A Lyapunov Function for Economic Optimizing Model Predictive Control

Moritz Diehl, Rishi Amrit, and James B. Rawlings

***Abstract*—Standard model predictive control (MPC) yields an asymptotically stable steady-state solution using the following procedure. Given a dynamic model, a steady state of interest is selected, a stage cost is defined that measures deviation from this selected steady state, the controller cost function is a summation of this stage cost over a time horizon, and the optimal cost is shown to be a Lyapunov function for the closed-loop system. In this technical note, the stage cost is an arbitrary economic objective, which may not depend on a steady state, and the optimal cost is *not* a Lyapunov function for the closed-loop system. For a class of nonlinear systems and economic stage costs, this technical note constructs a suitable Lyapunov function, and the optimal steady-state solution of the economic stage cost is an asymptotically stable solution of the closed-loop system under economic MPC. Both finite and infinite horizons are treated. The class of nonlinear systems is defined by satisfaction of a strong duality property of the steady-state problem. This class includes linear systems with convex stage costs, generalizing previous stability results [1] and providing a Lyapunov function for economic MPC or MPC with an unreachable setpoint and a linear model. A nonlinear chemical reactor example is provided illustrating these points.**

***Index Terms*—Asymptotic stability, economic cost function, model predictive control (MPC), unreachable setpoint.**

I. Introduction

Model predictive control (MPC) is a widespread feedback control technique whose increasing popularity is based on its ability to treat constrained and multiple-input-multiple-output systems, and because it allows incorporation of general (economic) optimization criteria into the feedback control design. Though the latter reason is frequently mentioned by MPC practitioners who often use economic cost functionals within their heuristic MPC schemes and report good practical performance and stability [2]–[5], there is little theory supporting the use of such economic criteria [6]–[8]. Instead, classical MPC stability theory usually assumes a cost function that penalizes deviations from a desired steady state in order to prove stability of the closed-loop system [9]. In this technical note, we propose and analyze a class of MPC schemes that use an economic cost function, yet admit a Lyapunov function that establishes stability properties.

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II. Economic MPC Versus Tracking MPC

We regard a discrete time, constrained dynamic system

$$x_{k+1} = f(x_k, u_k), \quad g(x_k, u_k) \leq 0$$

$k \in \mathbb{I}_{\geq 0}$, and an economic stage cost function $l(x, u)$. We want to find a feedback control law $u_k = u_e(x_k)$ so that the system remains feasible and (approximately) minimizes the average cost

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{\infty} l(x_k, u_k)$$

We first define the steady-state optimization problem

$$\min_{x,u} l(x, u) \quad \text{s.t.} \quad x - f(x, u) = 0, \quad g(x, u) \leq 0 \tag{1}$$

and its solution (x_s, u_s) that we assume to be unique throughout this technical note. An "economic MPC scheme" addresses a cost function $l(x, u)$ for which there may exist other points (x, u) with $l(x, u) < l(x_s, u_s)$ and $g(x, u) \leq 0$, which, however, do not satisfy the steady-state constraint $x = f(x, u)$. So far, no MPC analysis has been proposed for this case that establishes nominal stability using a Lyapunov function. This is in stark contrast to "tracking MPC", for which a mature body of Lyapunov based stability theory exists, cf. [9] [10, ch. 2]. The paper [1] establishes asymptotic stability for the linear, stabilizable model with strictly convex quadratic cost and an unreachable setpoint, but does not find a Lyapunov function.

III. Nominal Stability of an Economic MPC Formulation

The optimizing MPC scheme we propose solves, for each current system state x, the following optimal control problem with terminal equality constraint:

$$V_N(x) = \min_{x_0, u_0, \ldots, x_N} \sum_{k=0}^{N-1} l(x_k, u_k) \tag{2}$$

$$\text{s.t.} \quad x_0 - x = 0 \tag{3}$$

$$x_{k+1} - f(x_k, u_k) = 0, \quad k \in \mathbb{I}_{0:N-1} \tag{4}$$

$$g(x_k, u_k) \leq 0, \quad k \in \mathbb{I}_{0:N-1} \tag{5}$$

$$x_N - x_s = 0. \tag{6}$$

The solution is denoted $(x_0^*, x_1^*, \ldots, u_0^*, u_1^*, \ldots)$, which also depends on the initial state x, and the feedback control law is denoted $u_e(x) := u_0^*$.

An input sequence $\mathbf{u} = (u_0, u_1, \ldots u_{N-1})$ is termed *feasible* for initial state x if the input sequence and corresponding state sequence generated by the model $x^+ = f(x, u)$ with initial condition $x_0 = x$ together satisfy the constraints of the optimal control problem. We define the admissible set $\mathbb{Z}_N$ as this set of $(x_0, \mathbf{u})$ pairs, i.e.

$$\mathbb{Z}_N = \{(x_0, \mathbf{u}) | \exists x_1, \ldots, x_N : x_{k+1} = f(x_k, u_k), \\ g(x_k, u_k) \leq 0, \forall k \in \mathbb{I}_{0:N-1}, \quad x_N = x_s\}.$$

The set of admissible states $\mathcal{X}_N$ is then defined as the projection of $\mathbb{Z}_N$ onto $\mathbb{R}^n$

$$\mathcal{X}_N = \{x | \exists \mathbf{u} \text{ such that } (x, \mathbf{u}) \in \mathbb{Z}_N\}.$$

We assume throughout that $f(\cdot)$ and $l(\cdot)$ are Lipschitz continuous on the admissible set, i.e., there exist Lipschitz constants $L_f, L_l > 0$ such that for all $(x, u), (x'', u'') \in \mathbb{Z}_N$

$$\begin{aligned} \left|f(x, u) - f(x'', u'')\right| &\leq L_f \left|(x, u) - (x'', u'')\right| \\ \left|l(x, u) - l(x'', u'')\right| &\leq L_l \left|(x, u) - (x'', u'')\right| \end{aligned}.$$

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We require some form of system controllability and make the following assumption.

Assumption 1 (Weak Controllability): There exists a K_∞ function $\gamma(\cdot)$ [11] such that for every $x \in \mathcal{X}_N$, there exists $\mathbf{u}$ such that $(x, \mathbf{u}) \in \mathbb{Z}_N$ and

$$\sum_{k=0}^{N-1} |u_k - u_s| \leq \gamma(|x - x_s|). \tag{7}$$

Assumption 1 is weaker than a controllability assumption, but it bounds the cost of steering an initial state x to x_s. It confines attention to those initial states that can be steered to x_s in N steps while satisfying the control and state constraints, and requires that the cost of the input sequence doing so is not too large. This assumption is satisfied, for example, if the system is linear and stabilizable. Note also that because of the terminal constraint, the set $\mathcal{X}_N$ is forward invariant, i.e. $x \in \mathcal{X}_N$ implies $f(x, u_e(x)) \in \mathcal{X}_N$.

The key additional assumption for MPC with an economic stage cost function $l(x, u)$ is the following.

Assumption 2 (Strong Duality of Steady-State Problem): There exists a multiplier λ_s so that (x_s, u_s) uniquely solves

$$\min_{x,u} \; l(x, u) + [x - f(x, u)]' \lambda_s \quad \text{s.t.} \quad g(x, u) \leq 0. \tag{8}$$

Moreover, there exists a K_∞-function β such that the "rotated" stage cost function[1]

$$L(x, u) := l(x, u) + [x - f(x, u)]' \lambda_s - l(x_s, u_s) \tag{9}$$

satisfies

$$L(x, u) \geq \beta(|x - x_s|) \tag{10}$$

for all (x, u) satisfying $g(x, u) \leq 0$.

See Assumption 4.2 (ii) in [12] for a similar assumption and further general discussion of solving infinite horizon optimal control with unbounded costs. Assumption 2 holds for linear systems with convex constraints and strictly convex costs (if (1) also satisfies a Slater condition), but might exclude some practically relevant settings for nonlinear systems. Its value lies in allowing us to prove the following result.

Theorem 1 (Lyapunov Function): If Assumptions 1 and 2 hold, the steady-state solution of the closed-loop system $x^+ = f(x, u_e(x))$ is asymptotically stable with $\mathcal{X}_N$ as the region of attraction, and admits a Lyapunov function $\tilde{V}_N$ such that $\tilde{V}_N(x^+) \leq \tilde{V}_N(x) - L(x, u_e(x))$. The Lyapunov function is given by

$$\tilde{V}_N(x) := V_N(x) + [x - x_s]' \lambda_s - N l(x_s, u_s). \tag{11}$$

Proof: In order to prove the theorem, we make use of the following lemma, which compares the optimizing MPC law with a similar scheme that uses L as stage cost, i.e., we introduce the following "rotated" MPC problem:

$$\tilde{V}_N(x) = \min_{x_0, u_0, \ldots, x_N} \sum_{k=0}^{N-1} L(x_k, u_k) \tag{12}$$

$$\text{s.t.} \quad x_0 - x = 0 \tag{13}$$

$$x_{k+1} - f(x_k, u_k) = 0 \tag{14}$$

$$g(x_k, u_k) \leq 0, \quad k \in \mathbb{I}_{0:N-1} \tag{15}$$

$$x_N - x_s = 0. \tag{16}$$

■

[1]Note that modified cost is simply $l(x, u) - l(x_s, u_s)$. We termed this rotated cost in [1]. We prefer now to reserve the term rotated cost for (9).

Lemma 2: Problems (2) and (12) deliver the same solution, and indeed $\tilde{V}_N(x) = V_N(x) + [x - x_s]' \lambda_s - N l(x_s, u_s)$.

Proof: We transform the objective function of (12) as follows:

$$\sum_{k=0}^{N-1} L(x_k, u_k) + N l(x_s, u_s) \tag{17}$$

$$= \sum_{k=0}^{N-1} l(x_k, u_k) + [x_k - f(x_k, u_k)]' \lambda_s \tag{18}$$

$$= \sum_{k=0}^{N-1} l(x_k, u_k) + [x_{k+1} - f(x_k, u_k)]' \lambda_s + [x_0 - x_N]' \lambda_s \tag{19}$$

$$= \sum_{k=0}^{N-1} l(x_k, u_k) + [x_{k+1} - f(x_k, u_k)]' \lambda_s + [x_0 - x]' \lambda_s + [x_s - x_N]' \lambda_s + [x - x_s]' \lambda_s \tag{20}$$

$$= \sum_{k=0}^{N-1} l(x_k, u_k) + [x - x_s]' \lambda_s. \tag{21}$$

The last equality holds for any feasible solution, as all the dropped terms contain the equalities of the optimization problems (2) and (12). Thus, the rotated objective differs only by a constant term, and the two solutions are equal. Eq. (11) follows from collecting all terms. ■

Using the above lemma, it is straightforward to note that both MPC schemes deliver the same control law and to use the rotated MPC scheme to prove stability. Stability follows from the fact that $L(x, u)$ is uniquely minimized by (x_s, u_s) by assumption, its optimal value is zero, so that $L(x, u)$ is nonnegative, and non-zero if $x \neq x_s$. Next we show $\tilde{V}_N$ satisfies the properties of a Lyapunov function. First we have that for all $x \in \mathcal{X}_N$

$$\tilde{V}_N(f(x, u_e(x))) - \tilde{V}_N(x) \leq -L(x, u_e(x)) \tag{22}$$

$$\leq -\beta(|x - x_s|). \tag{23}$$

We also have for all $x \in \mathcal{X}_N$

$$\beta(|x - x_s|) \leq \tilde{V}_N(x) \leq \alpha(|x - x_s|) \tag{24}$$

in which $\alpha(\cdot) := (|\lambda_s| + L_l L_F)(\cdot) + L_l(1 + L_F)\gamma(\cdot)$ is a K_∞-function. The first inequality follows from (10) and the second is established in the Appendix. The steady-state solution (x_s) of the closed-loop system $x^+ = f(x, u_e(x))$ is therefore asymptotically stable with $\mathcal{X}_N$ as a region of attraction.

IV. Infinite Horizon Closed Loop Costing

In a second approach we assume that there exists a control law $\kappa(x)$, with $\kappa(x_s) = u_s$, that stabilizes the point x_s (with sufficient rate of convergence), and is feasible and positive invariant on some set containing x_s in its interior. We then formulate the following infinite horizon problem with only a finite number of free optimization variables, $u_0, \ldots, u_{N-1}$:

$$V_\infty(x) := \min_{x_0, u_0, \ldots} \sum_{k=0}^{\infty} [l(x_k, u_k) - l(x_s, u_s)] \tag{25}$$

$$\text{subject to}$$

$$x_0 - x = 0 \tag{26}$$

$$x_{k+1} - f(x_k, u_k) = 0 \tag{27}$$

$$g(x_k, u_k) \leq 0, \quad k \in \mathbb{I}_{\geq 0} \tag{28}$$

$$u_k - \kappa(x_k) = 0, \quad k \in \mathbb{I}_{\geq N}. \tag{29}$$

Note that we subtract the steady-state costs directly from l in order to avoid having to deal with infinite costs. In order to reduce the problem

again to a tracking MPC problem, we proceed analogously as before. We regard again a rotated objective

$$\tilde{V}_\infty(x) := \min_{x_0,u_0,\dots} \sum_{k=0}^{\infty} L(x_k,u_k) \tag{30}$$

subject to

$$x_0 - x = 0 \tag{31}$$

$$x_{k+1} - f(x_k,u_k) = 0 \tag{32}$$

$$g(x_k,u_k) \le 0, \quad k \in \mathbb{I}_{\ge 0} \tag{33}$$

$$u_k - \kappa(x_k) = 0, \quad k \in \mathbb{I}_{\ge N} \tag{34}$$

and first show the following two lemmata:

Lemma 3: If (x_k,u_k) with $k \in \mathbb{I}_{\ge N}$ is the closed-loop system dynamics for the stabilizing auxiliary control law $u_k = \kappa(x_k)$ starting at state x_N, then

$$\underbrace{\sum_{k=N}^{\infty} L(x_k,u_k)}_{=:V_f(x_N)} = \sum_{k=N}^{\infty} [l(x_k,u_k) - l(x_s,u_s)] + [x_N - x_s]'\lambda_s.$$

Proof: For each finite positive integer K we have

$$\begin{aligned}
&\sum_{k=N}^{N+K} L(x_k,u_k) \\
&= \sum_{k=N}^{N+K} [l(x_k,u_k) - l(x_s,u_s)] + [x_N - x_{N+K}]'\lambda_s \\
&= \sum_{k=N}^{N+K} [l(x_k,u_k) - l(x_s,u_s)] + [x_N - x_s]'\lambda_s \\
&+ [\underbrace{x_s - x_{N+K}}_{\to 0}]'\lambda_s
\end{aligned}$$

■

Lemma 4: In problems (25) and (30), the same solutions are obtained and

$$\tilde{V}_\infty(x) := V_\infty(x) + [x - x_s]'\lambda_s. \tag{35}$$

Proof: As before, the cost function of problem (30) can be shown to be

$$\sum_{k=0}^{N-1} L(x_k,u_k) + V_f(x_N) \tag{36}$$

$$= \sum_{k=0}^{N-1} [l(x_k,u_k) - l(x_s,u_s)] + [x_0 - x_N]'\lambda_s + V_f(x_N) \tag{37}$$

$$= \sum_{k=0}^{N-1} [l(x_k,u_k) - l(x_s,u_s)] + [x_0 - x_N]'\lambda_s + \sum_{k=N}^{\infty} [l(x_k,u_k) - l(x_s,u_s)] + [x_N - x_s]'\lambda_s \tag{38}$$

$$= \sum_{k=0}^{\infty} [l(x_k,u_k) - l(x_s,u_s)] + [x_0 - x_s]'\lambda_s. \tag{39}$$

■

From this lemma, we can conclude that the infinite horizon closed-loop costing MPC scheme is stabilizing and admits $\tilde{V}_\infty$ as a Lyapunov function.

Remark: We call a problem with linear f and convex l and g, a *convex* MPC problem. In the convex case, it is straightforward to show that Assumption 2 is satisfied if the steady-state problem (1) is feasible, satisfies a Slater condition, and if l is strictly convex in (x,u). Assumption 2 is in particular satisfied if the constraints are linear (then a Slater condition is not necessary) and the cost is quadratic with positive definite weighting matrices.

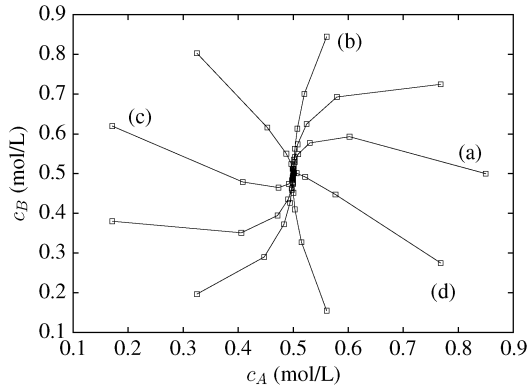


Fig. 1. Phase plot of the closed-loop trajectories from different initial concentrations. All trajectories converge to the optimal steady state.

V. Isothermal CSTR

Consider a single first-order, irreversible chemical reaction in an isothermal CSTR

$$A \longrightarrow B \qquad r = kc_A$$

in which k is the rate constant. The material balances are:

$$\frac{dc_A}{dt} = \frac{Q}{V_R}(c_{Af} - c_A) - k_r c_A$$

$$\frac{dc_B}{dt} = \frac{Q}{V_R}(c_{Bf} - c_B) + k_r c_A$$

in which c_A and c_B are the molar concentrations of A and B respectively, $c_{Af} = 1\ \mathrm{mol/L}$ and $c_{Bf} = 0$ are the feed concentrations of A and B, and Q is the flow through the reactor. The volume of the reactor V_R is fixed at 10 L. The rate constant is 1.2 L/(mol min). The available manipulated variable is the feed flowrate. Non-negativity constraints are imposed on the feed rate. An upper bound of 20 L/min is imposed on the flow rate. This model is nonlinear because of the presence of the bilinear terms Qc_A and Qc_B.

We formulate the economics of the problem based on the price of product B and a separation cost, which is assumed to be directly proportional to the flow rate. Hence the cost function is

$$L(c_A,c_B,Q) = -(2Qc_B - (1/2)Q). \tag{40}$$

The optimal steady state for this cost is $c_A = c_B = 1/2\ \mathrm{mol/L}$, $Q = 12\ \mathrm{L/min}$. Since this steady-state problem does not satisfy strong duality, we add a small quadratic penalty to make the steady-state problem strongly dual

$$\begin{aligned}
L(c_A,c_B,Q) = &-(2Qc_B - (1/2)Q) \\
&+ |c_A - 1/2|^2_{\bar{Q}_A} + |c_B - 1/2|^2_{\bar{Q}_B} \\
&+ |Q - 12|^2_{\bar{R}}
\end{aligned} \tag{41}$$

where $\bar{Q}_A = \bar{Q}_B = 0.505$ and $\bar{R} = 0.505$. The system is initialized at nine different states, and a horizon of $N = 30$ is chosen with the terminal constraint formulation. The sample time is 0.5 minutes. The cost functions (both original and rotated) are reported along the closed-loop profiles. Fig. 1 shows the different closed-loop state

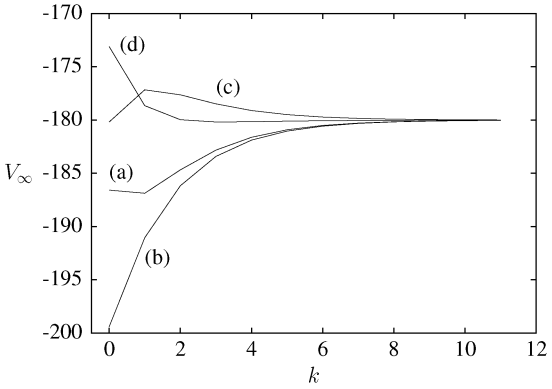


Fig. 2. Original (shifted) cost function profiles along selected closed-loop trajectories.

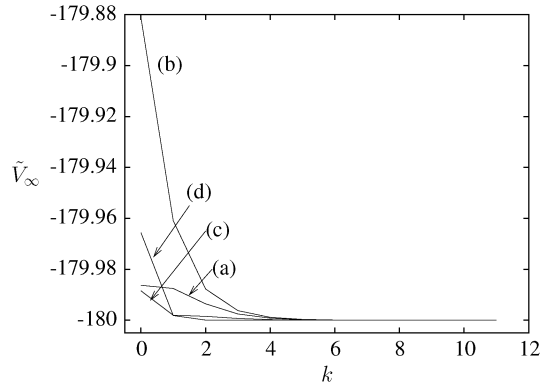


Fig. 3. Rotated cost function profiles along selected closed-loop trajectories.

trajectories converging to the optimal steady state. It is seen that while the original cost function is not strictly decreasing along the closed-loop trajectory (Fig. 2), the rotated cost is strictly decreasing as expected (Fig. 3).

VI. Conclusion

In this work, for the class of economic MPC problems that satisfy strong duality of the steady-state problem, a Lyapunov function was proposed, which allows us to establish asymptotic stability of the closed-loop system. The Lyapunov function was shown to be valid for both terminal control and terminal constraint problems. A nonlinear example that satisfies strong duality of the steady-state problem was presented showing the decrease in the proposed Lyapunov function, even when the economic cost function does not decrease.

Appendix

We wish to establish the upper bounding inequality in (24)

$$\tilde{V}_N(x) \leq \alpha\left(|x - x_s|\right)$$

for $x \in \mathcal{X}_N$. With a slight abuse of notation define the *nonoptimal* rotated cost function

$$\tilde{V}_N(x, \mathbf{u}) = (x - x_s)'\lambda_s + \sum_{k=0}^{N-1} l(x_k, u_k) - l(x_s, u_s)$$

which depends on initial state $x_0 = x$ and input sequence $\mathbf{u} = (u_0, u_1, \ldots, u_{N-1})$. Taking norms and using the triangle inequality and Lipschitz continuity of $l(\cdot)$ gives

$$\tilde{V}_N(x, \mathbf{u}) \leq |\lambda_s||x - x_s| + \sum_{k=0}^{N-1} L_l\left(|x_k - x_s| + |u_k - u_s|\right). \quad (42)$$

Using the Lipschitz continuity of $f(\cdot)$ and the system model, we have that for all $(x, \mathbf{u}) \in \mathbb{Z}_N$ and $k \in \mathbb{I}_{\geq 0}$

$$|x_k - x_s| \leq L_f^k|x - x_s| + L_f^k|u_0 - u_s| + L_f^{k-1}|u_1 - u_s| + \cdots + L_f|u_{k-1} - u_s|.$$

Summing this inequality gives

$$\sum_{k=0}^{N-1} |x_k - x_s| \leq L_F|x - x_s| + L_F \sum_{k=0}^{N-1} |u_k - u_s|$$

in which $L_F = 1 + L_f + \cdots + L_f^{N-1}$. Assumption 1 ensures that for all $x \in \mathcal{X}_N$, there exists $(x, \mathbf{u}) \in \mathbb{Z}_N$ such that

$$\sum_{k=0}^{N-1} |u_k - u_s| \leq \gamma\left(|x - x_s|\right).$$

We then have for such an $(x, \mathbf{u}) \in \mathbb{Z}_N$

$$\sum_{k=0}^{N-1} L_l\left(|x_k - x_s| + |u_k - u_s|\right) \leq L_l L_F|x - x_s| + L_l(1 + L_F)\gamma\left(|x - x_s|\right).$$

Substituting this result into (42) and defining the K_∞-function $\alpha(\cdot) = (|\lambda_s| + L_l L_F)(\cdot) + L_l(1 + L_F)\gamma(\cdot)$ gives for this $(x, \mathbf{u}) \in \mathbb{Z}_N$

$$\tilde{V}_N(x, \mathbf{u}) \leq \alpha\left(|x - x_s|\right).$$

Optimization over $\mathbf{u}$ then gives an optimal value that satisfies for all $x \in \mathcal{X}_N$

$$\tilde{V}_N(x) \leq \alpha\left(|x - x_s|\right)$$

and the result is established.

Appendix
Checking Strong Duality

Strong duality of the steady-state problem is checked numerically by comparing the solutions of the problems (1) and (8) in the CSTR example. Obtaining the same solution for the two problems establishes strong duality. The steady-state problem with modified cost function (41) is

$$\begin{aligned} \min_{c_A, c_B, Q} \quad & L = -(2Qc_B - (1/2)Q) + |c_A - 1/2|^2_{Q_A} \\ & + |c_B - 1/2|^2_{Q_B} + |Q - 12|^2_R \\ \text{s.t.} \quad & \frac{Q}{V_R}(c_{Af} - c_A) - k_r c_A = 0 \\ & \frac{Q}{V_R}(c_{Bf} - c_B) + k_r c_A = 0 \\ & 0 \leq Q \leq 20. \end{aligned} \quad (43)$$

After eliminating c_A and c_B using the equality constraints

$$c_A = \frac{Q c_{Af}}{Q + k_r V_R} \quad c_B = c_{Bf} + \frac{k_r V_R c_{Af}}{Q + k_r V_R}$$

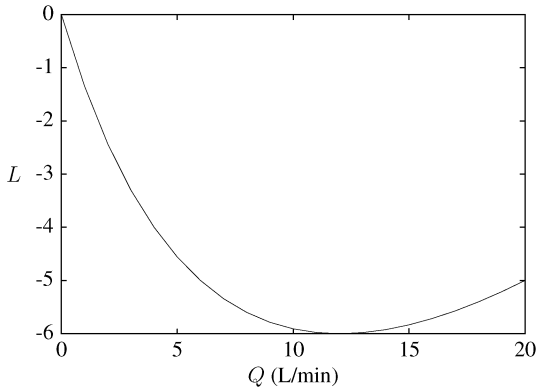


Fig. 4. Steady-state cost as a function of Q after eliminating c_A and c_B.

the cost can be plotted as a function of Q (Fig. 4). $Q = 12\ \mathrm{L/min}$ minimizes the cost which corresponds to $c_A = c_B = 1/2\ \mathrm{mol/L}$.

The second optimization problem (8) for the example is

$$\begin{aligned}\min_{c_A,c_B,Q}\quad & L = -(2Qc_B - (1/2)Q) + |c_A - 1/2|^2_{\tilde{Q}_A} \\ & + |c_B - 1/2|^2_{\tilde{Q}_B} + |Q - 12|^2_{\tilde{R}} \\ & + \lambda_1\left(\frac{Q}{V_R}(c_{Af} - c_A) - k_r c_A\right) \\ & + \lambda_2\left(\frac{Q}{V_R}(c_{Bf} - c_B) + k_r c_A\right) \\ & 0 \le Q \le 20 \end{aligned} \tag{44}$$

where $\lambda_1 = -10$ and $\lambda_2 = -20$ are the optimal Lagrange multipliers of the respective equality constraints in the original steady-state problem (43). This is a quadratic cost with a positive definite Hessian

$$H = \begin{bmatrix} 0.505 & 0 & 0.5 \\ 0 & 0.505 & 0 \\ 0.5 & 0 & 0.505 \end{bmatrix} \quad \mathrm{eig}(H) = \begin{bmatrix} 0.005 \\ 0.505 \\ 1.005 \end{bmatrix}.$$

The minimizer of this problem (44) is $c_A = c_B = 1/2\ \mathrm{mol/L}$, $Q = 12\ \mathrm{L/min}$, and the minimizer is unique. Hence, the solutions of (43) and (44) agree, and the CSTR example satisfies strong duality.

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Comments on "Distributed Control of Spatially Invariant Systems"

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I. INTRODUCTION

The paper [1] was awarded the George S. Axelby Outstanding paper award in 2004 and it has stimulated much new research into spatially invariant systems, a special class of infinite-dimensional systems. A central theme of the paper is that the theory of spatially invariant partial differential systems on an infinite domain can be reduced to the study of a family of finite-dimensional systems parameterized by $\omega \in \mathbb{R}$, which can then be analyzed pointwise. The main motivating examples in [1] were partial differential equations on an infinite domain with a spatial invariance property which can be formulated as state linear systems $\Sigma(A, B, C, D)$ on a Hilbert space Z as in [2], where A generates a strongly continuous semigroup on Z and $B \in \mathcal{L}(U, Z), C \in \mathcal{L}(Z, Y)$, $D \in \mathcal{L}(Z, Y)$ and U, Y are also Hilbert spaces, typically $\mathbf{L}_2(\mathbb{R})^n$ for some integer n. Under mild assumptions one can take Fourier transforms to obtain the isometrically isomorphic state linear system $\Sigma(\mathcal{F}A\mathcal{F}^{-1}, \mathcal{F}B\mathcal{F}^{-1}, \mathcal{F}C\mathcal{F}^{-1}, \mathcal{F}D\mathcal{F}^{-1}) := \Sigma(\check{A}, \check{B}, \check{C}, \check{D})$ with the state space $\mathcal{F}Z$, input space $\mathcal{F}U$ and output space $\mathcal{F}Y$, typically $\mathbf{L}_2(\jmath\mathbb{R})^n$. The new system operators are now matrices of multiplication operators and these are simpler to analyze. In [1], Section II-B, these are seen as infinitely many finite-dimensional systems $\Sigma(\check{A}(\jmath\omega), \check{B}(\jmath\omega), \check{C}(\jmath\omega), \check{D}(\jmath\omega))$ parameterized by $\omega \in \mathbb{R}$, and they suggest pointwise tests for various system theoretic properties of the original system $\Sigma(A, B, C, D)$.

In the introduction and in Section VI.A of [1] the impression is given that the approach is applicable to many distributed control problems including those involving fluid flow. The main theme of the present

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