Efficient Computation of Symbol Statistics from Bit a Priori Information in Turbo Receivers

Jianfeng Liu, Hilde Vanhaute, Marc Moonen, *Fellow, IEEE,*
André Bourdoux, *Member, IEEE,* and Hugo De Man, *Fellow, IEEE*

***Abstract*—In this paper, an efficient computational scheme is proposed to calculate the symbol mean and variance from bit a priori information, when a so-called multilinear mapping is employed. The multilinear mapping is exploited to reduce the number of the terms needed for the calculation of the symbol mean and variance.**

***Index Terms*—MMSE, iterative detection, turbo receivers.**

I. Introduction

THE concept of turbo processing has been successfully applied to many communication problems [1], such as equalization [2], [3], multi-user detection [4], and MIMO detection [5]. The design of the equalizer or detector part of a turbo receiver is based on two major approaches, namely soft interference cancellation and maximum a posteriori probability (MAP) estimation. The design of the soft interference cancelation receiver attracts special interest due to its reduced implementation complexity.

In the turbo soft interference cancelation receiver, the bit a priori information from the decoder has to be converted to symbols statistics. The symbol mean can be used as an initial estimate to cancel interference from other symbols when estimating the current symbol. Both the symbol mean and the symbol variance are exploited in the minimum mean square error (MMSE) turbo receiver. If the constellation size is $M = 2^Q$, where Q is the number of bits per symbol, the evaluation of the symbol mean and variance comprises a summation of M terms, and each of them involves a nonlinear function $\tanh(x/2)$. Although the principles of the turbo soft interference cancelation receivers have been well studied [1], the issue of how to calculate the symbol statistics from the bit a priori information has received only little attention so far.

In this paper, for the class of so-called multilinear mappings, low complexity formulas are obtained for the computation of the symbol statistics (mean and variance) from the bit a priori information

The structure of the paper is as follows: in Section II, the multilinear mapping is described, and the efficient symbol mean and variance calculation for such a multilinear mapping is derived. Section III draws the conclusion. The proof of Theorem 1 in Section II is included in the Appendix

II. Computing Symbol Statistics from bit a Priori Information

Assume that Q binary bits $\mathbf{b} = (b_1, \ldots, b_Q)$ are mapped into a symbol s and that the a priori bit log-likelihood ratios (LLR) for the bits $\mathbf{b}$ are $(\Lambda(b_1), \ldots, \Lambda(b_Q))$. The symbol s is drawn from a constellation $\{\mathcal{S} \mid s_1, \ldots, s_M\}$, where each of the $M = 2^Q$ constellation points s_i corresponds to the Q bits $\mathbf{b}_i = (b_{i,1}, \ldots, b_{i,Q})$. The formulae for calculating the symbol mean and variance for a general mapping are respectively:

$$\bar{s} = \sum_{s_i \in \mathcal{S}} s_i \cdot P(s = s_i) \tag{1}$$

$$\sigma_s^2 = \left(\sum_{s_i \in \mathcal{S}} |s_i|^2 \cdot P(s = s_i) \right) - |\bar{s}|^2 . \tag{2}$$

If we assume that the bits $\mathbf{b}$ are independent due to interleaving, the symbol probability is

$$P(s = s_i) = \prod_{j=1}^{Q} P(b_j = b_{i,j}) \tag{3}$$

$$= \prod_{j=1}^{Q} \frac{e^{\tilde{b}_{i,j}\Lambda(b_j)}}{1 + e^{\tilde{b}_{i,j}\Lambda(b_j)}} \tag{4}$$

$$= \prod_{j=1}^{Q} \frac{1 + \tilde{b}_{i,j}\xi_j}{2} \tag{5}$$

where $\tilde{b}_{i,j} \triangleq 2b_{i,j} - 1 \in \{-1, 1\}$ and $\xi_j \triangleq \tanh(\Lambda(b_j)/2)$ is called the soft bit, which is equal to ± 1 in the case of perfect a priori information.

It has been demonstrated [6] that the method of mapping bits into symbols, i.e. the assignment of bits into constellation points, is essential to realize the gain of the turbo processing. However, computing (1) and (2) for a general mapping requires $(Q + 1) \cdot 2^Q$ real multiplications, which may be a significant impediment. To efficiently compute the symbol statistics in (1) and (2), it is desirable that the symbol mapping has some structure.

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J. Liu was with the Department of Electrical Engineering (ESAT), K.U.Leuven, Arenberg 10, 3001 Heverlee, Belgium, and IMEC vzw, Kapeldreef 75, 3001 Heverlee, Belgium. He is now with Samsung Electronics Co. Ltd., San #24 Nongseo-Dong, Giheung-Gu, Yougin-city, Gyeonggi-Do, Korea 446-711 (e-mail: jf.liu@samsung.com).

H. Vanhaute and M. Moonen are with the Department of Electrical Engineering (ESAT), K. U. Leuven, Kasteelpark Arenberg 10, 3001 Leuven, Belgium (e-mail: {Hilde.Vanhaute, Marc.Moonen}@esat.kuleuven.be).

A. Bourdoux and H. De Man are with IMEC vzw, Kapeldreef 75, B-3001 Leuven, Belgium (e-mail: {Andre.Bourdoux, Hugo.DeMan}@imec.be).

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TABLE I
EXAMPLES OF MAPPING FUNCTIONS $\phi(\mathbf{b})$.

mapping	symbol mapping $s=\phi(\mathbf{b})$	symbol variance σ_s^2
Gray	$\tilde{b}_1(2-\tilde{b}_2)+\imath\cdot(\tilde{b}_3(2-\tilde{b}_4))$	$10-4(\xi_2+\xi_4)-\lvert\bar{s}\rvert^2$
Mixed	$\tilde{b}_1(1-2\tilde{b}_2)+\imath\cdot(\tilde{b}_3(1-2\tilde{b}_4))$	$10-4(\xi_2+\xi_4)-\lvert\bar{s}\rvert^2$
MSP	$\tilde{b}_3(1+2\tilde{b}_1\tilde{b}_2\tilde{b}_4)+\imath\cdot\tilde{b}_3\tilde{b}_4(1+2\tilde{b}_1)$	$10+4\xi_1(\xi_2\xi_4+1)-\lvert\bar{s}\rvert^2$
M16^a	$-\frac{\tilde{b}_4}{2}\left(\tilde{b}_1(1-\tilde{b}_3)+\tilde{b}_2(1+\tilde{b}_3)\right)+2\tilde{b}_1\tilde{b}_2\tilde{b}_3$ $+\imath\cdot\left(\frac{1}{2}\left(\tilde{b}_1(1+\tilde{b}_3)+\tilde{b}_2(1-\tilde{b}_3)\right)-2\tilde{b}_1\tilde{b}_2\tilde{b}_3\tilde{b}_4\right)$	$10-4\xi_3\xi_4(\xi_1+\xi_2)-\lvert\bar{s}\rvert^2$

Definition 1 (Multilinear mapping): Let $\tilde{\mathbf{b}}_i=2\mathbf{b}_i-1$, then a multilinear mapping function ϕ maps Q input bits $\mathbf{b}_i$ into a symbol s_i as follows:

$$s_i=\phi(\tilde{\mathbf{b}}_i)=\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}a_{q,l}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\quad i=1,\ldots,M \tag{6}$$

where $\mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)$ is the lth product combination out of $\binom{Q}{q}$ combinations of q bits out of Q bits $\tilde{\mathbf{b}}_i$, with $\mathcal{C}_{0,1}^{Q}(\tilde{\mathbf{b}}_i)=1$. Finally, $a_{q,l}$ are constant coefficients.

Some examples of multilinear mappings for 16-QAM are shown in **Table I**, including Gray mapping [7], Mixed mapping [7], modified set partitioning (MSP) [7] and the optimized mapping based on ideal a priori information (M16^a) [8]. Although in general, the mapping ϕ in equation (6) consists of $M=2^Q$ terms, many coefficients $a_{q,l}$ are zero in practice, which leads to simple expressions (the number of terms in the expression is much smaller than M) such as those in Table I.

Theorem 1 (Symbol statistics for the multilinear mapping): When the a priori information $(\Lambda(b_1),\ldots,\Lambda(b_M))$ is available, the symbol mean of the multilinear mapping (6) is

$$\bar{s}=\phi(\boldsymbol{\xi}) \tag{7}$$

where $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_Q)$. The symbol variance is

$$\sigma_s^2=\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}g_{q,l}\ \mathcal{C}_{q,l}^{Q}(\boldsymbol{\xi})-\lvert\bar{s}\rvert^2 \tag{8}$$

where $g_{q,l}$ are constant coefficients computed from $a_{q,l}$, among which many are again equal to zero. The number of terms left in equation (8) depends on $a_{q,l}(q=\{1,\ldots,Q\},\ l=\{1,\ldots,\binom{Q}{q}\})$.

The proof of the above theorem is given in the APPENDIX.

In Theorem 1, the multilinear mapping enables the mean to be calculated by replacing the hard bits $\tilde{b}_j\,(j=1,\ldots,Q)$ in the multilinear mapping function by the soft values ξ_j. In practice, the symbol variance can also be formulated in a simple expression as a large portion of the $g_{q,l}$'s are zero. Some examples of computing the symbol variance are given in Table I. A comparison of operation counts[1] for computing the mean and variance of one 16-QAM symbol is given in **Table II**. As shown in Table II, the computational complexity is greatly reduced by employing multilinear mappings.

[1]In counting the number of operations, two kinds of operations are omitted, including flipping the sign of a variable and multiplying/dividing a variable with 2^k, where k is an integer. In addition, we assume that $\xi_j\ (j=1,\ldots,Q)$ are already available.

TABLE II
OPERATION COUNT FOR COMPUTING THE MEAN AND VARIANCE OF ONE 16-QAM SYMBOL

Operators	Gray		Mixed		MSP		M16^a		a general mapping	
	$\bar{s}$	σ_s^2	$\bar{s}$	σ_s^2	$\bar{s}$	σ_s^2	$\bar{s}$	σ_s^2	$\bar{s}$ by (1)	σ_s^2 by (2)
Real multiplications	2	1	2	1	5	3	10	3	80	64
Real additions	2	3	2	2	2	3	8	3	63	64

III. CONCLUSIONS

In this paper, we have proposed an efficient procedure to calculate the symbol mean and variance from the bit a priori information when a multilinear mapping is employed. The multilinear mapping is exploited to reduce the number of the terms needed for the calculation of the symbol mean and variance.

APPENDIX

Proof of Theorem 1

The mapping is represented by a multilinear function as given in equation (6), repeated here for convenience

$$s_i=\phi(\tilde{\mathbf{b}}_i)=\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}a_{q,l}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\quad i=1,\ldots,M\,. \tag{9}$$

First, it is observed that $P(s=s_i)$ can also be written as a multilinear function of the vector $\boldsymbol{\xi}$,

$$P(s=s_i)=\prod_{j=1}^{Q}\frac{1+\tilde{b}_{i,j}\xi_j}{2} \tag{10}$$

$$=\frac{1}{M}\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}d_{q,l}^{(i)}\mathcal{C}_{q,l}^{Q}(\boldsymbol{\xi}) \tag{11}$$

with

$$d_{q,l}^{(i)}=\mathcal{C}_{q,l}(\tilde{\mathbf{b}}_i)=\pm1\,. \tag{12}$$

Equation (11) is obtained from (10) by explicitly multiplying the factors $(1+\tilde{b}_{i,j}\xi_j)$ and with $M=2^Q$.

By substituting equation (11) and (9) into equation (1), it can be shown that the symbol mean can again be written as a multilinear function of the vector $\boldsymbol{\xi}$,

$$\bar{s}=\sum_{i=1}^{M}P(s=s_i)\phi(\tilde{\mathbf{b}}_i) \tag{13}$$

$$=\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}f_{q,l}\ \mathcal{C}_{q,l}^{Q}(\boldsymbol{\xi}) \tag{14}$$

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where

$$f_{q,l} = \frac{1}{M}\sum_{i=1}^{M} d_{q,l}^{(i)}\ \phi(\tilde{\mathbf{b}}_i) \tag{15}$$

$$= \frac{1}{M}\sum_{i=1}^{M} \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\left(\sum_{r=0}^{Q}\sum_{m=1}^{\binom{Q}{r}} a_{r,m}\ \mathcal{C}_{r,m}^{Q}(\tilde{\mathbf{b}}_i)\right) \tag{16}$$

$$= a_{q,l}\left(\frac{1}{M}\sum_{i=1}^{M}(\mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i))^2\right) + \frac{1}{M}\underbrace{\sum_{r=0}^{Q}\sum_{m=1}^{\binom{Q}{r}}}_{(r,m)\neq(q,l)} a_{r,m}\ \left(\sum_{i=1}^{M}\mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\mathcal{C}_{r,m}^{Q}(\tilde{\mathbf{b}}_i)\right) \tag{17}$$

$$= a_{q,l}\,. \tag{18}$$

Equation (18) is obtained by using the fact that

$$\frac{1}{M}\sum_{i=1}^{M}(\mathcal{C}_{q,l}(\tilde{\mathbf{b}}_i))^2 = 1 \tag{19}$$

$$\sum_{i=1}^{M}\mathcal{C}_{q,l}(\tilde{\mathbf{b}}_i)\mathcal{C}_{r,m}(\tilde{\mathbf{b}}_i) = 0\,. \tag{20}$$

In addition,

$$\sum_{i=1}^{M}\phi(\tilde{\mathbf{b}}_i) = \sum_{i=1}^{M}\sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}} a_{q,l}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i) \tag{21}$$

$$= \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}} a_{q,l}\left(\sum_{i=1}^{M}\mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\right) \tag{22}$$

$$= Ma_{0,1}\,, \tag{23}$$

therefore,

$$a_{0,1} = \frac{1}{M}\sum_{i=1}^{M}\phi(\tilde{\mathbf{b}}_i)\,. \tag{24}$$

According to the theorem of translation invariance [9, Chapter 1], $a_{0,1} \triangleq 0$ without loss of generality.

From equation (14) (18) and (6), we have

$$\bar{s} = \phi(\boldsymbol{\xi})\,. \tag{25}$$

For the calculation of the symbol variance, by using equation (9), we first obtain

$$|s_i|^2 = \left|\phi(\tilde{\mathbf{b}}_i)\right|^2 = \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}\sum_{r=0}^{Q}\sum_{m=1}^{\binom{Q}{q}} \Re\{a_{q,l}\}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\Re\{a_{r,m}\}\ \mathcal{C}_{r,m}^{Q}(\tilde{\mathbf{b}}_i) + \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}}\sum_{r=0}^{Q}\sum_{m=1}^{\binom{Q}{q}} \Im\{a_{q,l}\}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i)\Im\{a_{r,m}\}\ \mathcal{C}_{r,m}^{Q}(\tilde{\mathbf{b}}_i) \tag{26}$$

where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ are the real and imaginary parts of a complex number respectively.

By explicitly expanding the sum over q, l, r, m in equation (26) and using the fact that $|\tilde{b}_{i,j}|^2 = 1$, this expression simplifies to

$$|s_i|^2 = \left|\phi(\tilde{\mathbf{b}}_i)\right|^2 = \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}} g_{q,l}\ \mathcal{C}_{q,l}^{Q}(\tilde{\mathbf{b}}_i) \tag{27}$$

with $g_{q,l}$ a sum of terms which are all products of two $\Re\{a_{r,m}\}$-coefficients and products of two $\Im\{a_{r,m}\}$-coefficients. However, these sums can not be written in a closed form expression for a general mapping function ϕ.

It is important to note that we have written $|s_i|^2$ again as a general multilinear function. Similarly to the derivation of equation (11), (18), and (24), we have

$$\sum_{s_i\in\mathcal{S}}|s_i|^2\cdot P(s=s_i) = \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}} g_{q,l}\ \mathcal{C}_{q,l}^{Q}(\boldsymbol{\xi}) \tag{28}$$

and

$$g_{0,1} = \frac{1}{M}\sum_{i=1}^{M}\left|\phi(\tilde{\mathbf{b}}_i)\right|^2. \tag{29}$$

Using equation (2), we finally obtain

$$\sigma_s^2 = \sum_{q=0}^{Q}\sum_{l=1}^{\binom{Q}{q}} g_{q,l}\ \mathcal{C}_{q,l}^{Q}(\boldsymbol{\xi}) - |\bar{s}|^2\,. \tag{30}$$

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