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SINGULAR VALUE DECOMPOSITION : A POWERFUL CONCEPT AND TOOL IN SIGNAL PROCESSING

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Abstract

The increased computational capabilities of modern computers, workstations and VLSI technology on the one hand and the low cost of sensors and data acquisition equipment on the other hand, make it more and more attractive and feasible to use the Singular Value Decomposition (SVD) in signal processing. In this context it is argued that the SVD may become a widespread tool in signal processing, similar to the role played by FFT in digital signal processing in the seventies and eighties.

At the conceptual level the Singular Value Decomposition (SVD) and the Generalized SVD (GSVD) provide a unifying framework and a numerically robust approach for the formulation and computation of new concepts, such as oriented energy and oriented signal-to-signal ratio's, angles between spaces and canonical correlation analysis. In our research group these concepts have been very instrumental in order to devise new algorithms for a wide variety of problems in signal processing and identification.

Keywords

SVD, GSVD, oriented energy, signal-to-signal ratio, signal processing, signal separation

Running title

SVD IN SIGNAL PROCESSING

1 Introduction

Digital signal processing of multichannel, multisensor or 2D images, often includes filtering, signal separation, error correction, compensation, interpolation, decimation, transformation, monitoring, data compression, feature extraction, ... Many of these tasks require least squares approximation, low rank approximation, norm and condition number calculation ([12]). Although SVD was already well-known in numerical analysis as a reliable concept for these computations ([11], [13]), it has only recently been rediscovered in signal processing ([2], [3], [5], [10], [20]). In most signal processing applications, the SVD provides a unifying framework, in which one can describe at once the conceptual formulation of the problem, the practical application and an explicit solution that is guaranteed to be numerically robust. In this way, the SVD has become a fundamental vehicle for the formulation and derivation of new concepts such as angles between subspaces, oriented energy, oriented signal-to-signal ratio, canonical correlation analysis, ... ([6], [7], [15], [16], [20]) and for the reliable computation of the solutions to problems such as total linear least squares, realization and identification of linear state space models, source separation by subspace methods, etc. It is expected that SVD will become a standard tool on the workbench of a designer of signal processing systems, like FFT was since the mid seventies.

The benefits of using the (generalized) singular value decomposition are most pronounced in those signal processing applications :

- where essentially rank decisions and the computation of the corresponding subspaces determine the complexity and parameters of the model
- where numerical reliability is of crucial importance and the potential loss of numerical accuracy (caused e.g. by the squaring of matrices) cannot be tolerated.
- where a conceptual framework, such as the geometric notion of oriented signal-to-signal ratio, may provide unrevealed additional insight, such as in factor-analysis-like problems.
- where robustness analysis, conditioning and sensitivity optimization are crucial, and are linked together with geometrical insight and interpretation, for which the (G)SVD may provide meaningful quantification (condition numbers, principle angles, ...).

Reliable routines for the SVD have been worked out by GOLUB ([8], [9]) in the sixties and are now available in many libraries like NAG and packages like MATLAB. These avoid the dangerous squaring of matrices. Moderately sized (order of magnitude $50 - 70$) SVD's are

now feasible on workstations and PC's. Moreover the computational burden may be reduced, since in many digital signal processing applications only part of the SVD information is needed or a priori information on the SVD can be incorporated. Moreover parallelization and vectorization research on systolic, hypercube, multiprocessor, or dedicated architectures may lead to efficient SVD solvers for arbitrary or structured matrices.

In this paper we first describe in section 2 the role of SVD in the design of digital signal processing systems in order to situate the overall picture to the reader. In section 3 and 4 the (generalized) singular value decomposition and its basic properties are described along with concepts like oriented energy and oriented signal-to-signal ratios. The usefulness of these concepts in the design of digital signal processing algorithms is shown by describing signal separation principles, based on strength or relative strength of signals in section 5. Some practical applications will further motivate the reader. It is concluded that SVD is a useful concept and should be an effective tool on the workbench of a designer of a digital signal processing system.

2 The SVD as a useful tool in the design of digital signal processing systems

Designers of digital signal processing systems for medical, telecommunication, automation, automotive, consumer or fabrication applications, usually are given a variety of signals measured with a certain precision (say 10%) and are requested to extract in real time useful information from these data. Because of the limited accuracy of the measurements, the data should be handled with care, while on the other hand the speed is rather crucial because of the real time nature of the application. In the design process a number of tools, like FFT, are offered from applied mathematics, with which the designer can experiment in order to find a signal processing algorithm which meets the specifications. These tools can be used and evaluated at a high (software) level in the top-down design. Moreover, depending on the needs, these tools can be tuned or adapted to the specific constraints of the application, in order to obtain a cost-effective and reliable signal processing system. FFT has been such a tool in signal processing since the mid seventies and SVD is expected to become an additional one in the near future.

In recent years one can witness an enormous expansion of computational power at all levels: supercomputers, workstations, minicomputers and microcomputers, VLSI, etc. In addition the measurement power is increasing by cheap sensors, transducers and data acquisition equipment. However in practice it is much more difficult to increase the accuracy of the

measurements than to increase the volume of the measurements. Hence the need for methods and software which can extract more accurate information from the measured data is more acute even at the expense of Kflops of computations. In this situation we believe that the singular value decomposition (SVD) is a very important tool which allows to take profit of the expansion of the computational power in order to improve the accuracy. SVD is well known in linear algebra for its solid numerical qualities. However it is an important concept in digital signal processing. On the other hand its widespread use is still hampered by the computational burden. Hence for SVD, like for FFT, special efficient algorithms (software), coprocessors and application specific integrated circuits (ASIC's) (hardware) are needed.

3 SVD and oriented energy of a vector sequence

In this section we will introduce the concept of oriented energy of a vector sequence and show that the SVD of the corresponding matrix constitutes the main tool to compute extremal values for this concept.

Definition 1 : Oriented energy : ([6], [7], [15])

Consider a sequence of p-vectors $\{\mathbf{a}_i\}$, $i = 1, \ldots, q$ *and arrange them as the columns of a* $p \times q$ *matrix* A. *Then* $E_{\mathbf{e}}[A]$ *is called the energy of the vectorset in the direction of unit vector* $\mathbf{e} \in \Re^p$ *and is defined as :*

$$E_{\mathbf{e}}[A] = \sum_{i=1}^{q} (\mathbf{e}^t \mathbf{a}_i)^2 = \|\mathbf{e}^t A\|^2 \qquad (1)$$

with $\|\cdot\|$ *the Euclidean norm.*

In words, the oriented energy of a vector sequence $\{\mathbf{a}_i\}$ in some direction $\mathbf{e}$ is a sum of squared projections of the vectors $\mathbf{a}_i$ onto direction $\mathbf{e}$. More general, one can define the oriented energy of a vectorset in a subspace $Q \subset \Re^p$ as

$$E_Q[A] = \sum_{i=1}^{q} \|P_Q(\mathbf{a}_i)\|^2 \qquad (2)$$

with $P_Q(\mathbf{a}_i)$ the orthogonal projection of the vectorset onto subspace Q.

Figure 1 shows the polar plot of the oriented energy of a set of vectors in two dimensions. Such a polar plot is found by drawing a vector with length $E_{\mathbf{e}}[A]$ in the direction of a variable unit vector $\mathbf{e}$, and this for all possible directions $\mathbf{e}$ in the vectorspace $\Re^2$. The figure then shows in general a p-dimensional surface with p directions of extremal energy : one direction of maximal oriented energy, one of minimal energy and $p-2$ directions with a saddle point.

We will now present the basic SVD-theorem, for our purpose restricted to real matrices. Furthermore we preferred the "economical" factorization, where the dimensions are reduced as much as possible and therefore the pseudo-diagonal matrix is replaced by a square diagonal one.

Theorem 1 : Autonne-Eckhart-Young theorem ([9])

For any real $p \times q$ matrix A, there exists a real factorization

$$A_{p,q} = U_{p,p}\, \Sigma_{p,p}\, V^t_{p,q} \qquad (\text{for } p < q)$$

in which the matrix U is orthogonal ($UU^t = I_p = U^tU$), the matrix V contains p orthonormal columns ($V^tV = I_p$) and $\Sigma_{p,p}$ is a real diagonal matrix with nonnegative diagonal elements, called the singular values σ_i of the matrix A.

Proofs of the existence and uniqueness of the SVD of a matrix are found in [9] and references therein. Some properties, useful in the further outline of this contribution, are mentioned here without proof.

- The number of singular values, different from zero, equals the algebraic rank of the matrix A.
- Any matrix A can be written as the sum of $r = \text{rank}(A)$ rank one matrices (dyadic decomposition)

$$A = \sum_{i=1}^{r} \mathbf{u}_i\, \sigma_i\, \mathbf{v}_i^t \tag{3}$$

  where $\mathbf{u}_i$ ($\mathbf{v}_i$) is the i-th column of U (V) in the SVD of A and σ_i is the i-th singular value.
- Norms of the matrix A can be computed knowing its singular values, since

$$\|A\|_F^2 = \sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}^2 = \sigma_1^2 + \ldots + \sigma_r^2 \tag{4}$$

$$\|A\|_2 = \sigma_1 \tag{5}$$

- The best rank k approximation of a matrix $A_{p,q}$ with known SVD is given by the partial dyadic decomposition

$$A_k = \sum_{i=1}^{k} \mathbf{u}_i\, \sigma_i\, \mathbf{v}_i^t \qquad \text{with} \quad k < r \tag{6}$$

  then $\forall\ B_{p,q}$ with $\text{rank}(B) = k$:

$$\min \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1} \tag{7}$$

$$\min \|A - B\|_F^2 = \|A - A_k\|_F^2 = \sigma_{k+1}^2 + \ldots + \sigma_r^2 \tag{8}$$

- Each $\mathbf{u}_j$, column of the matrix U in the SVD of A has the following property

$$\begin{aligned} \forall\, \mathbf{x} \in \Re^p, \quad \text{with} \quad \mathbf{x}^t \mathbf{u}_i &= 0, \quad (i = 1, \ldots, j-1) \\ \frac{\|\mathbf{x}^t\, A\|^2}{\|\mathbf{x}\|^2} &\leq \|\mathbf{u}_j^t\, A\|^2 = \sigma_j^2 \end{aligned} \tag{9}$$

with σ_j the j-th singular value of A.

From this last property we establish the link between the oriented energy concept and the SVD in the following theorem

Theorem 2 : The relation between oriented energy and SVD :

Consider a $p \times q$ matrix A with SVD as defined in Theorem 1. Then

$$E_{\mathbf{u}_i}[A] = \|\mathbf{u}_i^t A\|^2 = \sigma_i^2 \tag{10}$$

$$\forall\ \mathbf{e} = \sum_{i=1}^{p} \gamma_i\, \mathbf{u}_i, \qquad E_{\mathbf{e}}[A] = \sum_{i=1}^{p} \gamma_i^2\, \sigma_i^2 \tag{11}$$

In words, the SVD of a $p \times q$ matrix A looks for directions in the vectorspace $\Re^p$, for which the oriented energy of A is extremal. For a more detailed analysis and more properties of the concept of oriented energy, we refer to [6], [7], [15], [17], [18], [20].

4 Oriented signal-to-signal ratio and GSVD.

We can now define the oriented signal-to-signal ratio (S-S ratio) of two vector sequences as the ratio of oriented energies, as follows

Definition 2 : Oriented signal-to-signal ratio : ([6], [7], [20])

The oriented signal-to-signal ratio $R_{\mathbf{e}}[A, B]$ of two sets of p-vectors $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_j\}$, $(i = 1, \ldots, q$ and $j = 1, \ldots, k)$, in the direction of unit vector $\mathbf{e} \in \Re^p$, is defined as :

$$R_{\mathbf{e}}[A, B] = \frac{E_{\mathbf{e}}[A]}{E_{\mathbf{e}}[B]} = \frac{\|\mathbf{e}^t\, A\|^2}{\|\mathbf{e}^t\, B\|^2} \tag{12}$$

A similar plot as Figure 1 can now be constructed by drawing a vector with length $R_{\mathbf{e}}[A, B]$ in the direction of a variable unit vector $\mathbf{e}$, and this for all possible directions $\mathbf{e}$ in the vectorspace $\Re^p$. In Figure 2 this is done in two dimensions ($p = 2$). Figure 2.a shows the oriented energy polar plot of two vectorsets $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$, while Figure 2.b is the polar plot for the oriented S-S ratio in all directions. Note that there are again 2 directions (in general p) of extremal S-S ratio, but in contrast with the oriented energy, these directions are not orthonormal in this case. Figure 3 shows the S-S ratio polar plots that arise for different configurations of the oriented energies of matrices A and B.

A very important property of the S-S ratio is its invariance through a basis transformation in $\Re^p$, characterized by the non-singular $p \times p$ matrix T ([6], [7]). Indeed, one can verify that

$$R_{\mathbf{e}}[A, B] = R_{\mathbf{e}'}[TA, TB] \qquad \text{for} \quad \mathbf{e}' = \frac{T^{-t}\mathbf{e}}{\|T^{-t}\mathbf{e}\|} \tag{13}$$

At this point we present the basic Generalized SVD theorem, again in the most "economical" notations, as follows

Theorem 3 : The Generalized SVD : ([9])

Let A be a $p \times q$ and B a $p \times k$ matrix ($p < q$ and $p < k$), then there exist matrices U_A ($q \times p$) and U_B ($k \times p$), both with p orthonormal columns, and a non-singular $p \times p$ matrix X such that

$$\begin{aligned} A &= X^{-t} D_A U_A^t \\ B &= X^{-t} D_B U_B^t \end{aligned} \tag{14}$$

where $D_A = \mathrm{diag}(\alpha_1, \ldots, \alpha_p)$ and $D_B = \mathrm{diag}(\beta_1, \ldots, \beta_p)$, $(\alpha_i, \beta_i \geq 0)$, are square diagonal $p \times p$ matrices and $\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \geq \ldots \geq \frac{\alpha_r}{\beta_r}$, $r = rank(B)$.

Proof see [9]. The elements of the set $\sigma(A, B) = \left(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_r}{\beta_r}\right)$ are referred to as the generalized singular values of A and B. Each $\mathbf{x}_i$, column of the matrix X in the GSVD of (A, B) has the following interesting property

$$\forall\, \mathbf{y} \in \Re^p, \quad \frac{\|\mathbf{y}^t A\|^2}{\|\mathbf{y} B\|^2} \leq \frac{\|\mathbf{x}_i^t A\|^2}{\|\mathbf{x}_i^t B\|^2} = \frac{\alpha_i^2}{\beta_i^2} \tag{15}$$

In analogy with the relation between the oriented energy concept and the SVD, a link between the oriented signal-to-signal ratio and the Generalized SVD (GSVD) exists :

Theorem 4 : The relation between oriented S-S ratio and GSVD :

Consider a $p \times q$ matrix A and a $p \times k$ matrix B with GSVD as defined in Theorem 3. Then

$$R_{\mathbf{e}}[A,B] = \left(\frac{\alpha_i}{\beta_i}\right)^2 \quad \text{for} \quad \mathbf{e} = \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|} \tag{16}$$

where $\mathbf{x}_i$ is the i-th column of matrix X in the GSVD of (A,B).

Proof : Define the linear transformation $T = D_B^{-1} X^t$, then, due to the invariance property of the S-S ratio

$$\begin{aligned} R_{\mathbf{e}}[A,B] &= R_{\mathbf{e}'}[TA,TB] \\ &= R_{\mathbf{e}'}[D_B^{-1} D_A U_A^t, U_B^t] \\ &= E_{\mathbf{e}'}[D_B^{-1} D_A U_A^t] \end{aligned} \tag{17}$$

for

$$\mathbf{e}' = \frac{T^{-t}\,\mathbf{e}}{\|T^{-t}\,\mathbf{e}\|} = \frac{D_B\,X^{-1}\,\mathbf{e}}{\|D_B\,X^{-1}\,\mathbf{e}\|}$$

The matrix $C = D_B^{-1} D_A U_A^t$ can be factorized as follows

$$C = U_c\,\Sigma_c\,V_c^t \quad \text{with} \quad \begin{aligned} U_c &= I_p \\ \Sigma_c &= D_B^{-1} D_A \\ V_c &= U_A \end{aligned}$$

such that

$$E_{\mathbf{u}_{ci}}[C] = \sigma_{ci}^2 = \left(\frac{\alpha_i}{\beta_i}\right)^2 \tag{18}$$

due to the relation between oriented energy and SVD of C. In other words, $E_{\mathbf{e}'}[C]$ is extremal for $\mathbf{e}' = \mathbf{u}_{ci}$, such that $R_{\mathbf{e}}[A,B]$ is extremal for $\mathbf{e} = \frac{\mathbf{t}_i}{\|\mathbf{t}_i\|} = \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|}$.

□

The GSVD thus provides the extrema of the oriented S-S ratio and also the directions in which those extrema occur : these are the columns of the matrix X, which need not to be orthonormal.

5 Signal separation principles

Often, in multichannel signal processing a set of measured, recorded, acquired, captured, ... signals is given as a linear combination of some original signals called "source" signals, and corrupted by additive noise. The model of such a measured signal $m_i(t)$ is then

$$m_i(t) = \sum_{j=1}^{r} t_{ij}\, s_j(t) + n_i(t) \qquad \text{for } i = 1, \ldots, p \tag{19}$$

with p the number of measured signals and r the number of source signals involved. The coefficients t_{ij} in the model are called transfer coefficients, since they essentially depend on the signal transfer from the source, that generates the signal, to measurement point, that picks up the signal.

For digital signal processing applications it is required that all the signals are sampled, a process by which a continuous-time signal is transformed into a series of digital numbers. If we consider p signals over a certain time-interval with q samples, the model can be written in matrix notation as follows

$$M_{p,q} = T_{p,r}\, S_{r,q} + N_{p,q} \qquad \text{with} \quad r < p \ll q \tag{20}$$

where the p rows of the datamatrix M contain the q samples of the p recorded signals. In this equation, only M is known and it is the goal of signal separation to obtain an estimate of the original "source" signals in matrix S.

5.1 SVD of the datamatrix M

Suppose that we partition the r source signals into two groups : r_d desired source signals and r_u undesired source signals ($r = r_u + r_d$). We can then write for matrix $S_{r,q}$

$$S_{r,q} = \begin{matrix} q \\ \begin{pmatrix} S_u \\ S_d \end{pmatrix} \end{matrix} \begin{matrix} r_u \\ r_d \end{matrix} \tag{21}$$

and partition the r columns of the transfer matrix T similarly

$$T_{p,r} = \begin{matrix} r_u & r_d \\ \begin{pmatrix} T_u & T_d \end{pmatrix} \end{matrix} p \tag{22}$$

These grouped columns of T form subspaces of the p-dimensional column space of $M_{p,q}$. Suppose that the SVD of $M_{p,q}$ can be partitioned as follows ($p_o = p - r_u - r_d$)

$$M = \begin{matrix} r_u & r_d & p_o \\ \begin{pmatrix} U_1 & U_2 & U_3 \end{pmatrix} \end{matrix} \begin{pmatrix} \Sigma_u & 0 & 0 \\ 0 & \Sigma_d & 0 \\ 0 & 0 & \Sigma_o \end{pmatrix} \begin{matrix} \begin{pmatrix} V_1^t \\ V_2^t \\ V_3^t \end{pmatrix} \end{matrix} \begin{matrix} r_u \\ r_d \\ p_o \end{matrix} \tag{23}$$

Recent work ([17], [18], [19], [21], [22]) showed that :

- the stronger the undesired signals are present in the recorded signals and
- the more orthogonal the subspaces T_u and T_d of the transfer matrix T are,

the better the subspaces described by U_1 and U_2 match with T_u and T_d respectively. These two conditions can be met by recording the signals $m_i(t)$ as follows ($p_1 + p_2 = p$)

- p_1 signals are recorded as a mixture of desired and undesired signals ($p_1 \geq r_d$)
- p_2 signals are recorded such that they pick up a combination of undesired signals only ($p_2 \geq r_u$)

Example 1 : **Fetal ECG extraction** ([1], [4], [17], [18], [19])

We will illustrate this signal separation principle for the separation of the cutaneously recorded maternal and fetal electrocardiogram. It has been verified (and there exists a physical explanation [14]) that sufficiently far from the adult heart, its electrical activity can be described by three independent signals only. In the case of a pregnant woman, the obstetrician is interested in the fetal electrocardiogram (FECG), recorded by only placing electrodes on the maternal skin. The main problem is however the very strong and undesired maternal electrocardiogram (MECG) that is omnipresent in these recordings. Suppose that we apply our signal separation principle to this biomedical application and we assume that the number of undesired signal $r_u = 3$ (MECG), the number of desired signals $r_d = 3$ (FECG) (some experiments show that it can be less than three, but this is a safe upper bound) and we suppose for simplicity that there are no other source signals involved.

We now record $p_1 = 3$ signals on the maternal abdomen, close to the fetal heart, such that they contain a mixture of maternal and fetal ECG (see Figure 4, the last three signals). Furthermore we record $p_2 = 3$ signals that pick up MECG only by placing electrodes on the arms and thorax of the mother, far from the fetal heart (see Figure 4, the first three signals). By constructing the matrix M in this way, the SVD finds 6 orthonormal directions of extremal oriented energy. The three directions, associated with the three largest singular values, correspond to a subspace U_M that is a good estimate of the subspace T_u (the maternal subspace). The two following ones correspond to a subspace U_F, that is a good estimate of the fetal subspace and the last direction corresponds to a subspace of dimension 1 for other much weaker source signals. Projection of the datamatrix M onto these orthonormal directions then results in the signals of Figure 5. These signals are orthogonal estimates for the original source signals in S.

5.2 GSVD of selected data intervals

In some signal processing applications it occurs that, over a specific time interval, the undesired signals contribute considerably more in the measured signals than the desired ones (e.g. for the pulse-like ECG signals the maternal QRS-peaks often appear in between two fetal QRS-peaks (see Figure 6)). In that case we can construct from the datamatrix M a matrix $A_{p,k}$ as a sequence of such intervals. This matrix contains in the p-dimensional column space of M mainly contributions from the undesired signals only.

From section 4 we know that the GSVD of the matrix pair (M, A) finds non-orthogonal directions $\mathbf{x}_i$ of extremal oriented signal (in M)-to-signal (in A) ratio. Let $\frac{\alpha_i}{\beta_i}$ be the generalized singular values of M and A, arranged in non-increasing order. Then the column $\mathbf{x}_i$ of X in the GSVD of (M, A) is a vector for which the oriented energy of matrix M is $\frac{\alpha_i}{\beta_i}$ times larger than the oriented energy of matrix A. The projection of the datamatrix M onto the directions $\mathbf{x}_i$, that correspond with the generalized singular values $\gg 1$, then results in the desired signals.

Example 2 : <u>Fetal ECG extraction</u> [4]

Figure 6 shows 5 abdominally recorded potential signals, containing both maternal and fetal ECG. After arranging the signals in a $p \times q$ datamatrix M, a matrix M_M is composed as a sequence of several maternal QRS-intervals, not coinciding with fetal complexes (some of those intervals are indicated on Figure 6). The two largest generalized singular values are

$$\frac{\alpha_1}{\beta_1} = 9.986 \quad \text{and} \quad \frac{\alpha_2}{\beta_2} = 5.342$$

and projection onto the corresponding columns $\mathbf{x}_1$ and $\mathbf{x}_2$ of X results in the two MECG-free fetal heart signals of Figure 7.

Example 3 : <u>Speech enhancement</u>

At the moment we are doing tests on speech signals, recorded using multiple microphone systems, and corrupted by stationary noise (e.g. a ventilator, a motor running at constant rotation speed, etc.). From the recorded speech signals in datamatrix M, some intervals are selected, where speech is absent and only the noise is present. These intervals are arranged in a special matrix B and again the GSVD is computed of the matrix pair (M, B). Up till now we found an enhancement of the speech-to-noise ratio of about a factor 2.

6 Conclusion

In this contribution the use of the (Generalized) Singular Value Decomposition for signal processing and specifically for signal separation purposes is advocated. The SVD and GSVD are not only excellent tools in formulating and describing new geometrical concepts (oriented energy, oriented signal-to-signal ratio), they are also extremely useful in efficiently computing reliable and elegant solutions for many signal processing problems. Due to the increased computational capabilities of modern computers and the development of dedicated algorithms with incorporation of on-line, adaptive ([2], [3]) and parallel techniques, the computational burden can be reduced substantially. Therefore the SVD should be an effective tool on the workbench of a designer of digital signal processing systems.

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Figure captions

Figure 1: Polar plot of the oriented energy of a vector sequence in two dimensions.

Figure 2: (a). Polar plot of the oriented energy of two vector sequences in two dimensions. (b). Polar plot of the corresponding oriented signal-to-signal ratio in two dimensions.

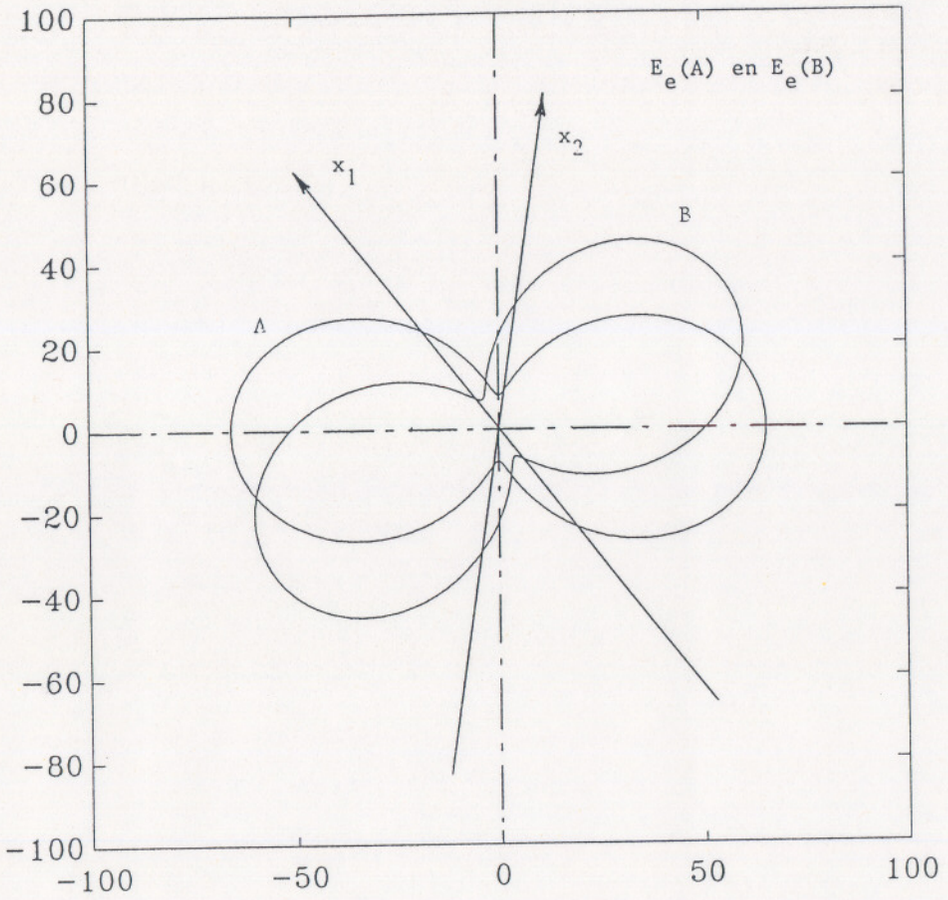
Figure 3: Polar plot of the oriented signal-to-signal ratio for several configurations of the oriented energies of two vectorsets.

Figure 4: Six-channel measurement set of 6 seconds long : the first three signals mainly contain signals from the maternal heart only, while the last three signals contain a mixture of maternal and fetal heart signals.

Figure 5: The six signals that result from a projection of the data of Figure 4 onto the U-matrix in the SVD of M. The fourth and fifth signals are fetal source signals.

Figure 6: A set of 5 recorded abdominal signals (interval of 6 seconds) containing a mixture of MECG and FECG.

Figure 7: The two MECG-free signals obtained with the GSVD-based signal separation method as a projection of the data signals onto the directions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively.


$E_e(A)$ en $E_e(B)$
x_1
x_2
A
B
100
80
60
40
20
0
-20
-40
-60
-80
-100
-100
-50
0
50
100

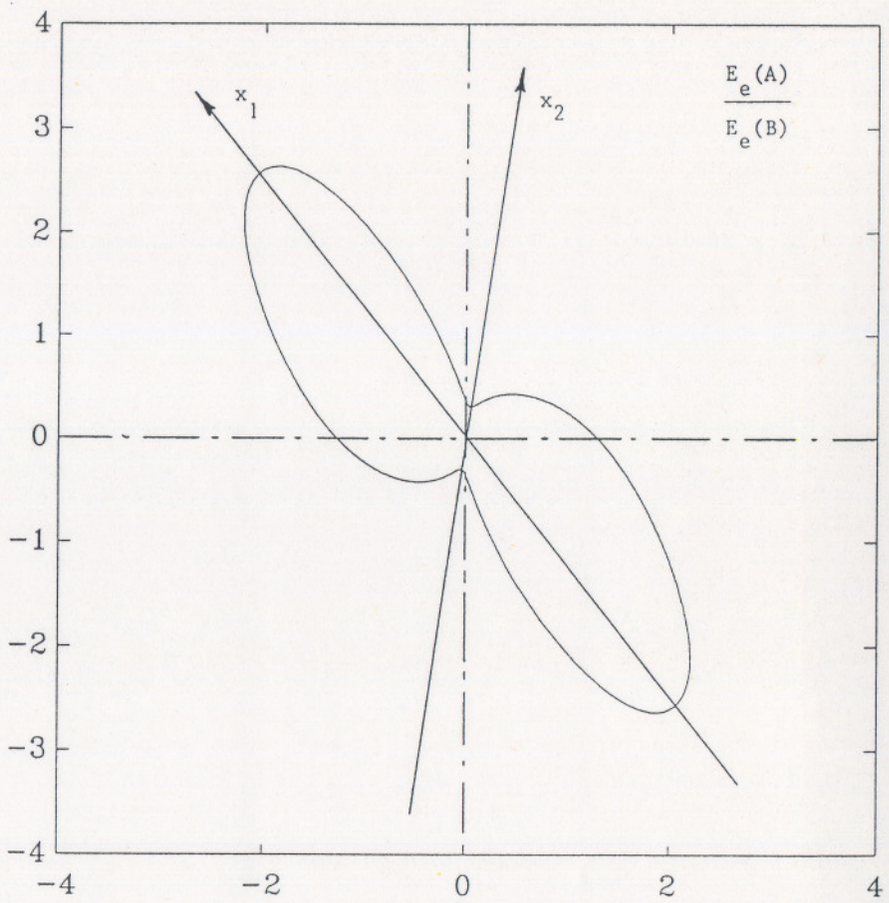
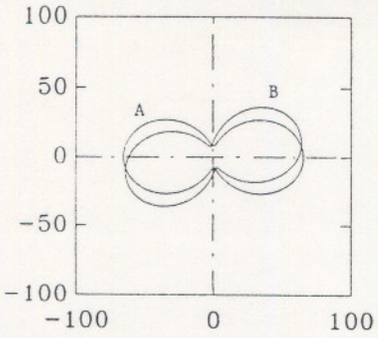
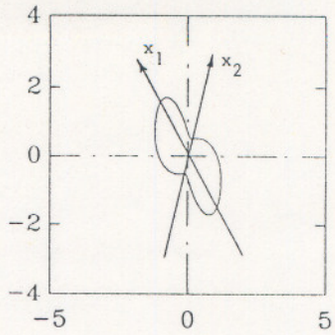
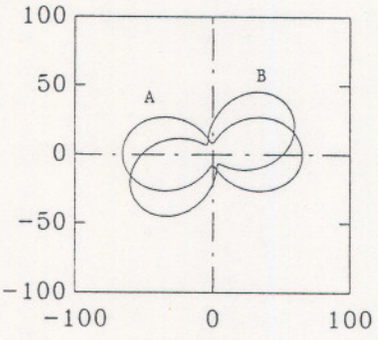

$\frac{E_e(A)}{E_e(B)}$
x_1
x_2
4
3
2
1
0
-1
-2
-3
-4
-4
-2
0
2
4

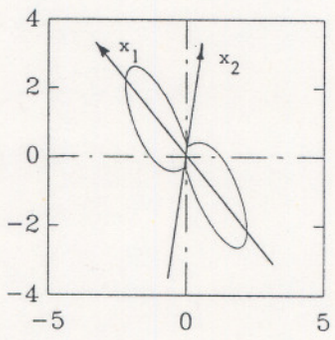
Fig 8

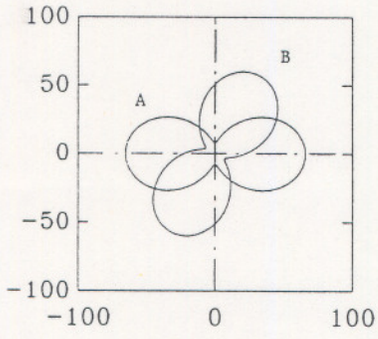
Fig 3

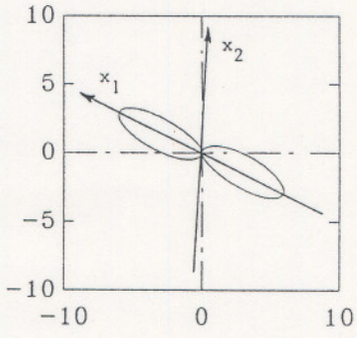

100
50
0
−50
−100
−100
0
100
A
B

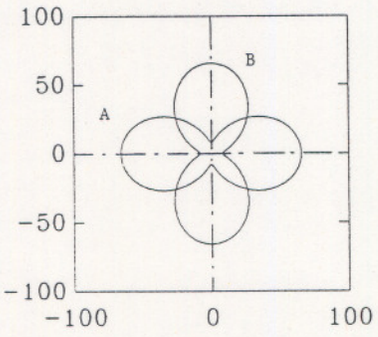

4
2
0
−2
−4
−5
0
5
x1
x2


100
50
0
−50
−100
−100
0
100
A
B


4
2
0
−2
−4
−5
0
5
x1
x2


100
50
0
−50
−100
−100
0
100
A
B


10
5
0
−5
−10
−10
0
10
x1
x2


100
50
0
−50
−100
−100
0
100
A
B

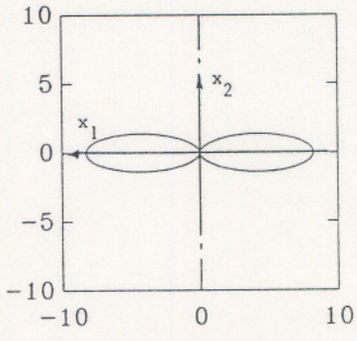
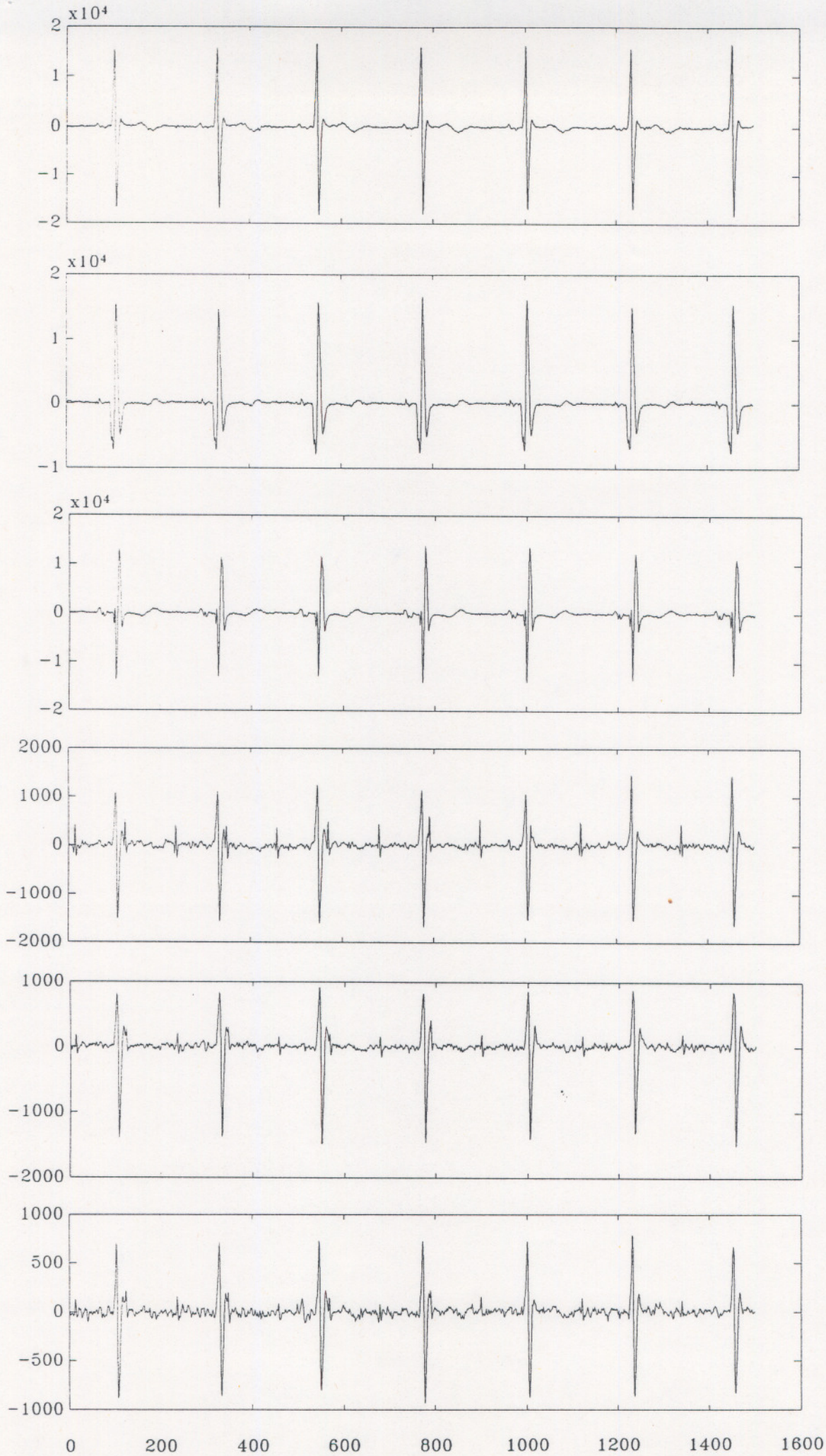
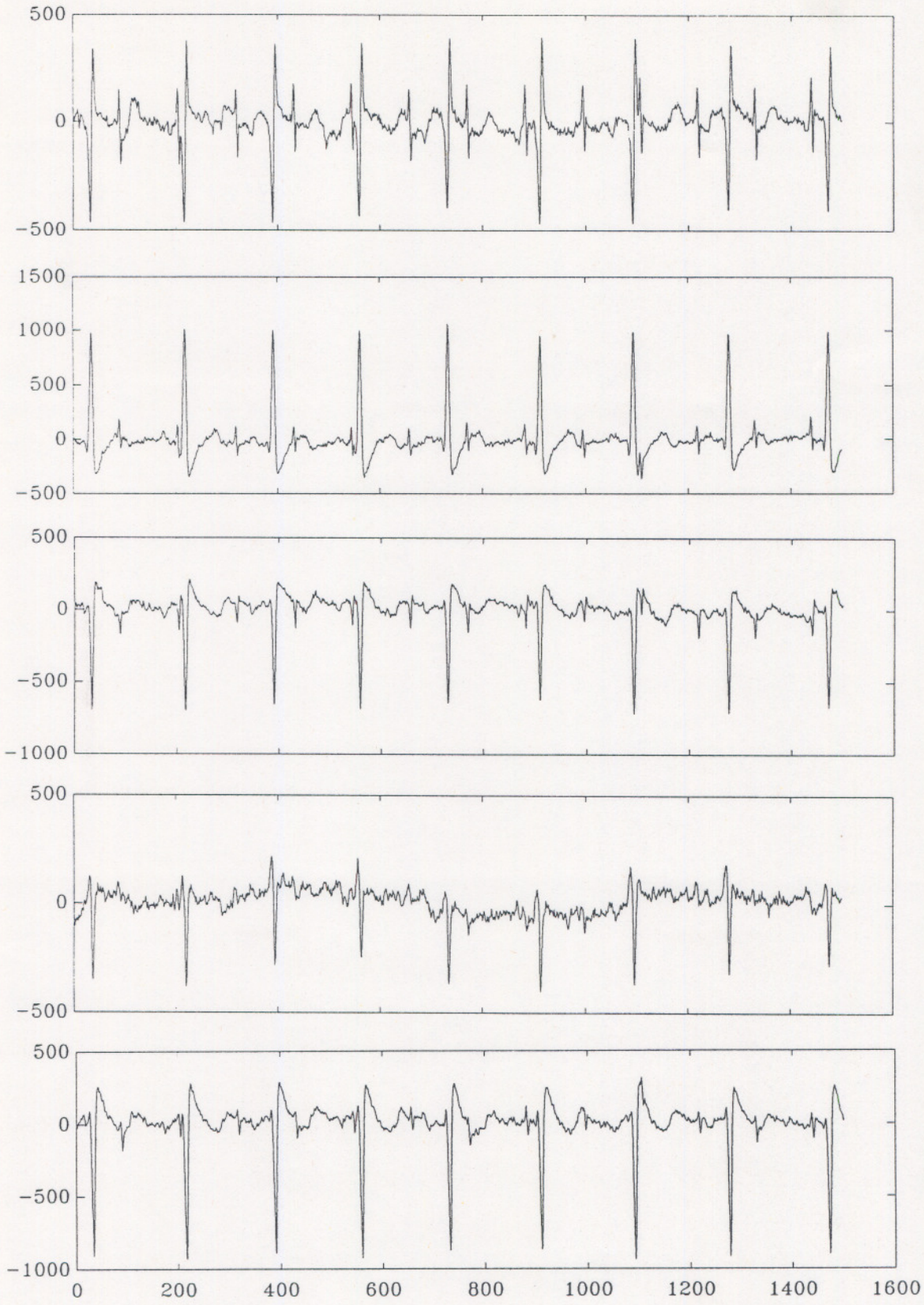

10
5
0
−5
−10
−10
0
10
x1
x2

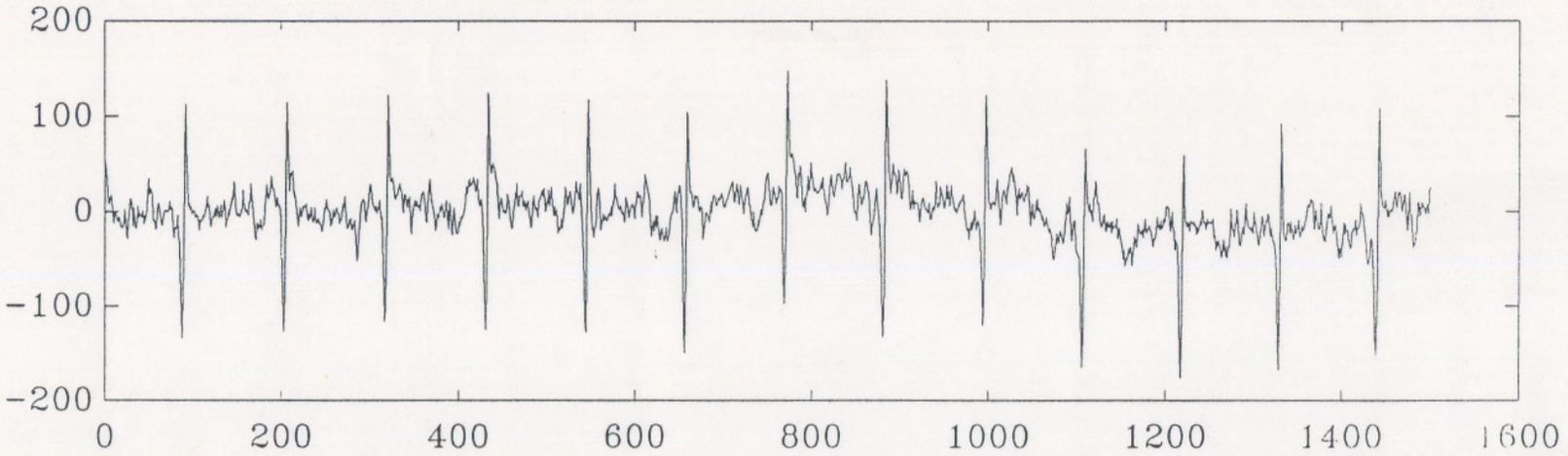
Fig 1

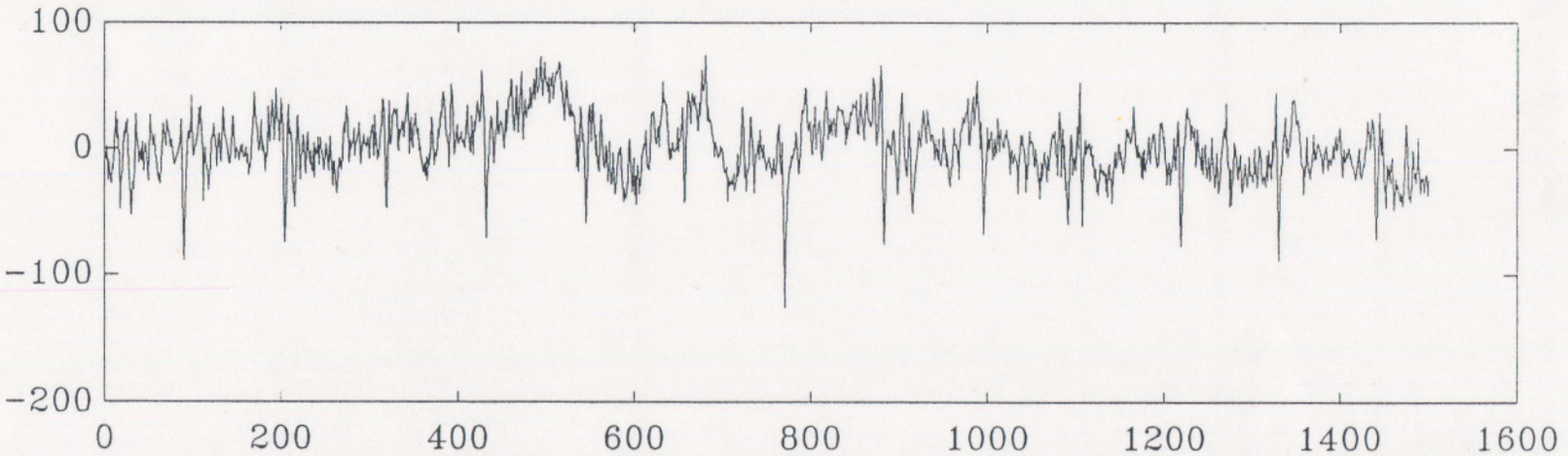

x10⁴
2
1
0
−1
−2
x10⁴
2
1
0
−1
x10⁴
2
1
0
−1
−2
2000
1000
0
−1000
−2000
1000
0
−1000
−2000
1000
500
0
−500
−1000
0
200
400
600
800
1000
1200
1400
1600


x10⁴
4
2
0
−2
−4
5000
0
−5000
−10000
5000
0
−5000
1000
500
0
−500
200
0
−200
−400
200
100
0
−100
−200
0
200
400
600
800
1000
1200
1400
1600

Fig 6


500
0
−500
1500
1000
500
0
−500
500
0
−500
−1000
500
0
−500
500
0
−500
−1000
0
200
400
600
800
1000
1200
1400
1600


200
100
0
−100
−200
0
200
400
600
800
1000
1200
1400
1600


100
0
−100
−200
0
200
400
600
800
1000
1200
1400
1600