Available at
www.ElsevierComputerScience.com
POWERED BY SCIENCE DIRECT®

Signal Processing 86 (2006) 1109–1115

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

Fast communication

The use of total least squares data fitting in the shape-from-moments problem

M. Schuermans[a,*], P. Lemmerling[a], L. De Lathauwer[a,b], S. Van Huffel[a]

[a]*Katholieke Universiteit Leuven, ESAT-SCD, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium*
[b]*ETIS (CNRS, ENSEA, UCP), UMR 8051, Cergy-Pontoise, France*

Received 13 May 2005; received in revised form 7 September 2005; accepted 13 September 2005
Available online 18 October 2005

Abstract

In this paper we discuss the problem of recovering the vertices of a planar polygon from its measured complex moments. Because the given, measured moments can be noisy, the recovered vertices are only estimates of the true ones. The literature offers many algorithms for solving such an estimation problem. We will restrict our discussion to the Total Least Squares (TLS) data fitting models HTLS and STLS and the matrix pencil method GPOF. We show the close link between the HTLS and the GPOF method. We use the HTLS method to compute starting values for the STLS method. We compare the accuracy of these three methods on simulated data.
© 2005 Elsevier B.V. All rights reserved.

Keywords: HTLS; STLS; GPOF; Matrix pencil; Rank reduction

1. Introduction

In this paper we discuss the problem to recover the vertices of a planar polygon from its measured complex moments. In [1–3], the importance and relevance of this problem is discussed. First of all, Milanfar et al. [1] introduced the problems of reconstructing a planar polygon from a set of its complex moments. Moreover, by exploiting the relationship of this shape-from-moments problem with similar problems in signal and array processing, a number of algorithms based on Prony's method were obtained, which could be applied to the reconstruction problem. Then, better numerical procedures, based upon matrix pencils, were proposed for the shape-from-moments reconstruction problem, described in [2]. Also, an analysis of the sensitivity of the shape-from-moments problem is presented in this reference. Later on, instead of concentrating mainly on the numerical aspects of the noiseless case, in [4] the treatment of the reconstruction problem is extended to a given noisy but longer set of moments. Because the given, measured moments can be noisy, the recovered vertices are only estimations of the true ones. The literature offers many algorithms for solving such an estimation problem. We will discuss the Hankel Total Least Squares (HTLS) method [5], the Structured TLS (STLS) method [6–8] and the Generalized Pencil of Function (GPOF) method [9] for the reconstruction of binary polygons from

*Corresponding author. Tel.: +32 16 321143; fax: +32 16 321970.
E-mail address: mieke.schuermans@esat.kuleuven.ac.be (M. Schuermans).

0165-1684/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.sigpro.2005.09.008

their given, estimated, noisy complex moments. Many applications from diverse areas can be cited, including computed tomography [1] and inverse potential theory [10].

The outline of the paper is as follows. In Section 2, the shape-from-moments problem is formulated. In Section 3 a short description is given of the GPOF method, the HTLS method and the STLS method. These methods can be used to solve the shape-from-moments problem. In this paper we will show the link between the HTLS and the GPOF method. We will compare the HTLS, the STLS and the GPOF method on simulated data and discuss their accuracy. This will be the topic of Section 4. It will become clear that it is useful to compute the HTLS solution in order to get starting values for the STLS method. The STLS method is an optimal method and we expect to get a better accuracy than with the GPOF method. Conclusions are drawn in Section 5.

2. The shape-from-moments problem

The reconstruction problem of a closed N-sided planar polygon from a set of its complex moments is defined as follows. Assume that $M+1$ complex moments τ_k with $k=0,\ldots,M$ are measured. We want to recover the polygon vertices z_n with $n=1,\ldots,N$ using the following relationship between moments and vertices:

$$\tau_k=\sum_{n=1}^{N} a_n z_n^k, \tag{1}$$

with a_n for $n=1,\ldots,N$ coefficients that only depend on the vertices. For more information about the derivation of Eq. (1), we refer to papers [1,2,4]. In [2], more details about the analysis of the sensitivity of the shape-from-moments problem can be found. In the presence of noise, Eq. (1) does not have an exact solution. In the rest of the paper we assume that the measured complex moments are perturbed by complex white Gaussian noise. Hence, Eq. (1) leads to an estimation problem.

3. Methods to solve the shape-from-moments problem

We briefly describe the GPOF, the HTLS and the STLS methods in this section. The literature offers many other algorithms to solve the shape-from-moments problem. We refer the interested reader to related work presented in [11–14]. In this paper, we will discuss the GPOF, the HTLS and the STLS methods. The GPOF and the HTLS methods are both non-iterative subspace-based parameter estimation methods. In this section, we will show that their approaches differ in the way of reducing the dimensionality of the shape-from-moments problem and in the way of solving the low-dimensional core problem. On the other hand, the STLS method is an iterative method that preserves the Hankel structure of the matrix formed by the given data sequence τ_k with $k=0,\ldots,M$. In Section 4, it will become clear that the vertices estimated via the STLS method are more accurate than the ones estimated via the GPOF method and the HTLS method.

3.1. GPOF

In this subsection a short description of the GPOF method is given. This method was presented originally by Hua and Sarkar [9] and used later on in [15–17]. From the data sequence τ_k for $k=0,\ldots,M$ a Hankel matrix H with $M-L+1$ rows, $(M+1)/2\geqslant L\geqslant N$, is constructed as follows:

$$H=\begin{bmatrix} \tau_0 & \tau_1 & \tau_2 & \cdots & \tau_L \\ \tau_1 & \tau_2 & \cdots & & \tau_{L+1} \\ \tau_2 & \cdots & & & \\ \vdots & & & & \\ \tau_{M-L} & \tau_{M-L+1} & \cdots & & \tau_M \end{bmatrix}. \tag{2}$$

The GPOF method makes use of two Hankel matrices T_0 and T_1 to compute the N vertices z_n. The matrix T_0 contains the first L columns of matrix H, expressed in (2), and matrix T_1 is equal to the submatrix of H that contains the last L columns. For the GPOF method, the following generalized eigenvalue problem needs to be solved:

$$(T_1-\lambda T_0)v=0. \tag{3}$$

In order to solve this rectangular generalized eigenvalue problem, the GPOF method makes use of the squaring idea of multiplying both sides of relationship (3) with T_0^+, where T^+ denotes the Moore–Penrose pseudo-inverse of T. From the obtained square $L\times L$ pencil, only N eigenvalues will correspond to the vertices we are looking for. The remaining $L-N$ eigenvalues correspond to the common null space of the matrices $T_0^+T_1$ and $T_0^+T_0$. So, the eigenvectors v of interest for the $L\times L$ pencil should be spanned by Y_1, where Y_1

consists of the first N columns of Y, defined by the Singular Value Decomposition (SVD) WDY^H of matrix T_0. Hence, the following $L \times N$ pencil is obtained:

$$(T_0^+ T_1 - \lambda T_0^+ T_0) Y_1 \alpha = 0. \tag{4}$$

Furthermore, the GPOF method shrinks the obtained $L \times N$ pencil into the following square $N \times N$ pencil in order to compute the N vertices:

$$(D_1^{-1} W_1^H T_1 Y_1 - \lambda I)\alpha = 0, \tag{5}$$

where W_1 is the truncation of W given by the first N columns of W and D_1 is equal to the $N \times N$ upper left part of D. The estimated vertices z_n, $n = 1, \ldots, N$, are equal to the eigenvalues of (5). For more details about the GPOF method, we refer to [4,9].

From Eq. (5), it is clear that the GPOF method has reduced the dimensionality of the shape-from-moments problem in Least Squares (LS)-sense [18], via computing the SVD of only one of the two matrices T_0 and T_1. Furthermore, by multiplying both sides of Eq. (3) with T_0^+ in order to square the generalized eigenvalue problem, the GPOF method also solves the shape-from-moments problem in LS-sense. In order to square pencil (3), the adjoint of T_0, denoted by T_0^H, can also be used. In this case, the final squared $N \times N$ pencil will be a generalized eigenvalue problem, instead of an eigenvalue problem as in (5).

In [19] exactly the same algorithm is called the LS–LS algorithm.

3.2. HTLS

In this subsection a short description of the HTLS method is given. The HTLS method makes direct use of the matrix H, expressed in (2), to search for good estimates of the vertices z_n, $n = 1, \ldots, N$. Note that matrix H is different from the augmented matrix $[T_0\ T_1]$ in the sense that it contains no duplicates of columns.

Using Eq. (1), Hankel matrix H can be written in terms of Vandermonde matrices:

$$H = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & & & \\ z_1^{M-L} & z_2^{M-L} & \ldots & z_N^{M-L} \end{bmatrix} \times \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & a_N \end{bmatrix} \begin{bmatrix} 1 & z_1 & \ldots & z_1^L \\ 1 & z_2 & \ldots & z_2^L \\ \vdots & & & \\ 1 & z_N & \ldots & z_N^L \end{bmatrix} \equiv SAC^{\mathrm{T}}. \tag{6}$$

From this Vandermonde decomposition, the vertices z_n can immediately be derived. In [12], an iterative method related to the Hankel–Vandermonde decomposition is presented. Instead of computing the decomposition of a Hankel matrix to Vandermonde form directly, we will search for another way to determine the vertices. From Eq. (6), it is clear that S satisfies the following shift invariance property:

$$\overline{S} = \underline{S} Z, \tag{7}$$

where $\overline{S}$ and $\underline{S}$ are submatrices of S, obtained by deleting the top and the bottom rows of S, respectively, and where Z is a diagonal matrix with the vertices z_n, $n = 1, \ldots, N$, on its diagonal. Using this shift invariance property and a basis transformation via the SVD of the matrix H, the vertices can be determined. The HTLS algorithm can be summarized as follows:

Step 1: Compute the SVD of H and truncate H to rank N: $H = U\Sigma V^H$, $H_{\mathrm{trun}} = U_N \Sigma_N V_N^H$, where U_N and V_N consist of the first N columns of U and V, respectively, and Σ_N is the $N \times N$ upper left submatrix of Σ.

Step 2: Solve the following overdetermined set of equations in Total Least Squares (TLS)-sense [20]:

$$\overline{V_N} \approx \underline{V_N} E^H, \tag{8}$$

where $\overline{V_N}$ and $\underline{V_N}$ are submatrices of V_N, obtained by deleting the first and the last rows of V_N, respectively. The eigenvalues of E are the N estimated vertices z_n.

These algorithmic steps form only a subpart of the HTLS algorithm. If also the estimated data sequence $\hat{\tau}_k$ for $k = 0, \ldots, M$ is of interest, the entire HTLS algorithm can be used. We refer to paper [5] for a description of the entire HTLS algorithm. In this paper we are only interested in the estimation of the N vertices z_n of a given N-sided planar polygon.

Instead of computing the SVD of the matrix T_0, which is the case in the GPOF method, the HTLS method computes the SVD of the matrix H. Hence, in the HTLS method, the information of both matrices T_0 and T_1 has been taken into account. So,

it is clear that the HTLS method has reduced the dimensionality of the shape-from-moments problem in TLS-sense. Moreover, also problem (8) in Step 2 of the HTLS algorithm is solved in TLS-sense.

In [19], a hybrid form of the HTLS algorithm is presented. It is called the TLS–LS algorithm because problem (8) is solved in LS-sense. Note moreover, that the TLS–Prony method, for the first time described by Pisarenko and later on also called LPTLS, solves the same problem as the HTLS method, but in a different way. A comparison of the two methods is presented in [5] and will not be discussed here.

The simulations in Section 4 will show that the estimated vertices, computed via the HTLS method, will not be more accurate than the ones computed by the GPOF method. So, by using the HTLS method, there is no gain in accuracy, at least for the shapes under investigation. The computationally most intensive part of the algorithm is the computation of the SVD of the $(M-L+1)\times(L+1)$ matrix H, which requires $\mathcal{O}((M-L+1)(L+1)^2+(L+1)^3)$ floating-point operations (where $\mathcal{O}(\cdot)$ denotes the order of magnitude). For the GPOF algorithm, the computational cost is mainly due to the computation of the SVD of the $(M-L+1)\times L$ matrix T_0, which requires $\mathcal{O}((M-L+1)L^2+L^3)$ floating-point operations.

So, the computational cost of the GPOF algorithm and the HTLS algorithm are comparable. Moreover, for the examples that we checked in Section 4, there is also no accuracy improvement for the HTLS algorithm. Nevertheless, by considering other shapes, the HTLS algorithm can perform better. In [5], it is shown that e.g. in the case of vertices with a small difference between their angles or situated near the origin, there will be a gain in accuracy by using the HTLS method.

However, for the investigated polygons, there is no improvement for the HTLS algorithm, despite the use of TLS instead of LS. The reason for this is the structure of the matrix H, which has not been taken into account. In the next section, we will present an algorithm that will take the structure into account.

3.3. STLS

Also useful for the shape-from-moments problem are techniques for solving the STLS problem. These methods preserve the structure of the Hankel matrix H, formed by the given data sequence τ_k with $k=0,\ldots,M$. The STLS problem is defined as follows [6–8].

Given an overdetermined set of m linear equations $Ax\approx b$ in $n\times 1$ unknowns x, find

$$\min_{\Delta A,\Delta b,x}\|[\Delta A\ \ \Delta b]\|_F^2 \quad \text{s.t.}\ \begin{cases}(A+\Delta A)x=b+\Delta b,\\ [A+\Delta A\ \ b+\Delta b] \text{ and } [A\ \ b]\\ \quad \text{have the same structure.}\end{cases} \tag{9}$$

No closed-form solution is known for problem (9), but several iterative approaches have been proposed. Possible approaches are the Constrained TLS approach [21], the Structured Total Least Norm (STLN) approach [22] and the Riemannian SVD approach [8]. Most of the existing STLS approaches are formulated in terms of real-valued data and parameters. A complex-valued version of the STLS framework is presented in [23]. For the shape-from-moments problem involving complex-valued data, we will focus on the STLN approach. There exist also other methods that exploit the Hankel structure, e.g., Cadzow's algorithm [11] and the Iterative Quadratic Maximum Likelihood algorithm [24], but they are suboptimal, as proven in [6,8].

In order to obtain the same structure in $[A+\Delta A\ b+\Delta b]$ as in $[A\ b]$, the STLN approach arranges the different errors on A in a vector α and the different errors on b in a vector β. The STLN problem formulation then is

$$\min_{\alpha,\beta,x}\left\|\begin{bmatrix}D_\alpha\alpha\\ D_\beta\beta\end{bmatrix}\right\|_F^2 \quad \text{s.t. } \hat{r}=0, \tag{10}$$

where $\hat{r}$ is the modified residue $b+\Delta b-(A+\Delta A)x$ and the matrices D_α and D_β are weighting matrices that account for the repetitions of elements in α and β.

From [4], it should be clear that STLS can be used to solve the reconstruction problem of an N-sided polygon from a set of its complex moments by computing the STLS solution of $Ax\approx b$ with A equal to the first N columns of matrix H, expressed in (2), b equal to the last column of H and $L=N$. The STLS solution $\hat{x}$ is a vector that contains the coefficients of a polynomial $P(z)=z^N+\sum_{n=1}^{N}\hat{x}_n z^{N-n}$ of which the roots are the N vertices. Note that this specific STLS problem is equivalent to the variable projection method [25], as described in [8], where through the use of a different

Table 1
RMSE for each method for the star-shaped polygon

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9	z_{10}	Overall error
GPOF	2.6e−6	1.2e−1	2.4e−6	1.1e−1	2.4e−6	1.3e−1	2.5e−6	1.1e−1	2.6e−6	1.8e−1	0.0932
HTLS	2.7e−6	1.3e−1	2.4e−6	1.2e−1	2.5e−6	1.5e−1	2.6e−6	1.2e−1	2.8e−6	1.9e−1	0.1029
STLS	2.4e−6	5.2e−2	2.3e−6	5.3e−2	2.1e−6	5.9e−2	2.1e−6	4.8e−2	2.3e−6	1.4e−1	0.0558

parameterization this STLS problem is reformulated into a nonlinear least squares minimization.

The STLS method has a higher computational cost than the two non-iterative methods GPOF and HTLS. Moreover, since a nonlinear optimization problem needs to be solved, the use of good starting values is of great importance for an STLS approach to converge within a reasonable amount of time. Nevertheless, from Section 4, it will become clear that a higher accuracy can be obtained for the estimated vertices by using the STLS method. The higher accuracy can be explained by the fact that STLS preserves the matrix structure thereby maintaining consistency of its solution and the maximum likelihood property in case of Gaussian measurement errors [6]. Recently, another related perturbation method has been presented [13]. For the case of $N = 1$, this perturbation method is equivalent to the classical unstructured TLS problem, so to the problem defined in (9) without the constraint that $[A + \Delta A\ b + \Delta b]$ and $[A\ b]$ have the same structure. For the case of larger N, this close relationship between the two approaches does not hold anymore. For more details, we refer to paper [13].

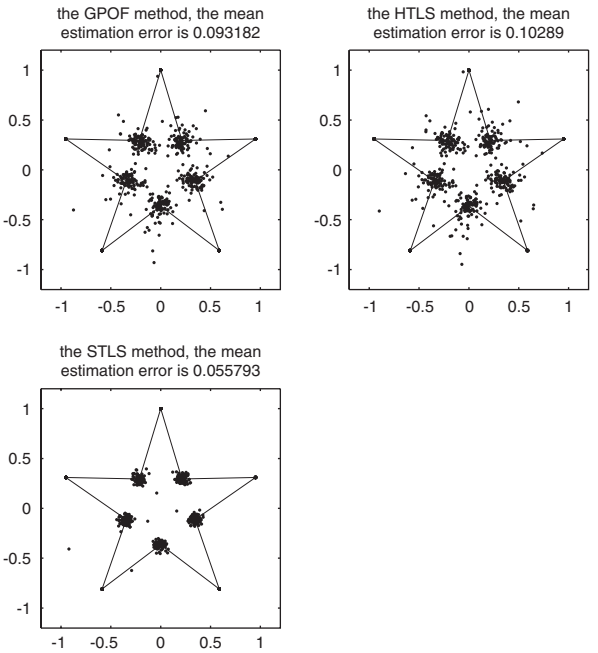


Fig. 1. Estimated vertices for the star-shaped polygon.

4. Comparison in performance between HTLS, STLS and GPOF

In this section we compare the performance of the GPOF-method and the methods HTLS and STLS from the TLS-family. For this comparison we use experiments that are presented in [4]. Here, we will not explain the setup of the experiments in detail. The reader is referred for more details to the paper of Elad et al. [4]. Put shortly, the experiments are constructed as follows: Firstly, a polygon is created and its complex moments are computed. Secondly, complex Gaussian white noise is added to the moments. Finally, the different algorithms—GPOF, HTLS and STLS with the HTLS solution as a starting value—are applied to this set of noisy complex moments in order to obtain the estimated vertices. The presented results are obtained by implementing the different algorithms in Matlab (version 6.1) on a PC i686 with 800 MHz and 256 MB memory.

Experiment 1: In this experiment we use a star-shaped polygon with 10 vertices. We assume that 101 noisy complex moments $\tau_0, \ldots, \tau_{100}$ are given with a noise variance of 2×10^{-4}. The results after applying the different methods to this noisy data set are presented in Table 1 and in Fig. 1. Table 1 contains, for each method, the average Root Mean Squared Error (RMSE) for each vertex z_n with $n = 1, \ldots, 10$ over 100 Monte Carlo simulations, as well as the overall error. The notation $\mathrm{e}-a$ is used instead of 10^{-a}, where a is an integer. In Fig. 1 the location of the 100 estimated sets of vertices for each method is presented.

Experiment 2: In this experiment we use a "C"-shaped polygon with eight vertices. We assume that 81 noisy complex moments are given with a noise

Table 2
RMSE for each method for the "C"-shaped polygon

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	Overall error
GPOF	6.2e−4	7.1e−5	5.7e−2	1.6e−1	1.7e−1	5.9e−2	7.1e−5	6.1e−4	0.0869
HTLS	6.8e−4	7.5e−5	6.5e−2	1.7e−1	1.9e−1	6.9e−2	7.2e−5	6.6e−4	0.0966
STLS	4.3e−4	5.9e−5	2.5e−2	4.9e−2	5.2e−2	2.5e−2	6.2e−5	4.4e−4	0.0282

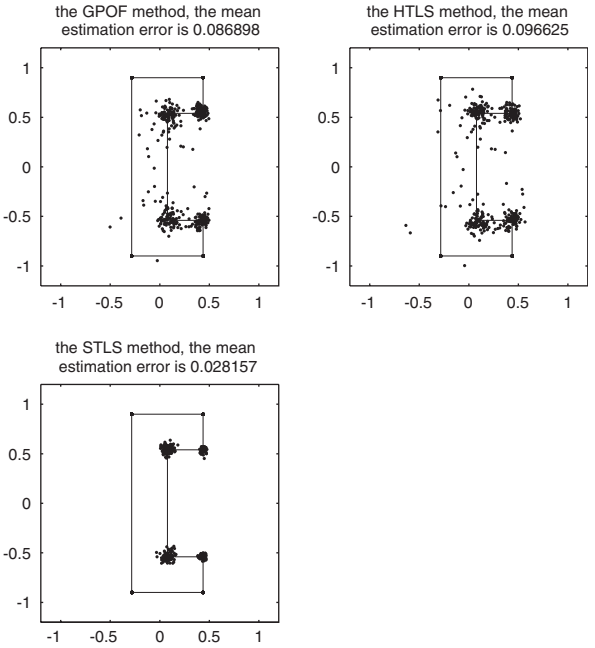


Fig. 2. Estimated vertices for the "C"-shaped polygon.

variance of 6×10^{-3}. The results after applying the different methods to this noisy data set are presented in Table 2 and in Fig. 2. Table 2 contains, for each method, the average RMSE for each vertex over 100 Monte Carlo simulations, as well as the overall error. In Fig. 2 the location of the 100 estimated sets of vertices for each method is presented.

From the tables and the figures it is clear that the estimated vertices computed via the STLS method are more accurate than the ones computed via the GPOF/HTLS method. The GPOF method and the HTLS methods perform quite similarly. Note from Table 1 that some vertices are much more accurate than others, e.g., $z_1, z_3, \ldots, z_9$. For an explanation of this behavior, we refer to [2].

When the noise variance is reduced, accuracy is improved for all the algorithms and the difference in accuracy between the three algorithms becomes weaker. On the other hand, all the algorithms fail when the noise level increases above a common threshold. The phenomenon of failure is related to problem (1) itself. When the noise increases, the number of moments τ_k lying above the noise level decreases. So, there are not enough "reliable" moments left in order to get meaningful estimates of the vertices. In these cases, unreliable results are not due to a wrong choice of particular algorithms, but to the ill-conditioning of the problem itself.

5. Conclusions

In this paper we discussed the problem of recovering the vertices of a planar polygon from its measured complex moments. Because the given moments can be noisy, the recovered vertices are estimations of the true ones. The literature offers many algorithms for solving such an estimation problem. We restricted our discussion to the HTLS, the STLS and the GPOF methods. We showed the close link between the HTLS and the GPOF methods. Through simulated data we compared the accuracy of the three mentioned methods. We found that the HTLS method and the GPOF method perform similarly. Of the three methods, the STLS method computes the most accurate vertex estimates.

Acknowledgements

Prof. Dr. Sabine Van Huffel is a full professor, Mieke Schuermans is a research assistant and Dr. Philippe Lemmerling was a postdoctoral researcher of the FWO (Fund for Scientific Research—Flanders) at the Katholieke Universiteit Leuven, Belgium. Dr. Lieven De Lathauwer holds a permanent research position with the French Centre National de la Reserche Scientifique (C.N.R.S.); he also holds a honorary research position with the Katholieke Universiteit Leuven, Belgium.

Our research is supported by

Research Council KUL: GOA-AMBioRICS, several PhD/postdoc and fellow grants; *Flemish Government*: *FWO*: PhD/postdoc grants, projects, G.0407.02 (support vector machines), G.0269.02 (magnetic resonance spectroscopic imaging), G.0270.02 (nonlinear Lp approximation), G.0360.05 (EEG, Epileptic), research communities (ICCoS, ANMMM);

IWT: PhD grants,

Belgian Federal Science Policy Office: IUAP V-22 (2002–2006): Dynamical Systems and Control: Computation, Identification and Modelling;

EU : PDT-COIL, BIOPATTERN, ETUMOUR.

The authors would like to thank Michael Elad for providing the source code that was used for the simulations and the referees for useful comments which improved the manuscript.

References

[1] P. Milanfar, G. Verghese, W. Karl, A. Willsky, Reconstructing polygons from moments with connections to array processing, IEEE Trans. Signal Process. 43 (February 1995) 432–443.

[2] G.H. Golub, P. Milanfar, J. Varah, A stable numerical method for inverting shape from moments, SIAM J. Sci. Comput. 21 (4) (December 1999) 1222–1243.

[3] B. Gustafsson, C. He, P. Milanfar, M. Putinar, Reconstructing planar domains from their moments, Inverse Problems 16 (4) (August 2000) 1053–1070.

[4] M. Elad, P. Milanfar, G.H. Golub, Shape from moments—an estimation theory perspective, IEEE Trans. Signal Process. 52 (July 2004) 1814–1829.

[5] S. Van Huffel, H. Chen, C. Decanniers, P. Van Hecke, Algorithm for time-domain nmr data fitting based on total least squares, J. Magnetic Resonance A 110 (October 1994) 228–237.

[6] P. Lemmerling, Structured total least squares: analysis, algorithms and applications, Ph.D. Thesis, Department of Electrical Engineering, Katholieke Universiteit Leuven, Belgium, 1999.

[7] P. Lemmerling, S. Van Huffel, Structured total least squares: analysis, algorithms and applications, in: S. Van Huffel, P. Lemmerling (Eds.), Total Least Squares and Errors-in-Variables Modelling, Kluwer Academic Publishers, Dordrecht, 2002, pp. 79–91.

[8] P. Lemmerling, L. Vanhamme, S. Van Huffel, B. De Moor, IQML-like algorithms for solving structured total least squares problems: a unified view, Signal Processing 81 (September 2001) 1935–1945.

[9] Y. Hua, T.K. Sarkar, Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise, IEEE Trans. Acoust. Speech Signal Process. 38 (May 1990) 814–824.

[10] V. Strakhov, M. Brodsky, On the uniqueness of the inverse logarithmic potential problem, SIAM J. Appl. Math. 46 (2) (1986) 324–344.

[11] J.A. Cadzow, Signal enhancement: a composite property mapping algorithm, IEEE Trans. Acoust. Speech Signal Process. 36 (1) (January 1988) 49–62.

[12] F. Luk, D. Vandevoorde, Decomposing a signal into a sum of exponentials, in: R. Chan, T. Chan, G. Golub (Eds.), Iterative Methods in Scientific Computing, Springer, Singapore, 1997, pp. 329–357.

[13] G. Boutry, M. Elad, G. Golub, P. Milanfar, The generalized eigenvalue problem for non-square pencils using a minimal perturbation approach, SIAM J. Matrix Anal. Appl. 27 (2) (2005) 582–601.

[14] A. Goldenshluger, V. Spokoiny, On the shape from moments problem and recovering edges from noisy radon data, Probability Theory and Related Fields 128 (1) (January 2004) 123–140.

[15] K.J.R. Liu, D.P. O'Leary, G.W. Stewart, Y.J. Wu, URV ESPRIT for tracking time-varying signals, IEEE Trans. Signal Process. 42 (December 1994) 3441–3448.

[16] Y. Li, K.J.R. Liu, J. Razavilar, A parameter estimation scheme for damped sinusoidal signals based on low-rank Hankel approximation, IEEE Trans. Signal Process. 45 (February 1997) 481–486.

[17] J. Razavilar, Y. Li, K.J.R. Liu, Structured low-rank matrix pencil for spectral estimation and system identification, Signal Processing 65 (March 1998) 363–372.

[18] G.H. Golub, C. Van Loan, Matrix Computations, third ed., The Johns Hopkins University Press, Baltimore, MD, 1996.

[19] A. van der Veen, E.F. Deprettere, SVD-based low-rank approximations of rational models, in: R. Vaccaro (Ed.), SVD and Signal Processing II, Elsevier, Amsterdam, 1991, pp. 431–454.

[20] S. Van Huffel, J. Vandewalle, The Total Least Squares Problem: Computational Aspects and Analysis, SIAM, Philadelphia, 1991.

[21] T. Abatzoglou, J. Mendel, G. Harada, The constrained total least squares technique and its application to harmonic superresolution, IEEE Trans. Signal Process. 39 (May 1991) 1070–1087.

[22] S. VanHuffel, H. Park, J.B. Rosen, Formulation and solution of structured total least norm problems for parameter estimation, IEEE Trans. Signal Process. 44 (10) (October 1996).

[23] A. Yeredor, Multiple delays estimation for chirp signals using structured total least squares, Linear Algebra and its Appl. 391 (November 2004) 261–286.

[24] Y. Bresler, A. Macovski, Exact maximum likelihood parameter estimation of superimposed exponential signals in noise, IEEE Trans. Acoust. Speech Signal Process. 34 (5) (October 1986) 1081–1089.

[25] G.H. Golub, V. Pereyra, Separable nonlinear least squares: the variable projection method and its applications, Inverse Problems 19 (2) (April 2003) R1–R26.