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Brief paper

Constrained linear MPC with time-varying terminal cost using convex combinations☆

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Abstract

Recent papers (IEEE Transactions on Automatic Control 48(6) (2003) 1092–1096, Automatica 38 (2002) 1061–1068, Systems and Control Letters 48 (2003) 375–383) have introduced dual-mode MPC algorithms using a time-varying terminal cost and/or constraint. The advantage of these methods is the enlargement of the admissible set of initial states without sacrificing local optimality of the controller, but this comes at the cost of a higher computational complexity. This paper delivers two main contributions in this area. First, a new MPC algorithm with a time-varying terminal cost and constraint is introduced. The algorithm uses convex combinations of off-line computed ellipsoidal terminal constraint sets and uses the associated cost as a terminal cost. In this way, a significant on-line computational advantage is obtained. The second main contribution is the introduction of a general stability theorem, proving stability of both the new MPC algorithm and several existing MPC schemes (IEEE Transactions on Automatic Control 48(6) (2003) 1092–1096, Automatica 38 (2002) 1061–1068). This allows a theoretical comparison to be made between the different algorithms. The new algorithm using convex combinations is illustrated and compared with other methods on the example of an inverted pendulum.
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1. Introduction

Stability of model-predictive control (MPC) has been intensively studied in the last decade, resulting in the identification of three ingredients to impose stability: a locally stabilizing terminal feedback controller, a terminal state constraint, and a terminal state cost. Several different stabilizing MPC schemes where shown to fit in this framework in Mayne, Rawlings, Rao, and Scokaert (2000). Recent publications have introduced the use of a time-varying terminal cost to impose stability for linear time-varying systems (Kim, 2002; Lee, Kwon, & Choi, 1998; Wan & Kothare, 2003) or to achieve improved feasibility and optimality for linear time-invariant systems (Bacic, Cannon, Lee, & Kouvaritakis, 2003; Bloemen, van den Boom, & Verbruggen, 2002).

The main disadvantage of MPC using a time-varying terminal cost is the increase in computational complexity induced by the on-line optimization of the terminal cost and constraint. In this paper, a new MPC scheme using a time-varying terminal cost and constraint is introduced for linear, time-invariant systems, further improving the computational advantage of the method proposed in Bacic et al. (2003). The method uses a discrete set of terminal costs and constraints that are calculated off-line in order to compute a time-varying terminal cost and constraint on-line. This can be considered as an extension to the use of convex combinations as introduced in Wan and Kothare (2003) in the context

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of robust MPC. Compared with the method in Bacic et al. (2003), the new method does not need an explicit decomposition of the terminal state, which leads to a decreased number of additional optimization variables. As a consequence, this leads to a further reduction in on-line computational complexity.

A general stability theorem is formulated, unifying MPC with fixed terminal cost and several MPC schemes using time-varying terminal cost, including the method introduced in this paper. The new theorem extends the well-known results presented in Mayne et al. (2000) and leads to additional insights in the different algorithms discussed.

This paper is organized as follows. In Section 2, the general notation used in this paper and some necessary background knowledge is briefly explained. Section 3 introduces the new MPC scheme and compares its computational complexity with the methods published in Bacic et al. (2003), Bloemen et al. (2002). Section 4 introduces a unifying stability theorem and Section 5 demonstrates the new method on a simple example.

2. Background

2.1. Model-predictive control

When referring to MPC for controlling a linear time-invariant system defined by

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, \ldots, \infty, \tag{1}$$

we will refer to a control scheme that solves at each time step k, given a value for $x(k) \in \mathbb{R}^{n_x}$, the following optimization problem:

$$\min_{\mathbf{x},\mathbf{u}} \; J_{n_h}(x(k), \mathbf{x}, \mathbf{u}) \tag{2a}$$

$$\text{s.t.} \quad x(k+i+1|k) = Ax(k+i|k) + Bu(k+i|k), \quad i = 0, \ldots, n_h - 1, \tag{2b}$$

with $J_{n_h}(x(k), \mathbf{x}, \mathbf{u}) = \sum_{i=0}^{n_h-1} \|u(k+i|k)\|_R^2 + \sum_{i=0}^{n_h} \|x(k+i|k)\|_Q^2$, where $\|x\|_Q^2 \triangleq x^T Q x$, after which $u(0|0) \in \mathbb{R}^{n_u}$ is applied to the plant. The scalars n_x and n_u, respectively, denote the number of states and inputs. $\mathbf{x} \in \mathbb{R}^{n_h \cdot n_x}$ and $\mathbf{u} \in \mathbb{R}^{n_h \cdot n_u}$ denote, respectively, the stacked vectors of within-horizon states and inputs, with $x(k+i|k)$ and $u(k+i|k)$, respectively, denoting the system state and inputs at time $k+i$ as predicted at time k. $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive-definite matrices denoting the state and input weighting matrices, while n_h denotes the horizon. $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$ define the linear state-space prediction model used by the controller. Additional state and input constraints are denoted by

$$x(k+i|k) \in \mathbb{X}, \quad u(k+i-1|k) \in \mathbb{U}, \quad i = 1, \ldots, n_h, \tag{2c}$$

with $\mathbb{X} \subset \mathbb{R}^{n_x}$ and $\mathbb{U} \subset \mathbb{R}^{n_u}$ convex sets. In the following sections, we call this the *standard MPC* algorithm.

2.2. Stability of MPC

In general, MPC stability is obtained by changing the cost term of the last state of the horizon into $F(x(k+n_h|k))$, where $F(\cdot)$ is a convex function ($\mathbb{R}^{n_x} \to \mathbb{R}$) and by imposing the terminal constraint $x(k+n_h|k) \in \mathbb{X}_{n_h}$. Asymptotic stability can be proved if there also exists a terminal state feedback controller $\kappa(\cdot)$ such that the following conditions are satisfied:

$$\text{(a)} \; \kappa(x) \in \mathbb{U}, \quad \forall x \in \mathbb{X}_{n_h}; \tag{3a}$$

$$\text{(b)} \; Ax + B\kappa(x) \in \mathbb{X}_{n_h}, \quad \forall x \in \mathbb{X}_{n_h}; \tag{3b}$$

$$\text{(c)} \; \mathbb{X}_{n_h} \subset \mathbb{X}; \tag{3c}$$

$$\text{(d)} \; F(x) - F(Ax + B\kappa(x)) \geqslant \|x\|_Q^2 + \|\kappa(x)\|_R^2, \quad \forall x \in \mathbb{X}_{n_h}. \tag{3d}$$

When dealing with a linear system (1), and in case the state and input constraints $\mathbb{X}$ and $\mathbb{U}$ are defined as

$$|x_i| \leqslant x_{i,\max}, \quad i = 1, \ldots, n_x, \tag{4a}$$

$$|u_i| \leqslant u_{i,\max}, \quad i = 1, \ldots, n_u. \tag{4b}$$

Then $F(\cdot)$, $\kappa(\cdot)$ and $\mathbb{X}_{n_h}$ can be chosen as

$$\kappa(x) = Kx, \tag{5a}$$

$$F(x) = x^T Q_{n_h} x, \tag{5b}$$

$$\mathbb{X}_{n_h} = \{x | x^T Z^{-1} x \leqslant 1\} \tag{5c}$$

and can be calculated by solving the LMIs stated in Kothare, Balakrishnan, and Morari (1996):

$$\min_{\gamma,Z,Y,X} \; \gamma \tag{6a}$$

s.t.

$$\begin{bmatrix} 1 & * \\ \bar{x} & Z \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} Z & * & * & * \\ Q^{1/2}Z & \gamma I & * & * \\ R^{1/2}Y & 0 & \gamma I & * \\ AZ + BY & 0 & 0 & Z \end{bmatrix} \geqslant 0, \tag{6b}$$

$$\begin{bmatrix} X & * \\ Y^T & Z \end{bmatrix} \geqslant 0, \quad \text{diag}(u_{\max}^2) \geqslant X, \tag{6c}$$

$$\begin{bmatrix} x_{i,\max}^2 Z & * \\ C_i Z & 1 \end{bmatrix} \geqslant 0, \quad i = 1, \ldots, n_x, \tag{6d}$$

with $C_i = [0 \; 0 \; \ldots \; 1 \; \ldots \; 0 \; 0]$ (ith component). Asterisks are used to denote the corresponding transpose of the lower block part of symmetric matrices. In the method proposed in Kothare et al. (1996) $\bar{x}$ represents the current state measurement and the above LMIs are recalculated at each time instant. In this context $\bar{x} \in \mathbb{X}$ is a state to be chosen by the user (called *canonical state*) determining the size of the resulting terminal constraint $\mathbb{X}_{n_h}$. The feedback matrix

K and the closed-loop Lyapunov function $F(x) = \|x\|_{Q_{n_h}} = x^T Q_{n_h} x$ can now be calculated as

$$K = YZ^{-1}, \quad Q_{n_h} = \gamma Z^{-1}. \tag{6e}$$

In the following sections, MPC using a terminal cost and controller (5), which is off-line calculated with the above optimization problem, is called *MPC with fixed terminal cost* or *F-MPC*. A related method was proposed in Chen and Allgöwer (1998).

2.3. MPC with time-varying terminal cost

As already indicated in Bloemen et al. (2002), a trade-off has to be made between feasibility (large $\|\bar{x}\|$) and local optimality (small $\|\bar{x}\|$). To solve this problem, an MPC scheme using the following optimization was introduced (Bloemen et al., 2002):

$$\min_{\mathbf{u},\gamma',X,Y,Z,\gamma} \gamma' + \gamma \quad \text{s.t.} \begin{cases} \begin{bmatrix} \gamma' - f^T\mathbf{u} & \mathbf{u}^T \\ \mathbf{u} & H^{-1} \end{bmatrix} \geqslant 0, \\ A_{\text{ineq}}\mathbf{u} \leqslant b_{\text{ineq}}, \end{cases} \tag{7}$$

and subject to (6b)–(6d) with $\bar{x} = \Psi_{n_h}\mathbf{u} + \psi_{n_h}$.

$\Psi_{n_h} = [A^{n_h-1}B \;\; A^{n_h-2}B \;\; \dots \;\; B]$ and $\psi_{n_h} = A^{n_h}x(0|0)$ define the dependence of the terminal state on the within-horizon inputs, while H and f represent the within-horizon control cost $\mathbf{u}^T H\mathbf{u} + f^T\mathbf{u}$. A_{ineq} and b_{ineq} denote the within-horizon input and state constraints. *In the following sections, we will refer to this method as MPC with full time-varying terminal cost or FTV-MPC*. See Bloemen et al. (2002) for details.

In Bacic et al. (2003) it was recognized that performing some of the optimization off-line and using on-line interpolation to calculate a time-varying terminal cost results in a significant computational advantage. The method proposed there uses a set of n off-line calculated terminal constraints, costs and controllers defined by Z_i, $Q_{n_h,i}$, K_i, satisfying (3a)–(3d). At each time step the terminal state $x(k+n_h|k)$ is decomposed into n different components $\hat{x}_i$ in the following way:

$$x(k+n_h|k) = \sum_{i=1}^{n} \hat{x}_i, \quad \begin{cases} \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geqslant 0, & \forall i, \\ x_i^T Z_i^{-1} x_i \leqslant 1, & \forall i, \\ \hat{x}_i = \lambda_i x_i, & \forall i. \end{cases} \tag{8}$$

These components are then subjected to the respective control laws K_i, resulting in the following control law:

$$u(k+n_h+j|k) = \sum_{i=1}^{n} K_i(A+BK_i)^j \hat{x}_i, \quad j \geqslant 0, \tag{9}$$

for which, through the use of an off-line convex optimization problem that is discussed in detail in Bacic et al. (2003), a quadratic cost function $T(x) = \tilde{x}^T V \tilde{x}$ with $\tilde{x} = [\hat{x}_1^T \;\; \hat{x}_2^T \;\; \dots \;\; \hat{x}_n^T]^T$ can be calculated. By using this cost function as a terminal cost, by using constraints (8) as a terminal constraint and by adding the $\hat{x}_i$ and λ_i as optimization variables, an MPC scheme with time-varying terminal cost, but with reduced computational complexity is obtained. *In the following sections, we refer to this method as MPC with time-varying terminal cost using state decomposition or SD-MPC*. See Bacic et al. (2003) for details.

3. Convex combinations

In this section, a different approach towards constructing a time-varying terminal cost and constraint is presented. We assume that we have a set of canonical states $\bar{x}_i$, $i = 1, \dots, n$, with $n > 0$ an integer chosen by the user. Assuming that we have solutions γ_i, X_i, Y_i and Z_i to (6b)–(6d) for these canonical states, it can be easily proved (based on the convexity of LMI's) that any convex combination

$$(\gamma, X, Y, Z) \equiv \sum_{i=1}^{n} \lambda_i(\gamma_i, X_i, Y_i, Z_i) \tag{10}$$

of these solutions, with $\lambda_i \geqslant 0, \sum \lambda_i \leqslant 1$, is a solution to (6b)–(6d) for $\bar{x} = \sum_{i=1}^{n} \lambda_i \bar{x}_i$, implying that the corresponding terminal controller, cost and constraint (using (6e) and (5a)–(5c)) also satisfies the MPC stability constraints (3a)–(3d). Note that in the strictest sense, convex combinations only allow $\sum_{i=1}^{n} \lambda_i = 1$, but since one can always find an arbitrarily small solution γ_0, X_0, Y_0, Z_0 to (6a)–(6d) for $\bar{x}_0 = 0$, the above is still valid, assuming this solution is incorporated in the convex combination with weight $\lambda_0 = 1 - \sum_{i=1}^{n} \lambda_i$. The idea is to calculate this discrete set of solutions off-line, while making the convex combinations on-line, thus eliminating the LMI's (6b)–(6d) from the on-line optimization problem. This leads to the following off-line algorithm for calculating the terminal constraints and terminal costs:

Algorithm 1 (*CC-MPC (off-line)*). *Given a model* (1), *state and input constraints* (4), *weighting matrices Q and R, horizon* n_h *and a positive integer n, calculate* X_i, Y_i, Z_i, γ_i, $i = 1, \dots, n$, *using any one of the following methods.*

(a) *Choose* $\bar{x}_i \in \mathbb{X}$ *and solve* (6a)–(6d) *for* $i = 1, \dots, n$.
(b) *Choose* $\bar{x} \in \mathbb{X}$, *for example* $\bar{x} = [0 \dots 1 \dots 0]^T$ (*jth component, with* $j \in \{1, \dots, n_x\}$), *choose* $\bar{x}_i = c_i\bar{x}$ *with* $c_{1\dots n} \in \mathbb{R}_0^+$ *and* $c_i < c_{i+1}$ *and solve* (6a)–(6d) *for* $i = 1, \dots, n$.
(c) *Calculate* $a_{1\dots n_x} \in \mathbb{R}_0^+$ *as* $\max_{a_i} a_i$ *subject to the constraints of* (7) *and* (6b)–(6d), *with* $x(0|0) = [0 \dots a_i \dots 0]^T$ (*ith component*). *Choose values* $c_{1\dots n} \in (0, 1)$ *with* $c_i < c_{i+1}$. *Calculate the* X_i, Y_i, Z_i, γ_i, $i = 1, \dots, n$ *as follows*:

$$\min_{\substack{\{u_j(k)\in\mathbb{R}^{n_u}\}_{j=1\dots n,k=0\dots n_h-1} \\ X,Y,Z,\gamma}} \gamma \tag{11a}$$

$$s.t.\ (\forall k = 0, \ldots, n_h - 1, \forall j = 1, \ldots, n):$$

$$x_j(k+1) = Ax_j(k) + Bu_j(k), \tag{11b}$$

$$u_j(k) \in \mathbb{U}, \tag{11c}$$

$$x_j(k) \in \mathbb{X}, \tag{11d}$$

$$s.t.\ (\forall j = 1, \ldots, n):$$

$$x_j(n_h)^T Z^{-1} x_j(n_h) \leqslant 1, \tag{11e}$$

$$x_j(0) = [0 \ \ldots \ c_i c_{\max} a_j \ \ldots \ 0]^T \tag{11f}$$

and subject to (6b)–(6d), *with* $c_{\max} \in (0, 1)$ *assigned the largest value resulting in a feasible optimization problem* (11) *and* (6b)–(6d) *with* $c_i = 1$.

Method (a) allows the user to choose the $\bar{x}_i$ freely, while method (b) reduces the choice of the $\bar{x}_i$ to the choice of $n+1$ positive scalars, which is more transparant to the user than method (a). Method (c) is a refinement of (b) and imposes that the terminal constraints have to be reachable in n_h time steps from n_x different initial states that have norms proportional to the c_i. Since in all three cases, the $X_i, Y_i, Z_i, \gamma_i, \quad i = 1, \ldots, n$, satisfy (6b)–(6d), they are valid to be used in the on-line part of the algorithm.

Algorithm 2 (*CC-MPC (on-line)*). *Given a set of terminal costs and constraints defined by* $X_i, Y_i, Z_i, \gamma_i, \ i = 1, \ldots, n$, *computed using Algorithm* 1, *solve at each time step k, given the current state* $x(k|k)$, *the following optimization problem*:

$$\min_{\mathbf{u}, \gamma, \lambda} \gamma + \sum_{i=1}^{n} \lambda_i \gamma_i \tag{12a}$$

s.t.

$$\begin{bmatrix} \gamma - f^T \mathbf{u} & \mathbf{u}^T \\ \mathbf{u} & H^{-1} \end{bmatrix} \geqslant 0, \tag{12b}$$

$$A_{\text{ineq}} \mathbf{u} \leqslant b_{\text{ineq}}, \tag{12c}$$

$$\begin{bmatrix} 1 & (\Psi_{n_h} \mathbf{u} + \psi_{n_h})^T \\ \Psi_{n_h} \mathbf{u} + \psi_{n_h} & \sum_{i=1}^{n} \lambda_i Z_i \end{bmatrix} \geqslant 0, \tag{12d}$$

$$\lambda_{1 \ldots n} \geqslant 0, \tag{12e}$$

$$\sum_{i=1}^{n} \lambda_i \leqslant 1, \tag{12f}$$

and apply $u(k)$ *to the system*.

Compared to SD-MPC, only $n+1$ additional variables (the variables γ and λ_i) have to be added to the standard MPC optimization problem, instead of $(n-1)(n_x+1)$ additional variables (the variables $\hat{x}_i$ and λ_i). In the following sections, we refer to this method as *MPC using convex combinations* or *CC-MPC*. Asymptotic stability of this method will be proved in Section 4. Two further refinements to this algorithm that at each time step considers only a subset of the total set of terminal constraints can be found in Pluymers, Roobrouck, Buijs, Suykens, and De Moor (2004a), Pluymers, Suykens, and De Moor (2004b).

Remark 1. It can easily be proved that the resulting terminal admissible set $\mathbb{X}_{n_h}$ is equal to the convex hull of the different terminal constraints defined by the Z_i, which is also the case for SD-MPC. The difference between the two methods will become clear in Section 4.

Remark 2. Using techniques from Lobo, Vandenberghe, Boyd, and Lebret (1998) both SD-MPC and CC-MPC can be reformulated as a *second-order cone program* (SOCP), which can be solved more efficiently. CC-MPC has the disadvantage that additional variables, similar to the terminal state components of SD-MPC, have to be introduced in order to construct the SOCP, but SD-MPC still has an additional $n \cdot n_x$-dimensional SOC constraint compared with CC-MPC. We refer to the extended internal report version of this paper Pluymers et al. (2004a) for more details.

4. Stability proof

Theorem 1. *Consider a linear state-space model given by* $x(k+1) = Ax(k) + Bu(k)$, *state and input constraints* $x(k) \in \mathbb{X}$ *and* $u(k) \in \mathbb{U}$ *and positive-definite weighting matrices Q and R. Furthermore, consider a parameterized set* $\kappa_\theta(\cdot) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}, \mathbb{X}_{n_h,\theta} \subset \mathbb{X}$ *and* $F_\theta(\cdot) : \mathbb{R}^{n_x} \to \mathbb{R}$ *with parameter* θ *that is defined over a user specified set* Θ *and assume that* $\forall \theta \in \Theta$ *stability conditions* (3a)–(3d) *are satisfied. The following MPC scheme, if feasible for* $k = 0$, *now ensures asymptotic stability of the closed-loop system if the time-varying set* $\Theta(k) \subset \Theta$, *chosen by the controller at each time step k before starting the optimization, satisfies* $\theta^o(k-1) \in \Theta(k)$, *with* $\theta^o(k-1)$ *the optimal value of* θ *at time step* $k-1$:

$$\min_{\mathbf{x}, \mathbf{u}, \theta} J^\star_{n_h}(x(k), \mathbf{x}, \mathbf{u}, \theta) = \sum_{i=0}^{n_h-1} \|u(k+i|k)\|^2_R + \sum_{i=0}^{n_h-1} \|x(k+i|k)\|^2_Q + F_\theta(x(k+n_h|k)) \tag{13a}$$

s.t.

$$x(k+i+1|k) = Ax(k+i|k) + Bu(k+i|k), \quad i = 0, \ldots, n_h - 1, \tag{13b}$$

$$u(k+i|k) \in \mathbb{U}, \quad i = 0, \ldots, n_h - 1, \tag{13c}$$

$$x(k+i|k) \in \mathbb{X}, \quad i = 1, \ldots, n_h - 1, \tag{13d}$$

$$x(k+n_h|k) \in \mathbb{X}_{n_h,\theta}, \tag{13e}$$

$$\theta \in \Theta(k). \tag{13f}$$

Proof. The main idea is to construct a feasible solution to the optimization problem at time step $k+1$ with associated cost $J^{\star,\mathrm{f}}_{n_\mathrm{h}}(x(k+1)) \equiv J^{\star}_{n_\mathrm{h}}(x(k+1), \mathbf{x}^{\mathrm{f}}(k+1), \mathbf{u}^{\mathrm{f}}(k+1), \theta^{\mathrm{f}}(k+1))$ using the optimal solution at time step k, with associated cost $J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k)) \equiv J^{\star}_{n_\mathrm{h}}(x(k), \mathbf{x}^{\mathrm{o}}(k), \mathbf{u}^{\mathrm{o}}(k), \theta^{\mathrm{o}}(k))$. We will prove that $J^{\star,\mathrm{f}}_{n_\mathrm{h}}(x(k+1)) < J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k)), \forall x(k) \neq 0$, which then leads to $J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k+1)) < J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k)), \forall x(k) \neq 0$ and implies that $J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k))$ is a Lyapunov function for the closed-loop system, thus establishing asymptotic stability. The feasible solution to (13) is constructed as follows:

$$\mathbf{x}^{\mathrm{f}}(k+1) = [(x^{\mathrm{o}}(k+2|k))^{\mathrm{T}}, \ldots, (x(k+n_\mathrm{h}|k)^{\mathrm{o}})^{\mathrm{T}}, f(x(k+n_\mathrm{h}|k)^{\mathrm{o}}, \kappa_{\theta^{\mathrm{o}}(k)}), (x(k+n_\mathrm{h}|k)^{\mathrm{o}}))^{\mathrm{T}}]^{\mathrm{T}}, \tag{14a}$$

$$\mathbf{u}^{\mathrm{f}}(k+1) = [(u^{\mathrm{o}}(k+1|k))^{\mathrm{T}}, \ldots, (u(k+n_\mathrm{h}-1|k)^{\mathrm{o}})^{\mathrm{T}}, \kappa_{\theta^{\mathrm{o}}(k)}(x(k+n_\mathrm{h}|k)^{\mathrm{o}})^{\mathrm{T}}]^{\mathrm{T}}, \tag{14b}$$

$$\theta^{\mathrm{f}}(k+1) = \theta^{\mathrm{o}}(k). \tag{14c}$$

Due to the restriction that $\Theta(k)$ has to be chosen in such a way that $\theta^{\mathrm{o}}(k) \in \Theta(k+1)$, the above solution is indeed feasible to (13) if (3a)–(3c) hold, which is the case $\forall\theta \in \Theta(k) \subset \Theta$. (3d), which also holds $\forall\theta \in \Theta$, then establishes $J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k+1)) < J^{\star,\mathrm{o}}_{n_\mathrm{h}}(x(k)), \forall x(k) \neq 0$, proving the theorem. □

This unifying theorem can be used to prove asymptotic stability of FTV-MPC, F-MPC, SD-MPC, CC-MPC and SCC-MPC, as follows:

- *FTV-MPC*: In case $\bar{x}$ is used as a parameter θ to parameterize the terminal cost and constraint, the terminal cost and constraint can be defined by $Q_{n_\mathrm{h},\theta} = \gamma Z^{-1}$ and $\mathbb{X}_{n_\mathrm{h},\theta} = \mathbb{X} \cap \{x | x^{\mathrm{T}} Z^{-1} x \leqslant 1\}$. When choosing the parameter admissible set as $\Theta = \{\bar{x} |$ (6b)–(6d) is feasible$\}$, it can be shown that $\bar{x}$ may be substituted with $x(k+n_\mathrm{h}|k)$ without influencing the optimization results and the above theorem then reduces to the FTV-MPC algorithm as introduced in Bloemen et al., 2002.
- *F-MPC*: F-MPC can be seen as a special case of FTV-MPC with $\Theta(k)$ defined as the singleton $\Theta(k) = \Theta = \{\bar{x}\}$. In this way stability of this method is trivially proven.
- *SD-MPC*: A slight reformulation also enables this method to fit into the above stability theorem. While the actual implementation uses both $\lambda_{i\ldots n}$ and $\hat{x}_{i\ldots n}$ as θ, in this reformulation we define $\theta = [\lambda_1, \ldots, \lambda_n]$ and define the terminal cost and constraint as $F_\theta(x) = \{\min_{\hat{x}_{1\ldots n}} \tilde{x}^{\mathrm{T}} V \tilde{x}$ s.t. (8)$\}$ and $\mathbb{X}_{n_\mathrm{h},\theta} = \{x(k+n_\mathrm{h}|k) | \exists \hat{x}_{1\ldots n}$ satisfying (8)$\}$. This constraint can easily be verified to be equivalent to $\mathbb{X}_{n_\mathrm{h},\theta} = \{x | x^{\mathrm{T}} (\sum_{i=1}^{n} \lambda_i Z_i)^{-1} x \leqslant 1\}$. The admissible parameter set used in this method can be defined by $\Theta(k) = \Theta = \{\theta | \sum_i \lambda_i = 1\}$. The controller obtained in this way can be written as a linear state feedback controller in terms of the terminal state components $\hat{x}_{1\ldots n}$, and can be shown (Bacic et al., 2003) to satisfy the stability conditions (3a)–(3d) for all $\theta \in \Theta$ in this generalized definition of the state, which validates the use of the above theorem to claim stability.
- *CC-MPC*: This method also corresponds to the choice $\theta = [\lambda_1, \ldots, \lambda_n]$, but uses as terminal cost and constraint $Q_{n_\mathrm{h},\theta} = (\sum_{i=1}^{n} \lambda_i \gamma_i)(\sum_{i=1}^{n} \lambda_i Z_i)^{-1}$ and $\mathbb{X}_{n_\mathrm{h},\theta} = \{x | x^{\mathrm{T}} (\sum_{i=1}^{n} \lambda_i Z_i)^{-1} x \leqslant 1\}$, while the parameter admissible set is defined as $\Theta(k) = \Theta = \{\lambda_{1\ldots n} | \lambda_{1\ldots n} \geqslant 0, \sum_i \lambda_i \leqslant 1\}$. It was shown in Section 3 that this terminal cost and constraint satisfies (3a)–(3d) for all $\theta \in \Theta$, and thus stability is guaranteed.

5. Example

This example deals with the control of an inverted pendulum, which consists of a cart that is driven by an electrical motor. A rod is connected to the cart by a joint that can rotate freely. The aim is to steer the cart to the desired position while keeping the rod in the upward position. The input of this system is the motor voltage V, the states are the cart position x_{cart}, the rod angle α and their time derivatives v_x and v_α. State and input constraints are defined by peak bounds $x_{\max} = [20\ 30\pi/180\ 10\ 10]^{\mathrm{T}}$ and $u_{\max} = 5$ with $x = [x_{\mathrm{cart}}\ \alpha\ v_x\ v_\alpha]^{\mathrm{T}}$ and $u = V$. The standard MPC algorithm with $n_\mathrm{h} = 3$, $Q = \mathrm{diag}([1\ 0.001\ 0.001\ 0.001])$ and $R = 0.01$ resulted in unstable behaviour for the initial states shown in the examples. An FTV-MPC controller and SD-MPC and CC-MPC controllers with ellipsoidal terminal

Table 1
Control performance (expressed as total simulation control cost) and computational complexity (expressed as maximum CPU time per iteration in seconds as measured on a P-4 2 GHz PC using MATLAB 6.5, SEDUMI 1.05R5 and MOSEK 3.0.1.18 Mosek of FTV-MPC (Bloemen et al., 2002), SD-MPC (Bacic et al., 2003), CC-MPC and SCC-MPC for different initial state values $x(0) = [b\ 0\ 0\ 0]^{\mathrm{T}}$ and a horizon of $n_h = 3$

	FTV-MPC	$n=3$		$n=9$		$n=17$	
		SD-MPC	CC-MPC	SD-MPC	CC-MPC	SD-MPC	CC-MPC
Control cost for $b=2$	52.4	53.5	55.3	52.5	52.8	52.5	52.7
Control cost for $b=4$	293.2	296.2	307.1	294.4	296.3	294.3	295.8
CPU times (LMI)	0.84	0.80	0.59	1.63	0.70	3.92	0.84
CPU times (SOCP)	—	0.16	0.16	0.18	0.17	0.53	0.17

All algorithms used a horizon of $n_\mathrm{h} = 3$.

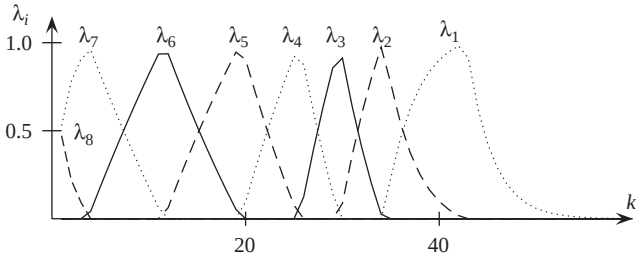


Fig. 1. λ_i values resulting from CC-MPC (with $n=9$) for the simulation with $b=4$ (see Table 1). The controller chooses a smaller terminal constraint as the state approaches the origin.

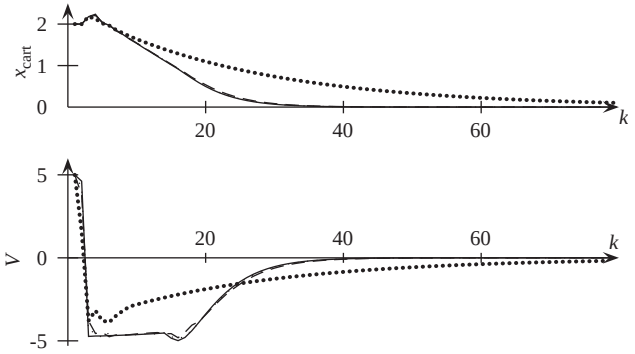


Fig. 2. Cart position (top) and input voltage (bottom) trajectories for FTV-MPC (solid), SD-MPC (dotted, $n=17$), CC-MPC (dashed, $n=17$) and MPC with fixed terminal cost (thick dotted, total simulation control cost: 71.7) for $a=2$ (see Table 1). All algorithms used a horizon of $n_h=3$.

constraint sets computed with Algorithm (1c) ($n=3, 9, 17$ and $c_i \in \{0.01, 0.02, 0.03, 0.04, 0.06, 0.07, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24, 0.26\}$) were also applied to the problem. Table 1 shows the results obtained when applying the different MPC algorithms to this system. FTV-MPC, SD-MPC and CC-MPC have very similar performance, especially for larger values of *n*, but CC-MPC still has a lower computational complexity than the two other algorithms. Fig. 1 shows that CC-MPC chooses a smaller terminal constraint and hence a more optimal terminal controller at each time step. Fig. 2 shows the cart position and input voltage for $b=2$ for FTV-MPC, CC-MPC, SD-MPC and F-MPC controllers. The first three algorithms have an almost identical control behaviour that is superior compared to the F-MPC controller (computed with $\bar{x}=[2\ 0\ 0\ 0]^T$). Note that an F-MPC controller with a smaller terminal constraint (and hence a locally more optimal terminal controller) resulted in on-line infeasibilities.

6. Conclusion

In this paper, a new MPC algorithm with time-varying terminal cost was introduced and demonstrated. The algorithm uses convex combinations of a set of precalculated terminal costs and constraints to calculate the terminal cost and has efficient LMI and SOCP formulations. The scheme is proved to be stabilizing within a stability framework unifying MPC with fixed terminal cost and other MPC schemes with time-varying terminal cost. The algorithm has a reduced computational complexity compared to similar algorithms, while achieving similar control performance levels. Due to the use of ellipsoidal invariant sets, the method can be extended to deal with model uncertainty, which is the subject of future research.

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