On Jacobi-Type Methods for Blind Equalization of Paraunitary Channels

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Abstract

In this paper a study of the cumulant-based blind equalization algorithms PAJOD and PAFA is conducted. Both algorithms assume that the data have been pre-whitened and hence the problem reduces to the estimation of paraunitary channels. The first contribution of this paper is an efficient implementation of the PAJOD algorithm called PAJOD2. The second contribution of this paper is a performance comparison between the PAJOD and PAFA algorithms.

Keywords: Blind Equalization, Blind Deconvolution, Tensors, Jacobi Method

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1. Introduction

Blind equalization of linear time-invariant MIMO channels refers to channel equalization techniques where only the observed signal is known. The observed signal is assumed to consist of an unknown convolutive mixture of input signals. The cumulant-based blind equalization algorithms Partial Approximate JOint Diagonalization (PAJOD) and PAraunitary FActorization (PAFA) were proposed in [1] and in [2], respectively. Both are based on contrast maximization and the working assumption for both algorithms is that the data have been pre-whitened. A method to perform pre-whitening has been proposed in [3].

Due to the pre-whitening the problem reduces to a search for a paraunitary equalizer. The PAFA algorithm looks for a paraunitary equalizer while the PAJOD algorithm only searches for a semi-unitary equalizer. The PAFA and PAJOD algorithms both consist of a Jacobi-type iteration where the Jacobi subproblem is solved by a computationally demanding resultant based procedure. The PAFA algorithm requires the rooting of a 56th degree polynomial in each Jacobi subproblem. PAJOD requires the rooting of either a 3rd or 24th degree polynomial in each of its Jacobi subproblems, as will be explained later.

The first contribution of this paper is a more efficient implementation of the PAJOD algorithm. Part of this work has been presented in [4]. The paraunitary equalizer PAFA fully takes the structure of the problem into account while the semi-unitary equalizer PAJOD only partially exploits the structure of the problem. However, the PAJOD algorithm is less computationally demanding than the PAFA algorithm. Hence, the second

contribution will be a comparison of the PAJOD and PAFA algorithms based on computer simulations.

The paper is organized as follows. First the notation used throughout the paper will be introduced. Since the algorithms are based on paraunitary filters and contrast optimization, a few basic notions about paraunitary filters and contrasts will be presented before discussing the system model. Next, in section 2 a brief review of the PAJOD and PAFA algorithms is given. Furthermore, a more efficient implementation of the PAJOD algorithm called PAJOD2 will be presented. Section 3 will compare the PAJOD and PAFA methods based on computer simulations. We end the paper with a conclusion in section 4.

1.1. Notation

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}[z]$ denote the set of natural, integer, real, complex numbers and the set of polynomials in z with coefficients in $\mathbb{C}$, respectively. Furthermore, let $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^\dagger$, $\mathrm{Re}\{\cdot\}$, $\mathrm{Im}\{\cdot\}$ and $\|\cdot\|_F$ denote the conjugate, transpose, conjugate-transpose, pseudo-inverse, real part, imaginary part and the Frobenius norm of a matrix, respectively. The operator $\mathrm{diag}(\cdot)$ sets all the off-diagonal elements of a matrix equal to zero. Let $\mathbf{I}_R \in \mathbb{C}^{R\times R}$ denote the identity matrix. Given $\mathbf{A} \in \mathbb{C}^{m\times n}$, then $\mathbf{A}_{ij}$ denotes the ith row-jth column entry of $\mathbf{A}$. Finally, let $\mathbf{H}(z) = \sum_n \mathbf{H}(n)z^{-n}$.

1.2. Paraunitary Filter

A filter $\mathbf{H}(z) = \sum_{l=0}^{L-1} \mathbf{H}(l)z^{-l} \in \mathbb{C}[z]^{R\times R}$ is paraunitary if $\mathbf{H}^H(\frac{1}{z^*})\mathbf{H}(z) = \mathbf{I}_R$. The paraunitary filter $\mathbf{H}(z)$ satisfies the properties [5]:

- The inverse filter $\mathbf{H}^{-1}(z) = \mathbf{H}^H(\frac{1}{z^*})$ is paraunitary.

- The channel impulse response matrix $\overline{\mathbf{H}} = [\mathbf{H}(0), \mathbf{H}(1), \ldots, \mathbf{H}(L-1)] \in \mathbb{C}^{R \times RL}$ is a semi-unitary matrix, i.e., $\overline{\mathbf{H}}\,\overline{\mathbf{H}}^H = \mathbf{I}_R$.

1.3. Contrast Optimization

The notion of contrast optimization was introduced in [6] and applied in the framework of MIMO equalization in [7]. Let $\mathcal{H}$ and $\mathcal{S}$ be the set of paraunitary filters and the set of transmitted symbol sequences, respectively. Furthermore, let $\mathcal{H} \cdot \mathcal{S}$ denote the set of recovered symbol sequences and $\mathcal{T}$ denote the set of paraunitary equalizers that do not violate the working assumptions on $\mathcal{S}$ specified below. Moreover, let $\mathbf{I}$ denote the identity operator, then a function $\mathcal{J}(\mathbf{H}; \mathbf{x})$ is called a contrast if it satisfies the properties [7]:

- **Invariance:** $\mathcal{J}(\mathbf{H}; \mathbf{x}) = \mathcal{J}(\mathbf{I}; \mathbf{x}),\ \forall \mathbf{H} \in \mathcal{T}, \forall \mathbf{x} \in \mathcal{H} \cdot \mathcal{S}$
- **Domination:** $\mathcal{J}(\mathbf{H}; \mathbf{x}) \leq \mathcal{J}(\mathbf{I}; \mathbf{x}),\ \forall \mathbf{H} \in \mathcal{H}, \forall \mathbf{x} \in \mathcal{S}$
- **Discrimination:** $\mathcal{J}(\mathbf{H}; \mathbf{x}) = \mathcal{J}(\mathbf{I}; \mathbf{x}),\ \forall \mathbf{x} \in \mathcal{S} \Rightarrow \mathbf{H} \in \mathcal{T}$

Under the assumption that there exists an equalizer that will fully recover the symbols, an equalizer corresponding to the global maximum of the contrast function is guaranteed to recover the symbol sequence.

System Model

Let $\mathbf{s}(n), \mathbf{x}(n) \in \mathbb{C}^R$ be the symbol and observation vector at time instant $n \in \mathbb{N}$, respectively. Assume that $\mathbf{s}(n)$ and $\mathbf{x}(n)$ are related via

$$\mathbf{x}(n) = \sum_{k=0}^{K-1} \mathbf{F}(k)\mathbf{s}(n-k),$$

where $\mathbf{F}(k) \in \mathbb{C}^{R\times R}$, $k \in [0, K-1]$, is the channel impulse response of the paraunitary filter $\mathbf{F}(z)$. The problem is to estimate the symbol sequence $\{\mathbf{s}(n)\}$ from the observation sequence $\{\mathbf{x}(n)\}$. This is done by the equalizer $\mathbf{H}(z)$ such that

$$\begin{aligned}\mathbf{y}(n) &= \sum_{l=0}^{L-1} \mathbf{H}(l)\mathbf{x}(n-l) \\ &= \sum_{l=0}^{L-1}\sum_{k=0}^{K-1} \mathbf{H}(l)\mathbf{F}(k)\mathbf{s}(n-l-k),\end{aligned}$$

where $\mathbf{H}(l) \in \mathbb{C}^{R\times R}$, $l \in [0, L-1]$ are the channel impulse response coefficients of $\mathbf{H}(z)$ and $\mathbf{y}(n)$ is the recovered symbol vector at time instant n. Since $\mathbf{F}(z)$ is paraunitary, we know that the equalizer $\mathbf{H}(z)$ is also paraunitary and that $K = L$. In the PAJOD and PAFA methods, the working assumptions on the transmitted signal sequences are:

- $\mathbf{s}_r(n)$ are mutually independent i.i.d., zero-mean processes with unit-variance for all $r \in [1, R]$.
- $\mathbf{s}(n)$ is stationary up to order 4 and hence the marginal cumulants of order 4 do not depend on n.
- At most one source has zero marginal cumulant of order 4.

2. Paraunitary Equalization Algorithms

The PAJOD and PAFA blind paraunitary equalization algorithms will be briefly reviewed and the computationally improved version of PAJOD algorithm called PAJOD2 will be introduced.

2.1. PAJOD

In [1], the cumulants of the observed data are stored in a set of $RL \times RL$ matrices $\mathbf{M}(\mathbf{b}, \gamma)$ in such a way that for a fixed pair $(\mathbf{b}, \gamma) = ([b_1, b_2], [\gamma_1, \gamma_2])$ we have the relation

$$\mathbf{M}_{\alpha_1 R+a_1, \alpha_2 R+a_2}(\mathbf{b}, \gamma) = \mathrm{Cum}[y_{a_1}(n-\alpha_1), y^*_{a_2}(n-\alpha_2), y_{b_1}(n-\gamma_1), y^*_{b_2}(n-\gamma_2)].$$

Moreover, in [1] it is shown that the function

$$\mathcal{J}_2^2 = \sum_{\mathbf{b},\gamma} \left\| \mathrm{diag}\left(\overline{\mathbf{H}} \mathbf{M}(\mathbf{b}, \gamma) \overline{\mathbf{H}}^H \right) \right\|_F^2 \tag{1}$$

is a contrast function, where $\|\mathrm{diag}\,(\mathbf{A})\,\|_F^2 = \sum_i |\mathbf{A}_{ii}|^2$ and $\overline{\mathbf{H}} = [\mathbf{H}(0), \mathbf{H}(1), \ldots, \mathbf{H}(L-1)] \in \mathbb{C}^{R \times RL}$. Due to the paraunitary assumption on $\mathbf{H}(z)$, the matrix $\overline{\mathbf{H}}$ is a semi-unitary matrix, i.e., $\overline{\mathbf{H}}\,\overline{\mathbf{H}}^H = \mathbf{I}_R$.

Jacobi procedure for semi-unitary matrices

To numerically find the semi-unitary matrix $\overline{\mathbf{H}}$ that will maximize the contrast (1) a Jacobi procedure was proposed in [1]. This procedure can be seen as a double extension of the JADE algorithm [8], [9]. First, the unknown matrix is semi-unitary instead of unitary. Second, only the R first diagonal entries are of interest.

A Jacobi procedure is based on the fact that any $RL \times RL$ unitary matrix with determinant equal to one can be parametrized as a product of Givens rotations [10]:

$$\mathbf{V} = \prod_{p=1}^{RL-1} \prod_{q=p+1}^{RL} \Theta[p, q]^H,$$

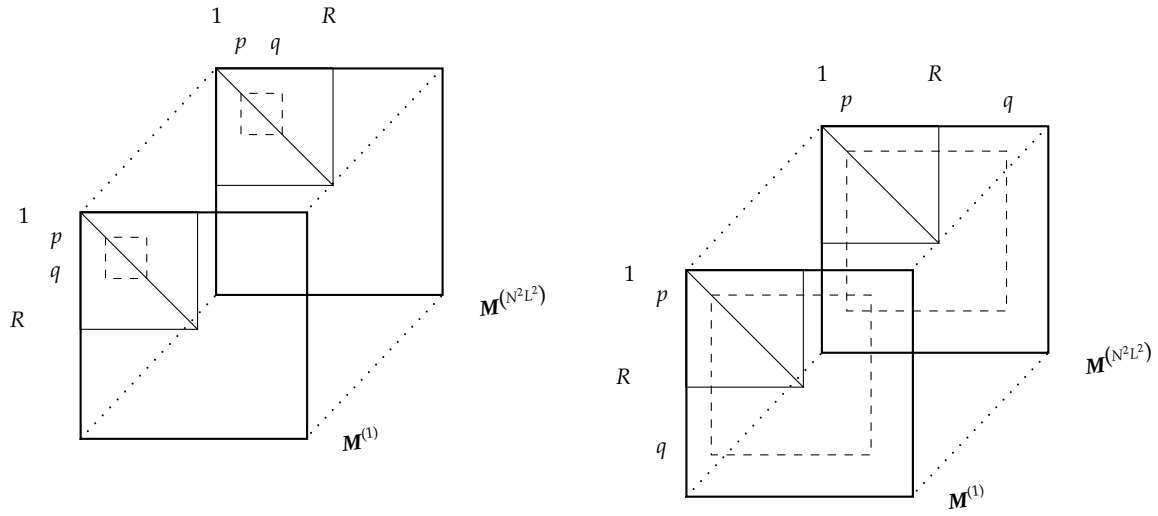
where $\Theta[p,q]$ is equal to the identity matrix, except for entries

$$\Theta_{pp}[p,q] = \Theta_{qq}[p,q] = \cos(\theta[p,q]),$$
$$\Theta_{qp}[p,q] = -\Theta_{pq}[p,q]^* = \sin(\theta[p,q])e^{i\phi[p,q]}, \quad \theta[p,q], \phi[p,q] \in \mathbb{R}.$$

Let $\mathbf{V}$ denote the product of Givens matrices with the initial value $\mathbf{V} = \mathbf{I}_{RL}$. The updating rules are given by $\mathbf{M}(\mathbf{b},\gamma) \leftarrow \Theta[p,q]^H \mathbf{M}(\mathbf{b},\gamma)\Theta[p,q]$ and $\mathbf{V} \leftarrow \Theta[p,q]^H \mathbf{V}$. In the PAJOD algorithm the semi-unitary matrix $\overline{\mathbf{H}}$ is determined as the first R rows of the unitary matrix $\mathbf{V}$. The Givens rotation matrix $\Theta[p,q]$ is chosen as the maximizer of

$$\begin{aligned}
\mathcal{J}_2^2(p,q) &= \sum_{\mathbf{b},\gamma} \sum_{k=1}^{R} \left| \left(\Theta[p,q]^H \mathbf{M}(\mathbf{b},\gamma)\Theta[p,q] \right)_{kk} \right|^2 \\
&= \sum_{\mathbf{b},\gamma} \sum_{k=1}^{R} \left| \sum_{\eta,\mu=1}^{RL} \Theta^*_{\eta k}[p,q]\Theta_{\mu k}[p,q]\mathbf{M}(\mathbf{b},\gamma)_{\eta\mu} \right|^2 \\
&= \sum_{\omega} \sum_{k=1}^{R} \left| \sum_{\eta,\mu=1}^{RL} \Theta^*_{\eta k}[p,q]\Theta_{\mu k}[p,q]\mathbf{M}^{(\omega)}_{\eta\mu} \right|^2 . \qquad (2)
\end{aligned}$$

Notice that we denote $\mathbf{M}(\mathbf{b},\gamma)$ by $\mathbf{M}^{(\omega)}$ for simplicity. Since plane rotations where $p > R$ do not have any effect on the first R rows of the matrices $\mathbf{M}^{(\omega)}$ only Givens rotations where $p \leq R$ are considered. Furthermore one has to distinguish between the cases where $q \leq R$ and $q > R$. The problems are illustrated in figure 1. When $q \leq R$, then the aim of the Givens rotation matrix $\Theta[p,q]$ is to jointly diagonalize the set of matrices $\{\mathbf{M}^{(\omega)}\}$ by maximizing the entries $\mathbf{M}^{(\omega)}_{pp}$ and $\mathbf{M}^{(\omega)}_{qq}$ for all ω as illustrated in figure 1(a). On the other hand, when $q > R$, then the aim of the Givens rotation matrix $\Theta[p,q]$ reduces to jointly diagonalize the set of matrices $\{\mathbf{M}^{(\omega)}\}$ by maximizing the entry $\mathbf{M}^{(\omega)}_{pp}$ for all ω as illustrated in figure 1(b).



(a) $q \leq R$ (b) $q > R$

Figure 1: The figures illustrates the PAJOD optimization problems for the case when **b** varies in $\{1, \ldots, R\}^2$ and γ in $\{0, \ldots, L-1\}^2$.

Let $\overline{\mathbf{M}}^{(\omega)} = \Theta[p,q]^H \mathbf{M}^{(\omega)} \Theta[p,q]$ and for notational convenience let $c = cos(\theta[p,q])$ and $s = \sin(\theta[p,q]) e^{i\phi[p,q]}$. Then for the case where $q \leq R$ equation (2) is equal to

$$\mathcal{J}_2^2(p,q)\Big|_{q \leq R} = \sum_{\omega} |\overline{\mathbf{M}}_{pp}^{(\omega)}|^2 + |\overline{\mathbf{M}}_{qq}^{(\omega)}|^2, \tag{3}$$

where

$$\overline{\mathbf{M}}_{pp}^{(\omega)} = \begin{bmatrix} c & s^* \end{bmatrix} \begin{bmatrix} \mathbf{M}_{pp}^{(\omega)} & \mathbf{M}_{pq}^{(\omega)} \\ \mathbf{M}_{qp}^{(\omega)} & \mathbf{M}_{qq}^{(\omega)} \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix}$$

and

$$\overline{\mathbf{M}}_{qq}^{(\omega)} = \begin{bmatrix} c & -s \end{bmatrix} \begin{bmatrix} \mathbf{M}_{pp}^{(\omega)} & \mathbf{M}_{pq}^{(\omega)} \\ \mathbf{M}_{qp}^{(\omega)} & \mathbf{M}_{qq}^{(\omega)} \end{bmatrix} \begin{bmatrix} c \\ -s^* \end{bmatrix}.$$

The maximization problem (3) is equivalent to the JADE diagonalization problem and therefore the JADE algorithm [8] can be applied to solve this problem.

When $q > R$ only the first diagonal term should be maximized, and equation (2) reduces to

$$\mathcal{J}_2^2(p,q)\Big|_{q>R} = \sum_p |\overline{\mathbf{M}}_{pp}^{(\omega)}|^2. \tag{4}$$

In [1] a resultant based [11] approach was taken to solve the maximization problem (4). It amounted to the rooting of a polynomial of degree 24 that has at most 8 real roots.

2.2. PAJOD2

This section will introduce the PAJOD2 algorithm, which is a computational improved version of the PAJOD algorithm. When $q \leq R$, then the PAJOD2 algorithm will apply the JADE algorithm to solve the Jacobi subproblem, just as in the PAJOD case.

When $q > R$ a more efficient eigenvector based approach will be proposed. In the derivation we will make use of the trigonometric identities

$$\begin{aligned}
2\cos^2(\theta) &= (1 + \cos(2\theta)) \\
2\sin^2(\theta) &= (1 - \cos(2\theta)) \\
2\cos(\theta)\sin(\theta) &= \sin(2\theta) \\
\cos(2\phi) &= \cos^2(\phi) - \sin^2(\phi) \\
\sin(2\phi) &= 2\cos(\phi)\sin(\phi)
\end{aligned}$$

Let $\theta = \theta[p,q]$, $\phi = \phi[p,q]$, $\hat{c} = \cos(2\theta)$, $\hat{s} = \sin(2\theta)$, $\alpha^{(\omega)} = \mathbf{M}_{pp}^{(\omega)} - \mathbf{M}_{qq}^{(\omega)}$ and

$\beta^{(\omega)} = \mathbf{M}_{pp}^{(\omega)} + \mathbf{M}_{qq}^{(\omega)}$, then equation (4) can be written as

$$\begin{aligned}\mathcal{J}_2^2(p,q)\Big|_{q>R} = \frac{1}{4}\sum_{\omega} \left|\beta^{(\omega)}\right|^2 + \left|\alpha^{(\omega)}\right|^2 \hat{c}^2 + 2\mathrm{Re}\left\{\beta^{(\omega)*}\alpha^{(\omega)}\right\}\hat{c} \\ + \left(\left|\mathbf{M}_{pq}^{(\omega)}\right|^2 + \left|\mathbf{M}_{qp}^{(\omega)}\right|^2 + 2\mathrm{Re}\left\{\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)}e^{i2\phi}\right\}\right)\hat{s}^2 \\ + 2\mathrm{Re}\left\{\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)*}e^{i\phi} + \mathbf{M}_{qp}^{(\omega)*}e^{-i\phi}\right)\right\}\hat{c}\hat{s} \\ - 2\mathrm{Re}\left\{\beta^{(\omega)*}\left(\mathbf{M}_{pq}^{(\omega)}e^{-i\phi} + \mathbf{M}_{qp}^{(\omega)}e^{i\phi}\right)\right\}\hat{s}. \qquad (5)\end{aligned}$$

By inspection of (5) we can identify the term independent of θ and ϕ as

$$k = \frac{1}{4}\sum_{\omega}\left|\beta^{(\omega)}\right|^2 = \frac{1}{4}\sum_{\omega}|\mathbf{M}_{pp}^{(\omega)} + \mathbf{M}_{qq}^{(\omega)}|^2. \qquad (6)$$

The linear terms in the variables $\hat{c}$ and $\hat{s}$ of (5) can be written as

$$\begin{aligned}l(\hat{c},\hat{s}) &= \frac{1}{2}\sum_{\omega}\mathrm{Re}\left\{\beta^{(\omega)*}\alpha^{(\omega)}\right\}\hat{c} - \mathrm{Re}\left\{\beta^{(\omega)*}\left(\mathbf{M}_{pq}^{(\omega)}e^{-i\phi} + \mathbf{M}_{qp}^{(\omega)}e^{i\phi}\right)\right\}\hat{s} \\ &= \frac{1}{2}\sum_{\omega}\mathrm{Re}\left\{\beta^{(\omega)*}\alpha^{(\omega)}\right\}\hat{c} - \mathrm{Re}\left\{\beta^{(\omega)*}\left(\mathbf{M}_{pq}^{(\omega)} + \mathbf{M}_{qp}^{(\omega)}\right)\right\}\hat{s}\cos(\phi) \\ &+ \mathrm{Re}\left\{i\beta^{(\omega)*}\left(\mathbf{M}_{pq}^{(\omega)} - \mathbf{M}_{qp}^{(\omega)}\right)\right\}\hat{s}\sin(\phi) \\ &= \sum_{\omega}\mathbf{g}^{(\omega)T}\mathbf{v}, \qquad (7)\end{aligned}$$

where

$$\mathbf{v} = \begin{bmatrix} \cos(2\theta[p,q]) \\ \sin(2\theta[p,q])\cos(\phi[p,q]) \\ \sin(2\theta[p,q])\sin(\phi[p,q]) \end{bmatrix}$$

$$\mathbf{z}^{(\omega)} = \frac{1}{2}\begin{bmatrix} \mathbf{M}_{pp}^{(\omega)} - \mathbf{M}_{qq}^{(\omega)} \\ -(\mathbf{M}_{pq}^{(\omega)} + \mathbf{M}_{qp}^{(\omega)}) \\ i(\mathbf{M}_{pq}^{(\omega)} - \mathbf{M}_{qp}^{(\omega)}) \end{bmatrix}$$

$$\mathbf{g}^{(\omega)} = \mathrm{Re}\left\{\left(\mathbf{M}_{pp}^{(\omega)} + \mathbf{M}_{qq}^{(\omega)}\right)^{*} \mathbf{z}^{(\omega)}\right\}$$

The quadratic term of (5) will now be written as

$$\begin{aligned}
q(\hat{c},\hat{s}) &= \frac{1}{4}\sum_{\omega} \left|\alpha^{(\omega)}\right|^2 \hat{c}^2 + 2\mathrm{Re}\left\{\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)*} e^{i\phi} + \mathbf{M}_{qp}^{(\omega)*} e^{-i\phi}\right)\right\}\hat{c}\hat{s} \\
&+ \left(\left|\mathbf{M}_{pq}^{(\omega)}\right|^2 + \left|\mathbf{M}_{qp}^{(\omega)}\right|^2 + 2\mathrm{Re}\left\{\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)} e^{i2\phi}\right\}\right)\hat{s}^2 \\
&= \frac{1}{4}\sum_{\omega} \left|\alpha^{(\omega)}\right|^2 \hat{c}^2 + \left(\left|\mathbf{M}_{pq}^{(\omega)}\right|^2 + \left|\mathbf{M}_{qp}^{(\omega)}\right|^2\right)\hat{s}^2\left(\cos^2(\phi) + \sin^2(\phi)\right) \\
&+ 2\mathrm{Re}\left\{\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)}\right\}\hat{s}^2\cos(2\phi) + 2\mathrm{Re}\left\{i\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)}\right\}\hat{s}^2\sin(2\phi) \\
&+ 2\mathrm{Re}\left\{\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)} + \mathbf{M}_{qp}^{(\omega)}\right)^{*}\right\}\hat{c}\hat{s}\cos(\phi) \\
&+ 2\mathrm{Re}\left\{i\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)} - \mathbf{M}_{qp}^{(\omega)}\right)^{*}\right\}\hat{c}\hat{s}\sin(\phi) \\
&= \frac{1}{4}\sum_{\omega} \left|\alpha^{(\omega)}\right|^2 \hat{c}^2 + \left(\left|\mathbf{M}_{pq}^{(\omega)}\right|^2 + \left|\mathbf{M}_{qp}^{(\omega)}\right|^2\right)\hat{s}^2\left(\cos^2(\phi) + \sin^2(\phi)\right) \\
&+ 2\mathrm{Re}\left\{\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)}\right\}\hat{s}^2\left(\cos^2(\phi) - \sin^2(\phi)\right) \\
&+ 4\mathrm{Re}\left\{i\mathbf{M}_{pq}^{(\omega)*}\mathbf{M}_{qp}^{(\omega)}\right\}\hat{s}^2\cos(\phi)\sin(\phi) \\
&+ 2\mathrm{Re}\left\{\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)} + \mathbf{M}_{qp}^{(\omega)}\right)^{*}\right\}\hat{c}\hat{s}\cos(\phi) \\
&+ 2\mathrm{Re}\left\{\alpha^{(\omega)}\left(\mathbf{M}_{pq}^{(\omega)} - \mathbf{M}_{qp}^{(\omega)}\right)^{*}\right\}\hat{c}\hat{s}\sin(\phi) \\
&= \mathbf{v}^T \sum_{\omega} \mathbf{G}^{(\omega)}\mathbf{v}, \qquad (8)
\end{aligned}$$

where $\mathbf{G}^{(\omega)} = \mathrm{Re}\left\{\mathbf{z}^{(\omega)}\mathbf{z}^{(\omega)H}\right\}$.

From the equations (6), (7) and (8), equation (4) can be reformulated as

$$\mathcal{J}_2^2(p,q)\Big|_{q>R} = \mathbf{v}^T\mathbf{G}\mathbf{v} + \mathbf{g}^T\mathbf{v} + k, \tag{9}$$

where $\mathbf{G} = \sum_\omega \mathbf{G}^{(\omega)}$ and $\mathbf{g} = \sum_\omega \mathbf{g}^{(\omega)}$. We should maximize (9) under the constraint $\|\mathbf{v}\|_F^2 = 1$.

Maximizing (9) subject to the constraint that $\|\mathbf{v}\|_F^2 = 1$ is a classical problem. In the context of source separation, it for instance appeared in [12]. Using the Lagrange multiplier method leads to

$$2\left(\mathbf{G} + \lambda\mathbf{I}_3\right)\mathbf{v} + \mathbf{g} = 0\,,\ \lambda \in \mathbb{R}. \tag{10}$$

Assuming that $\left(\mathbf{G} + \lambda\mathbf{I}_3\right)^{-1}$ exists, we have

$$\mathbf{v} = -\frac{1}{2}\left(\mathbf{G} + \lambda\mathbf{I}_3\right)^{-1}\mathbf{g}.$$

Given the eigenvalue decomposition $\mathbf{G} = \mathbf{E}\Lambda\mathbf{E}^T$, we have

$$\|\mathbf{v}\|_F^2 = 1 \Leftrightarrow \frac{1}{4}\sum_{n=1}^{3}\frac{\left(\mathbf{E}_n^T\mathbf{g}\right)^2}{\left(\Lambda_{nn} + \lambda\right)^2} = 1, \tag{11}$$

where $\mathbf{E}_n$ and Λ_{nn} denote the nth eigenvector and eigenvalue of $\mathbf{G}$, respectively. From (11) one can deduce that the problem amounts to rooting a polynomial of degree 6 and thereafter selecting the root of which the corresponding $\mathbf{v}$ maximizes $\mathcal{J}_2^2(p,q)$.

If $\left(\mathbf{G} + \lambda\mathbf{I}_3\right)^{-1}$ does not exist[2], which could occur if $\lambda = 0$ and $\mathbf{G}$ is singular or when $\lambda = -\Lambda_{nn}$ for some n, then we have to resort to (10) for

[2]This case has not been observed in our simulations.

the computation of $\mathbf{v}$ [12]:

$$\mathbf{v} = -\frac{1}{2}\left(\mathbf{G} - \Lambda_{nn}\mathbf{I}_3\right)^{\dagger}\mathbf{g} + c_n\mathbf{E}_n,$$

where c_n is a real constant chosen such that $\|\mathbf{v}\|_F^2 = 1$ and $\mathcal{J}_2^2(p,q)$ is maximum. If it exists, it is given by

$$c_n = \operatorname{sign}(\mathbf{E}_n^T\mathbf{g})\sqrt{1 - \|\left(\mathbf{G} - \Lambda_{nn}\mathbf{I}_3\right)^{\dagger}\mathbf{g}\|^2/4}.$$

2.3. PAFA

The PAFA algorithm is based on the fact that any FIR paraunitary filter $\mathbf{H}(z) \in \mathbb{C}[z]^{R\times R}$ of length L can be factorized as [5]:

$$\mathbf{H}(z) = \mathbf{Q}^{(M)}\mathbf{Z}(z)\mathbf{Q}^{(M-1)}\mathbf{Z}(z)\cdots\mathbf{Z}(z)\mathbf{Q}^{(0)}, \tag{12}$$

where M is the McMillan degree of $\mathbf{H}(z)$, $\mathbf{Q} \in \mathbb{C}^{R\times R}$ is unitary and $\mathbf{Z}(z) = \begin{bmatrix} \mathbf{I}_{R-1} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix}$.

The factorization (12) fully takes the paraunitary structure of the problem into account while PAJOD does not. An approach that cyclically estimates one unitary matrix $\mathbf{Q}^{(m)}$ while the other matrices are fixed was proposed in [2]. First (12) is reformulated as

$$\mathbf{H}(z) = \mathbf{A}(z)\mathbf{Q}^{(m)}\mathbf{B}(z), \tag{13}$$

where $\mathbf{A}(z) = \mathbf{Q}^{(M)}\mathbf{Z}(z)\cdots\mathbf{Z}(z)\mathbf{Q}^{(m+1)}$ and $\mathbf{B}(z) = \mathbf{Q}^{(m-1)}\mathbf{Z}(z)\cdots\mathbf{Z}(z)\mathbf{Q}^{(0)}$ are of length $l_A \leq L$ and $l_B \leq L$, respectively, which satisfy $l_A + l_B = L$. The relation between the signal vector after the equalization of $\mathbf{y}(n)$ and the observation vector $\mathbf{x}(n)$ can now be written as

$$\mathbf{y}_r(n) = \sum_{q,s,t}\sum_{u,p}\mathbf{A}_{rq}(u)\mathbf{Q}_{qs}^{(m)}\mathbf{B}_{st}(p)\mathbf{x}_t(n-p-u),$$

where the range of the variables is $u \in \{0, \ldots l_A\}$, $p \in \{0, \ldots l_B\}$ and $r, q, s, t \in \{1, \ldots R\}$. Let the output vector of the filter $\mathbf{B}(z)$ from equation (13) be denoted by $\mathbf{w}(n)$, then we also have the relation

$$\mathbf{y}_r(n) \;\; = \;\; \sum_{qs} \sum_{u} \mathbf{A}_{rq}(u) \mathbf{Q}^{(m)}_{qs} \mathbf{w}_s(n-u).$$

The fourth-order cumulants of $\mathbf{x}(n)$ will be denoted as

$$C_2^{2,\mathbf{x}}[r, \mathbf{k}] = \mathrm{Cum}[x_r(n-k_1), x_r^*(n-k_2), x_r(n-k_3), x_r^*(n-k_4)]$$
$$C_2^{2,\mathbf{x}}[\mathbf{r}, \mathbf{k}] = \mathrm{Cum}[x_{r_1}(n-k_1), x_{r_2}^*(n-k_2), x_{r_3}(n-k_3), x_{r_4}^*(n-k_4)].$$

where $\mathbf{r} = [r_1, r_2, r_3, r_4]$ and $\mathbf{k} = [k_1, k_2, k_3, k_4]$. Under the assumption that the signs $\epsilon = \pm 1$ of the kurtosis of the different signals are the same, it was shown in [7] that the function

$$\mathcal{J}_1^4 = \epsilon \sum_{r=1}^{R} C_2^{2,\mathbf{y}}[r, \mathbf{0}] \tag{14}$$

is a contrast, where

$$C_2^{2,\mathbf{y}}[r, \mathbf{0}] = \sum_{\mathbf{s}} \sum_{\mathbf{u}} \sum_{\mathbf{q}} \mathbf{A}_{rq_1}(u_1) \mathbf{A}^*_{rq_2}(u_2) \mathbf{A}_{rq_3}(u_3) \mathbf{A}^*_{rq_4}(u_4) \mathbf{Q}^{(m)}_{q_1 s_1} \mathbf{Q}^{(m)*}_{q_2 s_2} \mathbf{Q}^{(m)}_{q_3 s_3} \mathbf{Q}^{(m)*}_{q_4 s_4} C_2^{2,\mathbf{w}}[\mathbf{s}, \mathbf{u}],$$

$$C_2^{2,\mathbf{w}}[\mathbf{s}, \mathbf{u}] = \sum_{\mathbf{p}} \sum_{\mathbf{j}} \mathbf{B}_{s_1 j_1}(p_1) \mathbf{B}^*_{s_2 j_2}(p_2) \mathbf{B}_{s_3 j_3}(p_3) \mathbf{B}^*_{s_4 j_4}(p_4) C_2^{2,\mathbf{x}}[\mathbf{j}, \mathbf{u} + \mathbf{p}]$$

and $\mathbf{s}, \mathbf{q}, \mathbf{j} \in \{1, \ldots, R\}^4$, $\mathbf{u} \in \{0, \ldots, l_A\}^4$ and $\mathbf{p} \in \{0, \ldots, l_B\}^4$.

For the two-channel case, $R = 2$, the computation of the rotation matrix $\mathbf{Q}^{(m)}$ that will maximize (14) amounts to the rooting of a 56th degree polynomial [13] when a resultant based procedure is employed. This polynomial generally does not admit more than 16 real roots, see [2], [13] for further details. Hence, the PAFA algorithm must be considered as computationally more demanding than the PAJOD algorithm. For the case $R > 2$ no solution for the PAFA problem was given in [2].

3. Simulation Results

The algorithms PAJOD, PAJOD2 and PAFA will be tested on random 2-Input-2-Output channels of varying SNR. The transmitted data blocks consist each of a QPSK sequence of 512 symbols length. The paraunitary channel is generated, just as in [1], as follows:

$$\mathbf{F}(z) = \mathbf{R}(\phi_0, \theta_0) \prod_{m=1}^{L-1} \mathbf{Z}(z)\mathbf{R}(\phi_m, \theta_m)$$

where

$$\mathbf{Z}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \quad \mathbf{R}(\phi_m, \theta_m) = \begin{bmatrix} \cos(\phi_m) & -\sin(\phi_m)\mathrm{e}^{-i\theta_m} \\ \sin(\phi_m)\mathrm{e}^{i\theta_m} & \cos(\phi_m) \end{bmatrix}$$

and the parameters ϕ_m and θ_m are drawn from a uniform distribution over $[0, 2\pi)$. The filtered QPSK sequences of unit variance are perturbed by an additive white circular complex Gaussian noise with identity covariance matrix.

To see if the PAFA algorithm, which fully exploits the paraunitary structure of the channel, is better than the PAJOD algorithm, which only partially exploits the paraunitary structure of the channel, the performance will be based on the distance between the estimated and the true equalizer and the Symbol Error Rate (SER). Furthermore, the computational complexity of the algorithms will also serve as a basis of comparison.

Due to the inherent indeterminacies of the equalization problem, the equalizer can only be estimated up to a matrix of the form $\mathbf{D}(z) = \Lambda(z)\mathbf{P}$, where $\mathbf{P}$ is a permutation matrix and $\Lambda(z)$ is diagonal matrix with entries of the form λz^k, $k \in \mathbb{Z}$. Let $\mathbf{G}$ be the $R \times R(2L-1)$ matrix obtained by stacking

the global impulse response matrices after each other, then a measure of how far we are from perfect equalization is given by

$$\text{dist}(\mathbf{G}) = \min_{\mathbf{D}(z)} \|\mathbf{G}\mathbf{D}(z)\|_F.$$

A procedure to solve this problem was given in [14] and will not be repeated here.

3.1. Paraunitary Channel with Length $K = 3$.

In the first simulation, the paraunitary channel and equalizer are of length $K = L = 3$.

The mean and median dist values over 200 simulations can be seen in figure 2. From the figure it is observed that the PAFA algorithm seems to capture the channel structure better than the PAJOD algorithm.

A second measure of the performance is given by the SER. Mean SER and median SER results for the PAJOD and PAFA equalizer can be seen in figure 3. By inspection of the figure it is observed that above 10 dB the algorithms have similar performance and below 10 dB PAFA has a better performance.

To measure the elapsed time used to execute the algorithms in MATLAB, the built-in functions *tic*$(\cdot)$ and *toc*$(\cdot)$ were used. The mean and median time results can be seen in figure 4. By inspection of figure 4 it is clear that the PAJOD2 algorithm is cheaper than the PAJOD algorithm.

3.2. Paraunitary Channel with Varying Length K.

A second simulation was conducted in order to investigate the computation time of the algorithms as a function of the filter length. The

paraunitary channel and equalizer are of the same length $K = L$. The SNR was fixed to 10 while the filter and channel length varied from 2 to 10 with a hop factor of 1.

The mean computation time over 10 simulations result can be seen in figure 5. Here it is clear that the computational complexity of the PAFA algorithm grows exponentially as a function of the filter length.

4. Conclusion

In this paper the problem of blind equalization of paraunitary channels by means of Jacobi-type procedures was studied. After a review of the PAJOD and PAFA methods we proposed a computationally more efficient method called PAJOD2. The proposed method simplified the Jacobi-subproblem of the PAJOD method by going from the rooting of a 24th degree polynomial to the rooting of a polynomial of degree 6.

Simulation results indicate that for low SNR the PAFA method is more accurate that the PAJOD method. However, for high SNR both methods perform similarly. Such a comparison has not been conducted in the literature. Furthermore, computer simulations confirmed that the PAJOD2 method is faster than the PAJOD method.

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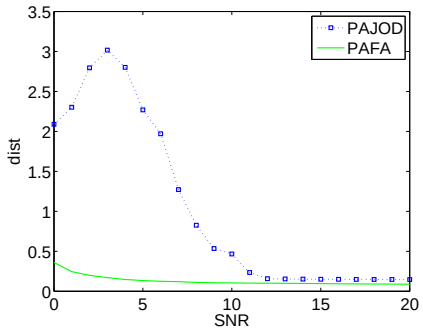
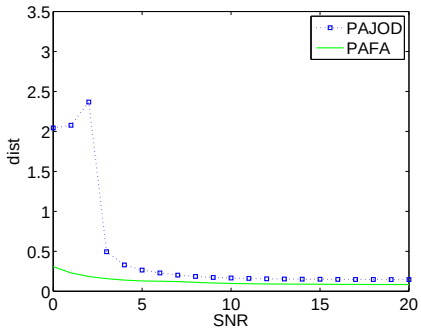
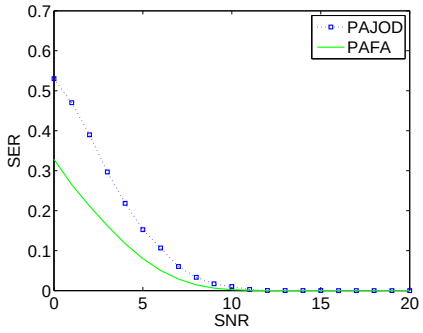


(a) Mean dist.

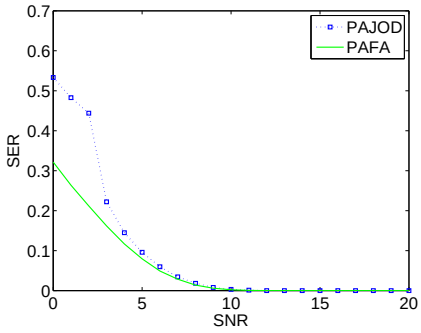


(b) Median dist.

Figure 2: Mean and median dist from perfect equalization in the first experiment.

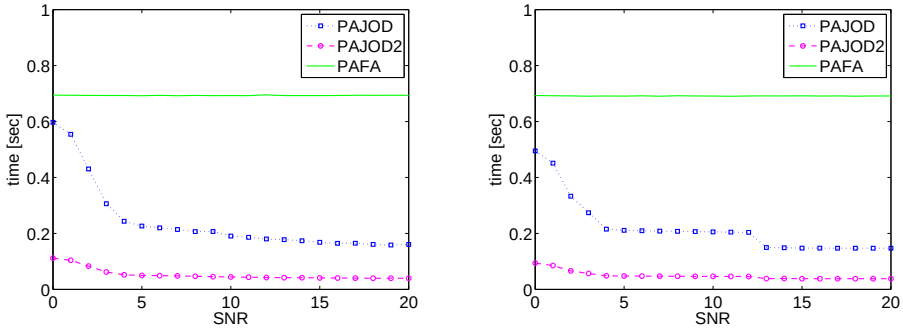


(a) Mean SER.



(b) Median SER.

Figure 3: Mean and median SER in the first experiment.



(a) Mean computation time. (b) Median computation time.

Figure 4: Mean and median values for the computation time for the first experiment as measured by MATLAB.

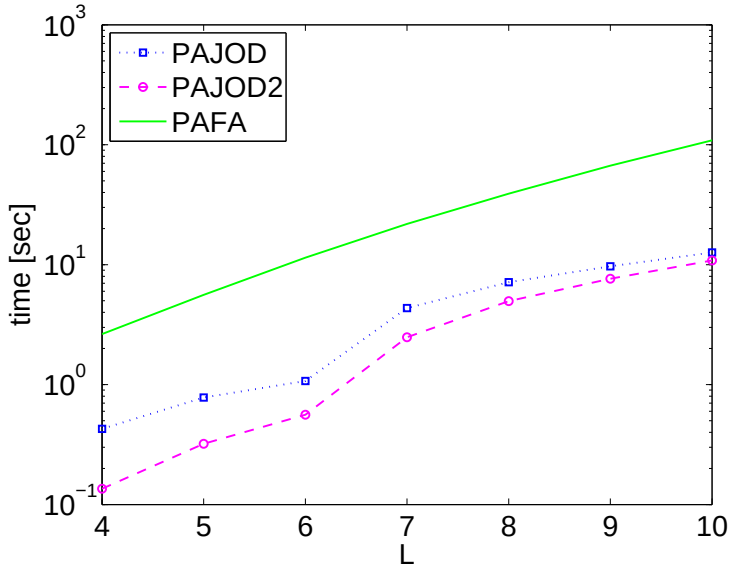


Figure 5: Computation time for filters and channels of varying length L in the second experiment.