\{ \zeta \in \mathbb{F}^X \mid A\zeta = 0 \right\}.$$

As usual $\mathrm{Im}(A)$ denotes the image of the matrix A.

Theorem 5.17. *If $\mathcal{F} = (X, I, \tau)$ is a frame and $A \in \mathbb{F}^{|I|\times|X|}$ an $\mathcal{F}$-matrix, then*

$$\dim Im(A) \geq |X| - \mu(\mathcal{F}) = |I| - \delta(\mathcal{F}).$$

Proof. Suppose that $\dim \mathrm{Im}(A) = k$. Then there exists a subset $B \subseteq X$ such that $|B| = k$ and the matrix $A_B = (a_{i,x})_{i\in I, x\in B}$ has rank k. It follows that for every $\zeta_{X\setminus B} \in \mathbb{F}^{X\setminus B}$ there exists a unique $\zeta_B \in \mathbb{F}^B$ such that

$$(A_B \; A_{X\setminus B}) \begin{pmatrix} \zeta_B \\ \zeta_{X\setminus B} \end{pmatrix} = 0.$$

From this it follows that the set $X \setminus B$ is a generating set. Due to Theorem 5.10, it follows that $|X \setminus B| \leq \mu(\mathcal{F})$. This gives $|B| \geq |X| - \mu(\mathcal{F})$. Applying (13) provides the inequality in the proposition. □

Remark 5.18. If $A \in \mathbb{F}^{n\times m}$ is a matrix, then the non-zero entries define a frame $\mathcal{F}(A)$ and $m - \mu(\mathcal{F}(A))$ is a lower bound for the dimension of $\mathrm{Im}(A)$.

Example 5.19. Consider the frame in the upper left of Fig. 21. An $\mathcal{F}$-matrix A has the following structure

$$\begin{pmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ * & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix},$$

where $* \in \mathbb{F}^*$ and the first column corresponds to a, the second to b and so forth. If $\mathbb{F} = \mathbb{F}_2$, the field with two elements, then every $*$ represents a 1. Since the rank of this matrix is 3, it follows that all sets shown in Fig. 21(C) are generating sets and form the bases of a matroid $\mathbf{M}_1$. By coincidence all these $\mathcal{F}$-independent sets are also $\Omega_{\mathcal{F}}$-primitive.

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Now consider the same frame again but with the $\mathcal{F}$-matrix with following values

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

where the entries are considered as elements of $\mathbb{F}_3$, the field with three elements. Then one computes that the rank of the matrix is equal to 4. Therefore, the generating sets have cardinality 2 and are formed by the complement of any 4-column selection with rank 4. It occurs that every pair of columns corresponds to a generating set, meaning that any set of two cells of the frame is generating. Also these form the bases of a matroid $\mathbf{M}_2$.

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