On the Diversity-Multiplexing Tradeoff Under Groupwise Successive Interference Cancelation

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***Abstract*—In this paper, we study the diversity-multiplexing tradeoff assuming groupwise successive interference cancelation (GSIC). GSIC with layered codes in multiple-input multiple-output (MIMO) can vary from the plain SIC to the maximum likelihood (ML) decoding according to the group sizes, thus is flexible in both complexity and performance. The maximum group size is a dominating factor for the computation complexity of GSIC when the processing complexity is polynomially dependent on the group size with a high order. In this paper, we show the tradeoff among the maximum group size, diversity gain, and multiplexing gain (GDMT) under the GSIC and under the GSIC combined with antenna selection (GSICAS).**

For a given group-size bound c, we show that the number of groups achieving the optimal DMT under the GSIC with α selected transmit antennas is equal to $\lceil \frac{\alpha}{c} \rceil$ and a larger group should have a higher priority in decoding order. The optimal grouping is obtained in a systematic way. Based on these results, we find a limited set of groupings that contains the grouping achieving the optimal GDMT under GSIC and under GSICAS. The optimal multiplexing gain allocation for the optimal grouping is then found systematically and efficiently within the limited set.

***Index Terms*—Multiple-input multiple-output (MIMO), diversity, spatial multiplexing, diversity-multiplexing tradeoff (DMT), successive interference cancelation (SIC).**

I. Introduction

A multiple-input multiple-ouput (MIMO) system provides two resources, diversity and spatial multiplexing. Diversity enhances the reliability of the transmitted data, whereas the spatial multiplexing increases the achievable data rate. There is a tradeoff between the diversity and multiplexing gains (DMT). The optimal DMT for the independently identically distributed (i.i.d.) Rayleigh fading channel with n transmit and m receive antennas, when the block length l is at least $n+m-1$, is derived in [1] by focusing on the high SNR regime. The DMT result also holds for a wider class of channels [2]. After the result of [1], the construction of a family of codebooks achieving the optimal DMT when l is at least n was done for $n=2$ case in [3], [4] and for all n in [5]. The optimal DMT in [1] was derived assuming the maximum likelihood (ML) decoding. The optimal DMT's under other decoding schemes are also studied in [6] and [7]. It was shown that the optimal DMT is also achievable by a class of lattice space time codes using a generalized minimum Euclidean distance lattice decoding [6]. The optimal DMT under a group detection is derived in [7].

A coding scheme with low decoding complexity is desired especially when m and n are large. In general, however, there is a tradeoff between complexity and performance. Groupwise successive interference cancelation (GSIC) [8] with layered codes achieves this tradeoff depending on how the grouping is done and can be efficient in trading off performance against complexity. According to the grouping, the complexity and performance of GSIC can vary from those of the plain SIC to the ML decoding. The maximum group size can dominantly affect the complexity of GSIC if the processing complexity of each group is polynomially dependent on its group size with a high order, e.g., roughly cubic complexity in the number of antennas discussed in [9]. Thus, the optimal DMT computed over all choices of the GSIC whose maximum group size is less than or equal to a given bound can be a rough measure of the tradeoff between the performance and complexity.

In this paper, for the i.i.d. Rayleigh fading channel with n transmit and m receive antennas, we show that for a group-size bound c the grouping that achieves the optimal GDMT under the GSIC should divide the transmit antennas into $\lceil \frac{n}{c} \rceil$ groups and moreover should give a higher priority in decoding order to the component code transmitted by a larger group. We also study GSIC combined with antenna selection (GSICAS) and show it can achieve a larger diversity gain than the plain GSIC in the low multiplexing gain regime. The optimal allocation of the multiplexing gain to the transmit antenna groups for a given grouping is derived systematically. From the characteristics of the optimal grouping in GDMT under the GSIC, we find out a very limited set of groupings which achieves the optimal GDMT.

In Section II, the system model, GSIC, and GSICAS are described. In Section III, the optimal DMT of a MIMO system under a fixed grouping is described. In Section IV we derive the optimal GDMT's under the GSIC and then under the GSICAS. In Section V, we conclude this paper.

II. System Model and Groupwise Successive Decoding

A. System Model

We consider a wireless communication system with n transmit and m receive antennas with arbitrarily distributed fading coefficients with unit variance. The fading coefficient from transmit antenna j to receive antenna i is denoted by $\mathbf{h}_{ij}$ and $\mathbf{H} = \left[\mathbf{h}_{ij}\right]$ is the corresponding channel matrix. $\mathbf{H}$ is

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assumed to be known to the receiver but not at the transmitter. $\mathbf{H}$ is assumed to remain constant for at least a block of l symbols. $\mathbf{X} = \left[\mathbf{x}_{ij}\right] \in \mathcal{C}$, an $n \times l$ matrix, is transmitted such that $\mathbf{x}_{ij}$ is transmitted from antenna i at time j, where $\mathcal{C}$ is the codebook of $\mathbf{X}$, which can be constructed as a Cartesian product of many smaller codebooks for each group as will be discussed later. $\mathbf{X}$ is normalized such that the average transmit power at each antenna in each symbol period is 1. Let $\mathbf{Y} = \left[\mathbf{y}_{ij}\right]$, an $m \times l$ matrix, be the received matrix where $\mathbf{y}_{ij}$ is the signal received from antenna i at time j. Then we have the following relation

$$\mathbf{Y} = \sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}\mathbf{X} + \mathbf{W} \tag{1}$$

where $\mathbf{W}$ is the additive noise with i.i.d. entries $\mathbf{w}_{ij} \sim \mathcal{CN}(0,1)$ and SNR is the average signal-to-noise ratio (SNR) at each receive antenna. $\mathcal{CN}(0,1)$ denotes the circularly symmetric complex normal distribution with zero mean and unit variance.

B. Groupwise Successive Interference Cancelation

The GSIC with multilayered code, which is found in [8], is described in this subsection. We use the following notations. The sub matrix composed of the elements in rows a through b and columns c through d of matrix $\mathbf{A}$ is denoted by $\mathbf{A}[a:b][c:d]$. When $\mathbf{v}$ is a column vector, the sub-vector composed of the elements in rows a through b of $\mathbf{v}$ is denoted by $\mathbf{v}[a:b]$. $\mathbf{A}_i$ represents the ith column of matrix $\mathbf{A}$.

The ML solution $\hat{\mathbf{X}}$ of $\mathbf{X}$ is the codeword in codebook $\mathcal{C}$ such that

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}\in\mathcal{C}} \left\| \mathbf{Y} - \sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}\mathbf{X} \right\|. \tag{2}$$

If $\mathbf{X}$ is equiprobable then $\hat{\mathbf{X}}$ is the optimal solution. But the computation complexity of ML is too high. The complexity can be reduced by layered coding and GSIC. Let B denote the number of layers or groups, which will be determined later based on how grouping is done. We let $I = (n_1, n_2, \ldots, n_B)$, which is an ordered set with $0 \le n_j \le n$, $\sum_{j=1}^{B} n_j = n$, and $m \ge n_1 + \cdots + n_{B-1} + 1$ and implies that the number of transmit antennas in the j th group is n_j. $\mathbf{X}$ is split according to I into B layers as $\mathbf{X}(j) = \mathbf{X}[i_{j-1}+1 : i_j][1:l]$ for $1 \le j \le B$ with $i_0 = 0$ and $i_j = \sum_{k=1}^{j} n_k$. Let $m_j \triangleq m - i_{j-1}$. We assume each $\mathbf{X}(j)$ is encoded separately using a DMT-optimal code, e.g., a random code as done in [1] or a lattice code as done in [6]. Therefore, $\mathbf{X}(j)$'s are independent from each other by design. Without loss of generality, $\mathbf{X}(j+1)$ is decoded before $\mathbf{X}(j)$. First, to decode $\mathbf{X}(B)$, nullify the signals transmitted from antennas 1 to i_{B-1} by multiplying (1) on the left by an $(m - n + n_B) \times m$ matrix $\Theta(B)$ whose rows are orthonormal vectors in the left null space of $\mathbf{H}[1:m][1:i_{B-1}]$. Then the effective channel for $\mathbf{X}(B)$ is

$$\mathbf{V}(B) = \mathbf{G}(B)\mathbf{X}(B) + \mathbf{Z}(B)$$

where $\mathbf{V}(B) \triangleq \Theta(B)\mathbf{Y}, \mathbf{G}(B) \triangleq \Theta(B)\sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}[1:m][i_{B-1}+1:n], \mathbf{Z}(B) \triangleq \Theta(B)\mathbf{W}$. Find

$$\hat{\mathbf{X}}(B) \triangleq \arg\min_{\mathbf{X}(B)} \|\mathbf{V}(B) - \mathbf{G}(B)\mathbf{X}(B)\|^2$$

and let $\mathbf{Y}(B-1) \triangleq \mathbf{Y} - \sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}[1:m][i_{B-1}+1:n]\hat{\mathbf{X}}(B)$. Then we have $\mathbf{Y}(B-1) = \sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}[1:m][1:i_{B-1}]\mathbf{X}[1:i_{B-1}][1:l] + \mathbf{W}$ and repeat the above groupwise nulling and canceling to have for $1 \le j \le B-1$

$$\mathbf{V}(j) = \mathbf{G}(j)\mathbf{X}(j) + \mathbf{Z}(j)$$
$$\hat{\mathbf{X}}(j) \triangleq \arg\min_{\mathbf{X}(j)} \|\mathbf{V}(j) - \mathbf{G}(j)\mathbf{X}(j)\|^2$$

where $\mathbf{V}(j) \triangleq \Theta(j)\mathbf{Y}(j), \mathbf{G}(j) \triangleq \Theta(j)\sqrt{\frac{\mathrm{SNR}}{n}}\mathbf{H}[1:m][i_{j-1}+1:i_j], \mathbf{Z}(j) \triangleq \Theta(j)\mathbf{W}$, $\Theta(1)$ is the $m \times m$ identity matrix, and, for $j \ne 1$, $\Theta(j)$ is an $m_j \times m$ matrix whose rows are orthonormal vectors in the left null space of $\mathbf{H}[1:m][1:i_{j-1}]$. $\begin{bmatrix}\hat{\mathbf{X}}(1)\\ \vdots \\ \hat{\mathbf{X}}(B)\end{bmatrix}$ is a suboptimal solution of (2). Here we assume $\hat{\mathbf{X}}(j)$ is found by an ML decoder. In practice, practical algorithms such as sphere decoding [9]–[13] can be used for finding $\hat{\mathbf{X}}(j)$.

The number of codewords in $\mathcal{C}$ is given by 2^R, i.e. R denotes the rate of $\mathcal{C}$, and $\mathcal{C}$ can be designed so that R increases with SNR. Let $\mathcal{C}_j$ denote the codebook for $\mathbf{X}(j)$ and let R_j denote the rate of $\mathcal{C}_j$, where R_j increases with SNR. Here, $\mathcal{C}$ is the Cartesian product of component codebooks $\mathcal{C}_j, 1 \le j \le B$ and $R = \sum R_j$.

C. Groupwise Successive Interference Cancelation Combined With Antenna Selection

GSIC with the grouping index set I in subsection II-B implies that all the n transmit antennas are selected and grouped into B groups and the data rate for each $\mathbf{X}(j)$ is positive. In the low multiplexing gain regime, transmitting using only a subset of transmit antennas can result in a better diversity gain than using all the transmit antennas as will be shown in subsection IV-B. In GSICAS, a subset of n transmit antennas is selected for transmission while the other antennas are silent. The grouping index set I for GSICAS is represented by $(n_1, n_2, \ldots, n_B)$ with $\sum_{j=1}^{B} n_j = \alpha \le n$ which means that among n transmit antennas only α antennas are selected, the α antennas are grouped into B groups, and the number of transmit antennas in the j th group is n_j. $\mathbf{X}(j)$ denotes the codeword transmitted by the j th group and the decoding process is the same as that described in subsection II-B.

III. Diversity and Multiplexing Gains Under a Fixed Grouping Index Set

Let P_{err} denote the error probability of the GSIC with a grouping index set I. Then the diversity and multiplexing gains are defined as in [14]:

Definition 1: $\mathcal{C}$ is said to achieve the multiplexing gain r and diversity gain d_I if

$$\lim_{\mathrm{SNR}\to\infty} \frac{R}{\log \mathrm{SNR}} \ge r$$

and

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log P_{err}(\mathrm{SNR})}{\log \mathrm{SNR}} \le -d_I.$$

For each r, we define $d_I^*(r)$ as the supremum of the diversity gain achieved over all transmit schemes under the GSIC with I when $\lim_{\mathrm{SNR}\to\infty}\frac{R}{\log \mathrm{SNR}}=r$. Conversely, for each diversity gain d, $r_I^*(d)$ is the supremum of the multiplexing gain achieved over all transmit schemes attaining diversity gain d under the GSIC with I. When $I=(n)$, i.e., a set containing only n, then the decoding with I is an ML decoding, and in this case we use notations d, d^*, and r^* instead of d_I, d_I^*, and r_I^* respectively.

The optimal DMT $d^*(r)$ under the ML decoding is known as follows.

Theorem 1: [1], [5] If the fading coefficients $\mathbf{h}_{ij}$ are i.i.d. circularly symmetric complex normal variables and l is assumed to be $l \geq n$, then $d^*(r)$ is the piecewise-linear function connecting the integer points $(k, d^*(k)), k=0,1,\ldots,\min\{m,n\}$, where

$$d^*(k)=(m-k)(n-k). \tag{3}$$

In particular the maximal diversity gain, $d_{\max}^*$, is mn and the maximal multiplexing gain, $r_{\max}^*$, is $\min\{m,n\}$.
In [1] the achievability part of Theorem 1 is verified for $l \geq m+n-1$ and in [5] for $l \geq n$.

Actually $d^*(r)$ can be represented in two alternative forms which are easy to handle.

Proposition 1: If the channel is the i.i.d. Rayleigh fading and $l \geq n$ then for $0 \leq r \leq \min\{m,n\}$

$$d^*(r)=(m-\lfloor r\rfloor)(n-\lfloor r\rfloor)-\bar{r}(m+n-2\lfloor r\rfloor-1) \tag{4}$$

and

$$d^*(r)=(m-r)(n-r)+\bar{r}-\bar{r}^2. \tag{5}$$

where $\lfloor r\rfloor$ is the largest integer not greater than r and $0 \leq \bar{r}=r-\lfloor r\rfloor<1$. For each diversity gain d, the largest achievable multiplexing gain $r^*(d)$ is

$$r^*(d)=r'(d)+\frac{\big(m-r'(d)\big)\big(n-r'(d)\big)-d}{m+n-2r'(d)-1} \tag{6}$$

where $r'(d)$ is the largest integer k such that $0 \leq k \leq \min\{m,n\}$ and $(m-k)(n-k) \geq d$.

Proof: (4) is the formula described in Theorem 1. (5) and (6) are derived directly from (4). ∎

We use notations $d^*(r;m,n)$ and $r^*(d;m,n)$ instead of $d^*(r)$ and $r^*(d)$ respectively when m and n are specified and $\mathbf{H}$ is the i.i.d. Rayleigh fading channel.

Let r_j denote the multiplexing gain achieved by $\mathcal{C}_j$. If $\lim_{\mathrm{SNR}\to\infty}\frac{R}{\log \mathrm{SNR}}=r$ and $\lim_{\mathrm{SNR}\to\infty}\frac{R_j}{\log \mathrm{SNR}}=r_j$ for every j, then we have $r=\sum_{j=1}^B r_j$. We have the following definitions for further development.

Definition 2: P_j is defined to be the error probability of $\mathbf{X}(j)$ on the condition that $\mathbf{X}(k)$ are decoded correctly for $j+1 \leq k \leq B$, i.e.

$$P_j=P(\hat{\mathbf{X}}(j)\neq \mathbf{X}(j)|\,\hat{\mathbf{X}}(k)=\mathbf{X}(k) \text{ for } j+1\leq k\leq B),$$

and $\mathbf{X}(j)$ is said to achieve diversity gain d_j when

$$\lim_{\mathrm{SNR}\to\infty}\frac{\log P_j}{\log \mathrm{SNR}} \leq -d_j.$$

$d_j(r_j)$ denotes an achievable diversity gain of $\mathbf{X}(j)$ when $\lim_{\mathrm{SNR}\to\infty}\frac{R_j}{\log \mathrm{SNR}}=r_j$, and $d_j^*(r_j)$ denotes the supremum of $d_j(r_j)$.

$\begin{pmatrix}\mathbf{X}(j)\\ \vdots \\ \mathbf{X}(k)\end{pmatrix}$ is simply denoted by $\mathbf{X}(j,\ldots,k)$.

The optimal DMT under the grouping with I is known as follows.

Theorem 2: [7] If $\lim_{\mathrm{SNR}\to\infty}\frac{R_j}{\log \mathrm{SNR}}=r_j$ and $\lim_{\mathrm{SNR}\to\infty}\frac{\log P_j}{\log \mathrm{SNR}} \leq -d_j$ for every $1\leq j\leq B$ then we have

(i) The multiplexing gain of $\mathbf{X}(j,\ldots,B)$ is $\sum_{k=j}^B r_k$. $\mathbf{X}(j,\ldots,B)$ achieves the diversity gain of $\min_{j\leq k\leq B} d_k(r_k)$, i.e.

$$\lim_{\mathrm{SNR}\to\infty}\frac{\log P(\hat{\mathbf{X}}(j,\ldots,B)\neq \mathbf{X}(j,\ldots,B))}{\log \mathrm{SNR}} \leq -\big(\min_{j\leq k\leq B} d_k(r_k)\big) \tag{7}$$

and the supremum of its diversity gain is $\min_{j\leq k\leq B} d_k^*(r_k)$ when $r_j,\ldots,r_B$ are fixed.

(ii) $\mathbf{X}$ achieves the diversity gain of $\min_{1\leq j\leq B} d_j(r_j)$ and

$$d_I^*(r)=\sup_{\sum_{j=1}^B r_j=r}\Big\{\min_{1\leq j\leq B} d_j^*(r_j)\Big\}. \tag{8}$$

Theorem 2 holds for any channel distribution but if the channel coefficients follow i.i.d. Rayleigh fading then $d_I^*(r)$ has the following form.

Corollary 1: If $\mathbf{H}$ is the i.i.d. Rayleigh fading channel and $l \geq \max_j\{i_j-i_{j-1}\}$ then, for each $1\leq j\leq B$, for $0\leq r_j\leq n_j$

$$d_j^*(r_j)=d^*(r_j;m_j,n_j)=(m_j-r_j)(n_j-r_j)+\bar{r}_j-\bar{r}_j^2 \tag{9}$$

where $\bar{r}_j \triangleq r_j-\lfloor r_j\rfloor$, and therefore

$$d_I^*(r)=\max_{\sum_{j=1}^B r_j=r}\Big\{\min_{1\leq j\leq B}\big((m_j-r_j)(n_j-r_j)+\bar{r}_j-\bar{r}_j^2\big)\Big\}. \tag{10}$$

Proof: $d_j^*(r_j)=d^*(r_j;m_j,n_j)=(m_j-r_j)(n_j-r_j)+\bar{r}_j-\bar{r}_j^2$ if $l\geq i_j-i_{j-1}$, by Proposition 1. (10) is obtained by (9) and Theorem 2 (ii). ∎

We can obtain (10) by the the following process. For a fixed index set I, $d_I^*(r)$ is the piecewise-linear curve connecting the points $(F_i,G_i), 0\leq i\leq M$ such that

$$\begin{aligned}
&F_0=0, G_0=\min_{1\leq j\leq B} m_jn_j\\
&F_1=\sum_{j=1}^B s_{1j}, G_1=G_0 \text{ where } s_{1j}=r^*(G_1;m_j,n_j)\\
&\mathcal{D}=\{G_2,G_3,\ldots,G_M\}\\
&\quad=\{(m_j-a_j)(n_j-a_j): 1\leq j\leq B, \lceil s_{1j}\rceil \leq a_j\leq n_j, a_j\in\mathbb{Z}\}\\
&F_i=\sum_{j=1}^B s_{ij} \text{ with } s_{ij}=r^*(G_i;m_j,n_j) \text{ for } 2\leq i\leq M
\end{aligned} \tag{11}$$

where $\mathcal{D}$ is arranged in decreasing order, i.e., $G_2>\cdots>G_M=0$, $M=|\mathcal{D}|+1$, and s_{ij} is calculated by (6) as $s_{ij}=k_{ij}+\frac{(m_j-k_{ij})(n_j-k_{ij})-G_i}{m_j+n_j-2k_{ij}-1}$ with the largest integer k_{ij} such that $0\leq k_{ij}\leq \min\{m_j,n_j\}$ and $(m_j-k_{ij})(n_j-k_{ij})\geq G_i$.

Here, $\lceil s_{1j} \rceil$ is the smallest integer greater than or equal to s_{1j}. For a multiplexing gain r, the multiplexing gain vector

$$\begin{aligned}\mathbf{r}^*_{(I,r)} &\triangleq \arg_{\mathbf{r}} d^*_I(r) \\ &= \arg_{\mathbf{r}} \max_{\{\mathbf{r}:\sum_j r_j = r\}} \min_j \{(m_j - r_j)(n_j - r_j) + \bar{r}_j - \bar{r}_j^2\}\end{aligned} \tag{12}$$

and $d^*_I(r)$ are calculated by the following simple process (P):

(1) if $r = 0$ then $d^*_I(r) = G_0, \mathbf{r}^*_{(I,r)} = (0, \ldots, 0)$; and if $r = n$ then $d^*_I(r) = 0, \mathbf{r}^*_{(I,r)} = (n_j)_{j=1}^B$

(2) if $0 < r < n$, we have

$$\begin{aligned} d^*_I(r) &= \frac{G_i - G_{i-1}}{F_i - F_{i-1}}(r - F_{i-1}) + G_{i-1} \\ \mathbf{r}^*_{(I,r)} &= \left(r^*\left(d^*_I(r); m_j, n_j\right)\right)_{j=1}^B. \end{aligned} \tag{13}$$

where $i \geq 1$ is the smallest integer such that $F_i \geq r$.

IV. Diversity and Multiplexing Gains With GSICAS

As seen in the previous section, the DMT curve under a GSICAS depends on its grouping index set $I = (n_1, \ldots, n_B)$. If the maximum group size of I, i.e., $\max_{1 \leq j \leq B} n_j$ is 1 then the corresponding decoder to I is the plain antenna-by-antenna SIC, and if the maximum group size of I is n then the corresponding decoder includes the ML decoding. In general, for a given bound on the maximum group size, one can try to find the best choices for selecting antennas, grouping antennas, and allocating rates across antenna groups for maximizing the diversity gain at a given multiplexing gain. Let $c(I) = \max_j\{n_j\}$. Let $\mathcal{I}(c,n) \triangleq \{I = (n_1, \ldots, n_B) : c(I) \leq c, \sum_{j=1}^B n_j \leq n\}$ and for each $I = (n_1, \ldots, n_B)$ let $A(r,I) \triangleq \{\mathbf{r} = (r_1, \ldots, r_B) : \sum_j r_j = r, r_j \leq n_j \text{ for } j = 1, \ldots, B\}$. Then the optimal GDMT of system (1) under the GSICAS, denoted by $d^*(r,c)$, is the supremum of diversity gain on the condition that the multiplexing gain of the transmit scheme is r and the constraint of group-sizes is less than or equal to c, i.e. we have

$$d^*(r,c) = \max_{I \in \mathcal{I}(c,n)} d^*_I(r). \tag{14}$$

Here $d^*_I(r) = \sup_{\mathbf{r} \in A(r,I)} d^*_I(\mathbf{r})$ where $d^*_I(\mathbf{r}) \triangleq \min_{1 \leq j \leq B} d^*_j(r_j)$. To obtain $d^*(r,c)$, the optimal grouping index set, denoted by $I(r,c)$, and the optimal multiplexing gain vector $\mathbf{r}^*(r,c)$ should be found for each pair (r,c). This is a complicated problem because the cardinality of $\mathcal{I}(c,n)$ increases rapidly with n and c, and moreover the multiplexing gain vector $\mathbf{r}$ maximizing the diversity gain for each I should be found.

$\mathcal{I}(c,n)$ is the disjoint union of $\mathcal{J}(c,\alpha) \triangleq \{I = (n_1, \ldots, n_B) : c(I) \leq c, \sum_{j=1}^B n_j = \alpha\}$ where $\alpha = 1, \ldots, n$. Note that $\mathcal{J}(c,\alpha)$ is never empty. Therefore, if we let

$$\bar{d}^*(r,c;\alpha) \triangleq \max_{I \in \mathcal{J}(c,\alpha)} d^*_I(r) \tag{15}$$

then

$$d^*(r,c) = \max_{1 \leq \alpha \leq n} \bar{d}^*(r,c;\alpha). \tag{16}$$

Here, $\bar{d}^*(r,c;\alpha)$ is the optimal GDMT under the GSIC when the system has α transmit and m receive antennas.

A. Optimal Tradeoff of Group-size Constraint, Diversity and Multiplexing Gains Under the GSIC

The optimal GDMT under the GSIC, $\bar{d}^*(r,c;n)$, is studied in this subsection. For any (r,c), we find a very limited subset of $\mathcal{J}(c,n)$ which includes the optimal grouping index set to get $\bar{d}^*(r,c;n)$ when $\mathbf{H}$ is the i.i.d. Rayleigh fading channel. The optimal multiplexing gain vector to obtain $d^*_I(r)$ for each pair of (I,r) is obtained analytically, and thus the optimal multiplexing gain vector achieving $\bar{d}^*(r,c;n)$ can be found.

If the channel is the i.i.d. Rayleigh fading and $l \geq n$ then

$$\bar{d}^*(r,c;n) = \max_{I \in \mathcal{J}(c,n)} \max_{\mathbf{r} \in A(r,I)} \min_{1 \leq j \leq B} \{(m_j - r_j)(n_j - r_j) + \bar{r}_j - \bar{r}_j^2\} \tag{17}$$

where B, m_j, n_j, and r_j depend on I. To obtain $\bar{d}^*(r,c;n)$, according to (17), $d^*_I(r) = \max_{\mathbf{r} \in A(r,I)} \min_{1 \leq j \leq B}\{(m_j - r_j)(n_j - r_j) + \bar{r}_j - \bar{r}_j^2\}$ should be calculated for all $I \in \mathcal{J}(c,n)$ and be compared each other. For each I, $d^*_I(r)$ can be obtained by the process below Corollary 1. Moreover, computing $d^*_I(r)$ and $\mathbf{r}^*_{(I,r)}$ of I's in only a small subset of $\mathcal{J}(c,n)$ is sufficient to find $\bar{d}^*(r,c;n)$ and the corresponding multiplexing gain vector $\mathbf{r}^*$. This is proved in the following. The optimal groupings for $c = 1$ and $c = n$ are trivially $(1, \ldots, 1)$ and (n) respectively. Therefore we assume that $1 < c < n$.

First, the optimal DMT curve $d^*(r; m, n)$ satisfies the following property.

Lemma 1: If $M_1 \geq M_2 + k$ and $N_1 \geq N_2 + k$ where k is a positive integer then for $0 \leq r \leq \min\{M_2, N_2\}$

$$d^*(r + k; M_1, N_1) \geq d^*(r; M_2, N_2). \tag{18}$$

The equality holds iff $M_1 = M_2 + k$ and $N_1 = N_2 + k$.

Proof: From Proposition 1

$$\begin{aligned} d^*(r + k; M_1, N_1) &= (M_1 - r - k)(N_1 - r - k) + \bar{r} - \bar{r}^2 \\ &\geq d^*(r; M_2, N_2). \end{aligned}$$

■

From Lemma 1, we have the following proposition which says that, to maximize DMT or GDMT, a component code transmitted by a group of more transmit antennas should have a higher priority in decoding order, that is, n_j is non-decreasing.

Proposition 2: For any $B \geq 2$ and $n_1 \leq n_2 \leq \cdots \leq n_B$, let $\mathcal{I}$ be the collection of the grouping index sets such that n transmit antennas are divided into B groups having $n_1, n_2, \ldots, n_B$ transmit antennas then $I = (n_1, n_2, \ldots, n_B)$ has the maximal DMT among $\mathcal{I}$.

Proof: See Appendix A. ■

The groupings in $\mathcal{I}$ have the same group-size distribution and $(n_1, n_2, \ldots, n_B)$ has the maximum GDMT among $\mathcal{I}$.

Proposition 3: For every $1 < c < n$, if $I = (n_1, n_2, \ldots, n_B)$ is a grouping index set such that $c(I) \leq c$ and $B > \left\lceil \frac{n}{c} \right\rceil$ then there exists a grouping index set $J = (u_1, u_2, \ldots, u_K)$ satisfying

$$K = \left\lceil \frac{n}{c} \right\rceil, \quad c(J) \leq c,$$

and

$$d^*_J(r) \geq d^*_I(r) \quad \text{for } 0 \leq r \leq n.$$

Proof: See Appendix B. ∎

From Proposition 3, for a group-size constraint c, the number of antenna groups of the grouping index set achieving the optimal GDMT under the GSIC is always equal to $\lceil \frac{n}{c} \rceil$. Let $\mathcal{F}(c,n)$ denote the collection of the grouping index sets I such that $c(I) \le c$, $B = \lceil \frac{n}{c} \rceil$ and n_j is non-decreasing, i.e.

$$\mathcal{F}(c,n) \triangleq \{I = (n_1, \ldots, n_{\lceil \frac{n}{c} \rceil}) : \max_{1 \le j \le \lceil \frac{n}{c} \rceil} n_j \le c, \sum_{j=1}^{\lceil \frac{n}{c} \rceil} n_j = n, n_1 \le \cdots \le n_{\lceil \frac{n}{c} \rceil}\}.$$

Then from Propositions 2 and 3, for each pair of multiplexing gain r and group-size limit c, the optimal grouping index set achieving $\bar{d}^*(r,c;n)$ is in $\mathcal{F}(c,n)$.

Theorem 3:

$$\bar{d}^*(r,c;n) = \max_{I \in \mathcal{F}(c,n)} d_I^*(r) \tag{19}$$

and if I^* is an optimal grouping index set found from (19) then an optimal multiplexing gain vector $\mathbf{r}^*$ is

$$\mathbf{r}^* = \left(r^*\left(d_{I^*}^*(r); m_j, n_j\right)\right)_j \tag{20}$$

where m_j, n_j are of I^*.

In (19), $\mathcal{F}(c,n)$ is very limited and has only a few members. For example, when $m = n = 10$, for all $1 \le c \le 10$, $\mathcal{F}(c,n)$ are directly followed as

$$\mathcal{F}(10,10) = (10),\ \mathcal{F}(9,10) = \{(i, 10-i) : i = 1,2,3,4,5\}$$
$$\mathcal{F}(8,10) = \{(i,10-i) : i = 2,3,4,5\},$$
$$\mathcal{F}(7,10) = \{(i,10-i) : i = 3,4,5\}$$
$$\mathcal{F}(6,10) = \{(i,10-i) : i = 4,5\},\ \mathcal{F}(5,10) = \{(5,5)\}$$
$$\mathcal{F}(4,10) = \{(2,4,4),\ (3,3,4)\},$$
$$\mathcal{F}(3,10) = \{(1,3,3,3),\ (2,2,3,3)\}$$
$$\mathcal{F}(2,10) = \{(2,2,2,2,2)\},$$
$$\mathcal{F}(1,10) = \{(1,1,1,1,1,1,1,1,1,1)\}.$$

To find $\bar{d}^*(r, c = 6; 10)$, the corresponding index set I^*, and multiplexing gain vector $\mathbf{r}^*$, comparing DMT over only two index sets is enough from Theorem 3, instead of comparing DMT over 412 index sets. As for $\bar{d}^*(r, c = 3; 10)$, instead of 194 index sets, comparing only two index sets is enough.

When $c \ge \frac{n}{2}$, $\mathcal{F}(c,n) = \left\{(i, n-i) : n - c \le i \le \frac{n}{2}\right\}$ and the grouping achieving the optimal GDMT is determined for each r by a formula derived in the following. For $r = 0$, if we compare only $d_I^*(0)$'s among I having only two groups, then according to (10) $(i^*, n-i^*)$ where

$$i^* \triangleq \arg\min_{1 \le i \le n} \max\{d_{(i,n-i)}^*(0)\} = \arg\min_{1 \le i \le n} \min\{|mi - (m-i)(n-i)|\}$$

has the maximum value of $d_I^*(0)$. We get $i^* = \lceil \frac{2m+n-\sqrt{4m^2+n^2}}{2} \rfloor$, which is the closest integer to $\frac{2m+n-\sqrt{4m^2+n^2}}{2}$. Furthermore, $(i^*, n-i^*)$ has the following property.

Proposition 4: We assume that $n > 1$. For all $n > i > i^*$,

$$d_{(i,n-i)}^*(r) \le d_{(i^*,n-i^*)}^*(r) \quad \text{for } 0 \le r \le n. \tag{21}$$

If $i^* \le i_1 < i_2 < n$ then

$$d_{(i_1,n-i_1)}^*(r) \ge d_{(i_2,n-i_2)}^*(r) \quad \text{for } 0 \le r \le n. \tag{22}$$

Proof: See Appendix C. ∎

Notice that $i^* \le \lceil \frac{n}{2} \rfloor$, and if we think about the optimal DMT curve of (1) under GSIC's with $\{(i, n-i) : 1 \le i < n\}$ only it is sufficient to consider the index sets $(i, n-i)$ with $i \le i^*$ from Propositions 2 and 4. Let I_r^2 be the index set grouping the transmit antennas into two groups that achieves $\max_{1 \le i \le n} d_{(i,n-i)}^*(r)$. Then I_r^2 varies according to r, and I_r^2 and $d_{I_r^2}^*(r)$ are derived in the following proposition.

Proposition 5: Let $a(i)$ be defined for $0 \le i \le i^*$ as

$$a(i) \triangleq \begin{cases} 0 & \text{if } i = i^* \\ s_1(i) + s_2(i) & \text{if } 1 \le i \le i^* - 1 \\ n & \text{if } i = 0 \end{cases}$$

where $s_1(i)$ and $s_2(i)$ are the multiplexing gains such that

$$d^*(s_1(i); m, i+1) = mi,\ d^*(s_2(i); m-i-1, n-i-1) = mi.$$

Then we have

$$I_r^2 = (i, n-i) \ \text{ for } a(i) \le r \le a(i-1). \tag{23}$$

Proof: See Appendix D. ∎

Thus, we have $d_{I_r^2}^*(r) = d_{(i,n-i)}^*(r)$ for $a(i) \le r \le a(i-1)$, and $d_{I_r^2}^*(r) \ge d_I^*(r)$ for $0 \le r \le n$ for any $I \in \{(n_1, n-n_1) : 1 \le n_1 \le n\}$. In Proposition 5, $a(i)$'s are easily calculated for given m and n and hence $d_{I_r^2}^*(r)$ is easily derived. In particular, when $m = n$, $a(i)$'s are given as

$$a(i) = m - i - \frac{i}{m+i} - \left\lceil \sqrt{mi} \right\rceil + \frac{\left\lceil \sqrt{mi} \right\rceil^2 - mi}{2\left\lceil \sqrt{mi} \right\rceil - 1}$$

for $1 \le i \le i^* - 1$. Now, from Propositions 4 and 5, we have the following result.

Corollary 2: If $\frac{n}{2} \le c \le n - i^*$ then

$$I^* = (n-c, c) \text{ for all } 0 \le r \le n;$$

and if $n - i^* < c < n$ then

$$I^* = \begin{cases} (j, n-j) & \text{if } i^* \ge j > (n-c) \text{ and } a(j) \le r \le a(j-1) \\ (n-c, c) & \text{if } r > a(n-c). \end{cases}$$

$\bar{d}^*(r,c;n)$ and $\mathbf{r}^*$ are calculated as $d_{I^*}^*(r)$ and $\mathbf{r}_{(I^*,r)}^*$ respectively by process (P).

Therefore, when $m = n = 10$, for all $0 \le r \le 10$ $I^* = (4,6)$ for $c = 6$ and $I^* = (5,5)$ for $c = 5$. Then $d_{I^*}^*(r)$ and $\mathbf{r}_{(I^*,r)}^*$ are computed by process (P). If $c = 7$ then

$$I^* = \begin{cases} (4,6) & \text{if } 0 \le r \le \frac{188}{143} \\ (3,7) & \text{if } \frac{188}{143} < r \le 10 \end{cases}$$

and $d_{I^*}^*(r)$ and $\mathbf{r}_{(I^*,r)}^*$ are computed by process (P).

$$\mathcal{G}_\alpha \triangleq \begin{cases} \left\{(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil}) \in \mathcal{F}(c,\alpha) : n_{\lceil \frac{\alpha}{c}\rceil} = c, \quad mn_1 > (m-(\lceil \frac{\alpha}{c}\rceil - 1)c)c\right\} & \text{if } c < \alpha \le \lfloor \frac{n}{c}\rfloor c \\ \mathcal{F}(c,\alpha) & \text{if } \alpha = c \text{ or } \lfloor \frac{n}{c}\rfloor c < \alpha \le n \end{cases} \tag{24}$$

B. Optimal Tradeoff for Group-size Constraint, Diversity and Multiplexing Gains Under the GSIC Combined With Antenna Selection

GSIC combined with antenna selection has more flexibility and is better in GDMT than GSIC, which is presented below and descriptively in examples. The optimal GDMT under the GSICAS, $d^*(r,c)$, is obtained systematically by applying Theorem 3. It is assumed that $n \ge c$, since if $n < c$ then obviously (n) is the optimal grouping and $d^*(r,c) = d^*(r;m,n)$. We define $\mathcal{G}_\alpha$ for $c \le \alpha \le n$ as in (24). We also define $\mathcal{G}(c,n)$ as

$$\mathcal{G}(c,n) \triangleq \bigcup_{c\le\alpha\le n} \mathcal{G}_\alpha. \tag{25}$$

The following shows the main result in this section, which says that we only need to consider $I(r,c)$ in $\mathcal{G}(c,n)$ for each (r,c) and that the optimal multiplexing gain vector is simply given by that achieved at $d^*_{I(r,c)}(r)$.

Theorem 4: The optimal GDMT under the GSICAS is given by

$$d^*(r,c) = \max_{I\in\mathcal{G}(c,n)} d^*_I(r) \tag{26}$$

and the optimal multiplexing gain vector $\mathbf{r}^*(r,c)$ is given by

$$\mathbf{r}^*(r,c) = \left(r^*\left(d^*_{I(r,c)}(r);m_j,n_j\right)\right)_j \tag{27}$$

where m_j and n_j are of $I(r,c)$.

Proof: From Theorem 3,

$$d^*(r,c) = \max_{1\le\alpha\le n} \max_{I\in\mathcal{F}(c,\alpha)} d^*_I(r). \tag{28}$$

If $\alpha \le c$ then $\mathcal{F}(c,\alpha) = \{(\alpha)\}$ and $d^*_{(\alpha)}(r) = d^*(r;m,\alpha) \le d^*_{(c)}(r)$. If $c < \alpha \le \lfloor \frac{n}{c}\rfloor c$ then for any $(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil}) \in \mathcal{F}(c,\alpha)$ such that $n_{\lceil \frac{\alpha}{c}\rceil} < c$ we can see that $d^*_{(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil}-1,c)}(r) \ge d_{(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil})}(r)$ and $(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil}-1,c) \in \mathcal{F}(c,\sum_{j=1}^{\lceil \frac{\alpha}{c}\rceil-1} n_j + c)$. If $(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil}) \in \mathcal{F}(c,\alpha)$, $mn_1 \le (m-(\lceil \frac{\alpha}{c}\rceil - 1)c)c$, and $c < \alpha \le \lfloor \frac{n}{c}\rfloor c$ then $I = (\underbrace{c,\ldots,c}_{\lceil \frac{\alpha}{c}\rceil \text{ c's}})$ has a better DMT than $(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil})$ i.e. $d^*_I(r) \ge d^*_{(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil})}(r)$; because for any multiplexing gain vector $(r_1,\ldots,r_{\lceil \frac{\alpha}{c}\rceil})$ of grouping $(n_1,\ldots,n_{\lceil \frac{\alpha}{c}\rceil})$ the multiplexing gain allocation $(r_2,\ldots,r_{\lceil \frac{\alpha}{c}\rceil},r_1)$ for I has larger diversity gain. In detail, $m+n_1 > m-(\lceil \frac{\alpha}{c}\rceil - 1)c + c$ and we have from (5) $d^*(r_1;m,n_1) \le d^*(r_1;m-(\lceil \frac{\alpha}{c}\rceil - 1)c,c)$ for any $0 \le r_1 \le n_1$. For $k = 1,\ldots,\lceil \frac{\alpha}{c}\rceil - 1$ we have $\sum_{j=1}^{k} n_j \ge (k-1)c$ and thus $d^*(r_{k+1};m-\sum_{j=1}^{k} n_j, n_{k+1}) \le d^*(r_{k+1};m-(k-1)c,c)$ for any $0 \le r_{k+1} \le n_{k+1}$. ∎

$\mathcal{I}(c,n)$ is a very large set and there are many groupings to compare their DMT's to obtain (14), but $\mathcal{G}(c,n)$ is very limited and the process to obtain the optimal GDMT becomes much simpler. Since each $d^*_I(r)$ is obtained analytically by (11) and comparison among $d^*_I(r)$'s can be done by using process (P), $d^*(r,c)$ can be obtained analytically. From Theorem 4, for each (r,c) the data rate for the jth antenna group of $I(r,c)$ can be determined as $r^*\left(d^*_{I(r,c)}(r);m_j,n_j\right)\log \mathrm{SNR}$ to achieve $d^*(r,c)$.

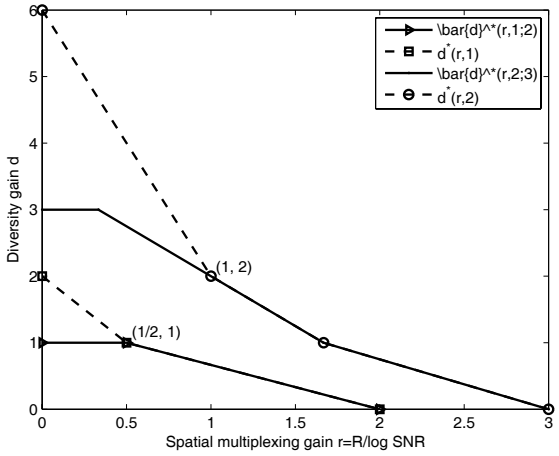


Fig. 1. $d^*(r,1)$ and $\bar{d}^*(r,1;2)$ for MIMO Rayleigh fading channel with $m=n=2$. $d^*(r,2)$ and $\bar{d}^*(r,2;3)$ for the MIMO Rayleigh fading channel with $m=n=3$. $d^*(r,1)$ is the line connecting points $(0,1),(\frac{1}{2},1),(2,0)$ and the curve of $\bar{d}^*(r,1;2)$ is the line connecting points $(0,2),(\frac{1}{2},1),(2,0)$; $d^*(r,2)$ is the line with $(0,3),(1/3,3),(1+2/3,1),(3,0)$ and $\bar{d}^*(r,2;3)$ is the line with points $(0,6),(1,2),(1+2/3,1),(3,0)$.

For some constant r_0 depending on m, n, and c, $d^*(r,c) > \bar{d}^*(r,c;n)$ for $r < r_0$ and $d^*(r,c) = \bar{d}^*(r,c;n)$ for $r \ge r_0$. From the following examples, we can see quantitatively the advantage of the GSICAS over the GSIC.

Example 1: When $m=n=2$ and $c=1$, then $\mathcal{F}(1,2) = \{(1,1)\}$, $\mathcal{G}(1,2) = \{(1),(1,1)\}$, and

$$d^*(r,1) = \begin{cases} d^*_{(1)}(r) & \text{for } 0 \le r < \frac{1}{2} \\ d^*_{(1,1)}(r) & \text{for } \frac{1}{2} \le r \le 2 \end{cases} \tag{29}$$
$$d^*_{(1)}(1/2) = d^*_{(1,1)}(1/2) = 1.$$

We have $\bar{d}^*(r,1;2) = d^*_{(1,1)}(r)$ for $0 \le r \le 2$. $d^*(r,1) > \bar{d}^*(r,1;2)$ for $0 \le r < \frac{1}{2}$ and $d^*(r,1) = \bar{d}^*(r,1;2)$ for $\frac{1}{2} \le r \le 2$.

For the case of $m=n=3$ and $c=2$, $\mathcal{F}(2,3) = \{(1,2)\}$, $\mathcal{G}(2,3) = \{(2),(1,2)\}$, and

$$d^*(r,2) = \begin{cases} d^*_{(2)}(r) & \text{for } 0 \le r < 1 \\ d^*_{(1,2)}(r) & \text{for } 1 \le r \le 3 \end{cases} \tag{30}$$
$$d^*_{(2)}(1) = d^*_{(1,2)}(1) = 2.$$

Here, $d^*(r,2) > \bar{d}^*(r,2;3)$ for $0 \le r < 1$ and $d^*(r,2) = \bar{d}^*(r,2;3)$ for $1 \le r \le 3$.

In Fig. 1, $d^*(r,c)$ and $\bar{d}^*(r,c;n)$ for the cases of $(m=n=2,c=1)$ and $(m=n=3,c=2)$ are compared visually. For larger m, n, and c, we consider the following cases.

Example 2: For $m=n=8$ and $c=5$, we have

$$\mathcal{F}(5,8) = \{(3,5)\},\ \mathcal{G}(5,8) = \{(5),(1,5),(2,5),(3,5)\},$$

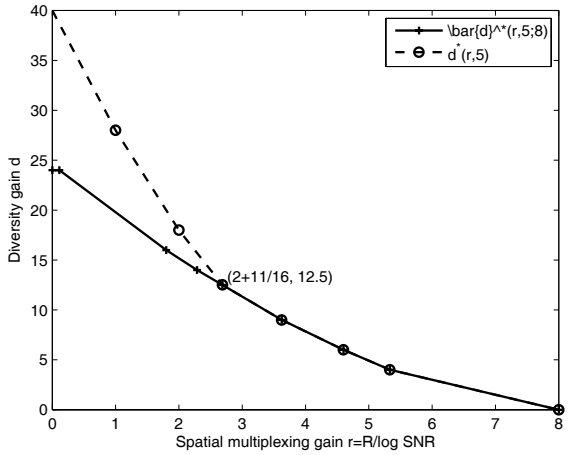


Fig. 2. $d^*(r,5)$ and $\bar{d}^*(r,5;8)$ for the MIMO Rayleigh fading channel with $m=n=8$. The curve of $d^*(r,5)$ is the line linking the following points $(0,24),(\frac{1}{9},24),(1.8,16),(2+\frac{2}{7},14),(3.625,9),(4.6,6),(5+1/3,4),(8,0)$, and the curve of $\bar{d}^*(r,5;8)$ is the line with points $(0,40),(1,28),(2,18),(2+\frac{11}{16},12.5),(3.625,9),(4.6,6),(5+1/3,4),(8,0)$.

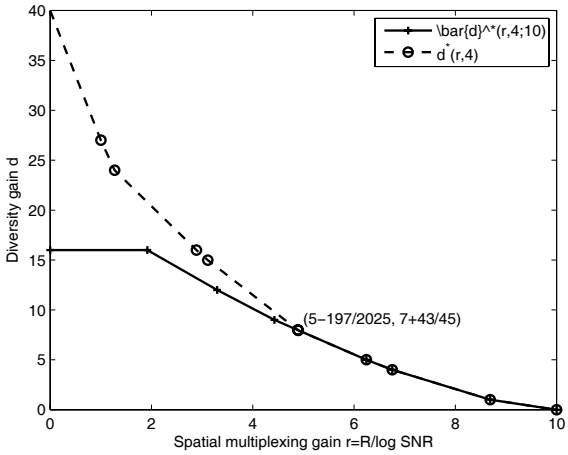


Fig. 3. $d^*(r,4)$ and $\bar{d}^*(r,4;10)$ for the MIMO Rayleigh fading channel with $m=n=10$. The curve of $d^*(r,4)$ is the line with points $(0,16),(1+\frac{91}{99},16),(3+\frac{23}{77},12),(4+\frac{3}{7},9),(6+\frac{11}{45},5),(6+\frac{34}{45},4),(8+\frac{31}{45},1),(10,0)$, and the curve of $\bar{d}^*(r,4;10)$ is the line with points $(0,40),(1,27),(1+\frac{3}{11},24),(2+\frac{8}{9},16),(3+\frac{1}{9},15),(4+\frac{8}{9},8),(5-\frac{197}{2025},7+\frac{43}{45}),(6+\frac{11}{45},5),(6+\frac{34}{45},4),(8+\frac{31}{45},1),(10,0)$.

and

$$d^*(r,5)=\begin{cases} d^*_{(5)}(r) & \text{for } 0\le r<2+\frac{11}{16} \\ d^*_{(3,5)}(r) & \text{for } 2+\frac{11}{16}\le r\le 8 \end{cases} \tag{31}$$

with $d^*_{(5)}(2+\frac{11}{16})=d^*_{(3,5)}(2+\frac{11}{16})=12.5$. Here $d^*(r,5)$ is larger than $\bar{d}^*(r,5;8)=d^*_{(3,5)}(r)$ for $0\le r<2+\frac{11}{16}$ and is much larger for near $r=0$.

Consider another case of $m=n=10$ and $c=4$, then $\mathcal{F}(4,10)=\{(2,4,4),(3,3,4)\}$ and $\mathcal{G}(4,10)=\{(4),(3,4),(4,4),(1,4,4),(2,3,4),(2,4,4),(3,3,4)\}$. We can easily see that $\bar{d}^*(r,4;10)=d^*_{(2,4,4)}(r)$ for all $0\le r\le 10$. $d^*(r,4)$, on the other hand, is

$$d^*(r,4)=\begin{cases} d^*_{(4)}(r) & \text{for } 0\le r<1+\frac{3}{11} \\ d^*_{(4,4)}(r) & \text{for } 1+\frac{3}{11}\le r<5-\frac{197}{2025} \\ d^*_{(2,4,4)}(r) & \text{for } 5-\frac{197}{2025}\le r\le 10 \end{cases} \tag{32}$$

with $d^*_{(4)}\big(1+\frac{3}{11}\big)=d^*_{(4,4)}\big(1+\frac{3}{11}\big)=24$ and $d^*_{(4,4)}\big(5-\frac{197}{2025}\big)=d^*_{(2,4,4)}\big(5-\frac{197}{2025}\big)=7+\frac{43}{45}$.

Figs. 2 and 3 show these two examples graphically. As shown in the above examples, the difference between $d^*(r,c)$ and $\bar{d}^*(r,c;n)$ becomes larger for small r, particularly near $r=0$, for larger m, n, and c.

V. Conclusion

The DMT under the GSICAS is studied with the constraint on the maximum group size of the transmit antenna groups. When the channel of a MIMO system is the i.i.d. Rayleigh fading, we showed analytic ways of obtaining the optimal GDMT under the GSIC and under the GSICAS. For a given group-size bound c, the grouping achieving the optimal GDMT under the GSIC is shown to split the n transmit antennas into $\lceil\frac{n}{c}\rceil$ groups and to give a higher priority in decoding order to the component code transmitted by a larger group. From this, for a given group-size bound, a very limited collection of grouping index sets is shown to achieve the optimal GDMT under the GSICAS. For a fixed grouping index set the optimal multiplexing gain allocation to the transmit antenna groups in diversity is obtained analytically, and thus the corresponding data rate allocation for the grouping is derived. Therefore, the optimal GDMT under the GSICAS, the corresponding grouping, and rate allocation are obtained systematically.

Appendix A
Proof of Proposition 2

First, for the case of $B=2$ with $\mathcal{I}=\{(n_1,n_2),(n_2,n_1)\}$ we show that

$$d^*_{(n_1,n_2)}(r)\ge d^*_{(n_2,n_1)}(r) \text{ for } 0\le r\le n\ . \tag{33}$$

From (10), we have $d^*_{(n_2,n_1)}(0)=\min\{mn_2,(m-n_2)n_1\}=(m-n_2)n_1$. Let s_1 and s_2 be the multiplexing gains such that

$$\begin{aligned} d^*_{(n_1,n_2)}(s_1+s_2)&=d^*(s_1;m,n_1)\\ &=d^*(s_2;m-n_1,n_2)=(m-n_2)n_1. \end{aligned}$$

Then $s_2=n_2-n_1$ and s_1 is some positive real number since $mn_1>(m-n_2)n_1$. For $0\le r\le n_2-n_1+s_1$ we have

$$d^*_{(n_1,n_2)}(r)\ge d^*_{(n_2,n_1)}(0)\ge d^*_{(n_2,n_1)}(r). \tag{34}$$

For $n_2-n_1+s_1<r\le n$, r is split into t_1 and t_2 such that $r=t_1+t_2$ and

$$d^*_{(n_1,n_2)}(r)=d^*(t_1;m,n_1)=d^*(t_2;m-n_1,n_2),$$

where $t_1>s_1$ and $t_2>n_2-n_1$. By Lemma 1

$$d^*(t_2;m-n_1,n_2)=d^*(t_2-(n_2-n_1);m-n_2,n_1)$$

and

$$\begin{aligned} d^*(n_2-n_1+t_1;m,n_2)&=d^*(t_1;m-(n_2-n_1),n_1)\\ &<d^*(t_1;m,n_1). \end{aligned}$$

Therefore $d^*_{(n_2,n_1)}(r) = d^*(\tilde{t}_2; m-n_2, n_1)$ where $\tilde{t}_2$ is some positive number such that $\tilde{t}_2 > t_2 - (n_2 - n_1)$ and

$$d^*_{(n_2,n_1)}(r) < d^*(t_2 - (n_2 - n_1); m - n_2, n_1) = d^*_{(n_1,n_2)}(r). \tag{35}$$

By (34) and (35) we conclude (33).

Then, for $B > 2$, $d^*_{(n_1,n_2,\ldots,n_B)}(r) = \max_{I\in\mathcal{I}} d^*_I(r)$ is proved by applying (33) repeatedly.

Appendix B
Proof of Proposition 3

We let

$$b(q) \triangleq B - \left\lceil \frac{n}{c} \right\rceil + q \quad \text{for } 1 \le q \le \left\lceil \frac{n}{c} \right\rceil$$

then $b(1) \ge 2$ and $b(\lceil \frac{n}{c} \rceil) = B$. There exist nonnegative integers t_q for $1 \le q \le \lceil \frac{n}{c} \rceil$ such that

$$n_{b(q)} + t_q \le c \text{ and } \sum_{q=1}^{\lceil \frac{n}{c} \rceil} (n_{b(q)} + t_q) = n.$$

Let $u_q \triangleq n_{b(q)} + t_q$ for $1 \le q \le \lceil \frac{n}{c} \rceil$ and $J \triangleq (u_1, \ldots, u_{\lceil \frac{n}{c} \rceil})$. Then

$$d^*_J(r) \ge d^*_I(r) \quad \text{for } 0 \le r \le n, \tag{36}$$

which is proved in the following.

Let $m_1 \triangleq m$ and $m_i \triangleq m - \left(\sum_{j=1}^{i-1} n_j\right)$ for $2 \le i \le B$, then

$$d^*_I(r) = \max_{\sum_{j=1}^B r_j = r} \min_{1\le j\le B} d^*(r_j; m_j, n_j).$$

Let $v_1 \triangleq m$ and $v_q \triangleq m - \left(\sum_{j=1}^{q-1} u_j\right)$ for $2 \le q \le \lceil \frac{n}{c} \rceil$, then

$$d^*_J(r) = \max_{\sum_{q=1}^{\lceil \frac{n}{c} \rceil} r_q = r} \min_{1\le q\le \lceil \frac{n}{c} \rceil} d^*(r_q; v_q, u_q).$$

Since $m_{b(q)} = v_q - \sum_{p=q}^{\lceil \frac{n}{c} \rceil} t_p$, $v_q \ge m_{b(q)} + t_q$ for $1 \le q \le \lceil \frac{n}{c} \rceil$. For any given $0 \le r \le n$, let $\mathbf{r} = (r_1, \ldots, r_B)$ be a multiplexing gain vector such that $\sum_{j=1}^B r_j = r$ and each r_j is allocated to the group of n_j transmit antennas then $\sum_{j=1}^{b(1)-1} r_j \le \sum_{q=1}^{\lceil \frac{n}{c} \rceil} t_q$, since $\sum_{j=1}^{b(1)-1} n_j = \sum_{q=1}^{\lceil \frac{n}{c} \rceil} t_q$. Therefore, there exist $s_1, \ldots, s_{\lceil \frac{n}{c} \rceil}$ such that $\sum_{j=1}^{b(1)-1} r_j = \sum_{q=1}^{\lceil \frac{n}{c} \rceil} s_q$ with $s_q \le t_q$ for $q = 1, \ldots, \lceil \frac{n}{c} \rceil$, and we have

$$\begin{aligned}
\min_{1\le j\le B} d^*(r_j; m_j, n_j) &\le \min_{1\le q\le \lceil \frac{n}{c} \rceil} d^*(r_{b(q)}; m_{b(q)}, n_{b(q)}) \\
&\le \min_{1\le q\le \lceil \frac{n}{c} \rceil} d^*(r_{b(q)} + t_q; v_q, u_q) \\
&\le \min_{1\le q\le \lceil \frac{n}{c} \rceil} d^*(r_{b(q)} + s_q; v_q, u_q)
\end{aligned}$$

where the second inequality follows from Lemma 1 and $\sum_{q=1}^{\lceil \frac{n}{c} \rceil} r_{b(q)} + s_q = r$. This concludes (36).

Appendix C
Proof of Proposition 4

For each i such that $n \ge i > i^*$ we have $mi > (m-i)(n-i)$ and $d^*_{(i,n-i)}(0) = (m-i)(n-i)$. Both mi^* and $(m-i^*)(n-i^*)$ are greater than or equal to $(m-i)(n-i)$ and

$$d^*_{(i^*,n-i^*)}(r) \ge d^*_{(i,n-i)}(r) \quad \text{for } 0 \le r \le i - i^* + s_1$$

where s_1 is the multiplexing gain such that $d(s_1; m, i^*) = (m-i)(n-i)$, and $d^*_{(i^*,n-i^*)}(i - i^* + s_1) = d(s_1; m, i^*) = d(i-i^*; m-i^*, n-i^*)$. For $i-i^*+s_1 < r \le n$, $d^*_{(i^*,n-i^*)}(r) = d^*(\tilde{s}_1; m, i^*) = d^*(i-i^*+\tilde{s}_2; m-i^*, n-i^*)$ for some $\tilde{s}_1 \ge s_1$ and $\tilde{s}_2 > 0$. Here $r = \tilde{s}_1 + i - i^* + \tilde{s}_2$. $d^*(\tilde{s}_2; m-i, n-i) = d^*(i - i^* + \tilde{s}_2; m - i^*, n - i^*)$ and $d^*(i - i^* + \tilde{s}_1; m, i) = d^*(\tilde{s}_1; m - i + i^*, i^*) < d^*(\tilde{s}_1; m, i^*)$ by Lemma 1. Hence

$$d^*_{(i,n-i)}(r) < d^*(\tilde{s}_2; m-i, n-i) = d^*_{(i^*,n-i^*)}(r).$$

Therefore we have (21).

Next, if $n > i_2 > i_1 > i^*$ then $mi_1 > (m-i_1)(n-i_1)$, $mi_2 > (m-i_2)(n-i_2)$, and $d^*_{(i_1,n-i_1)}(r) > d^*_{(i_2,n-i_2)}(r)$ for $0 \le r \le i_2 - i_1 + t_1$ where $t_1 = r^*((m-i_2)(n-i_2); m, i_1) > 0$. For $i_2 - i_1 + t_1 < r \le n$, by Lemma 1 $d^*_{(i_1,n-i_1)}(r) \ge d^*_{(i_2,n-i_2)}(r)$. Therefore $d^*_{(i_1,n-i_1)}(r) \ge d^*_{(i_2,n-i_2)}(r)$ for $0 \le r \le n$. Combining this with (21), (22) holds for $n > i_2 > i_1 \ge i^*$.

Appendix D
Proof of Proposition 5

For any i such that $2 \le i \le i^*$, we have for all $1 \le j < i$

$$d^*_{(i,n-i)}(r) \ge d^*_{(j,n-j)}(r) \quad \text{for } 0 \le r \le a(i-1), \tag{37}$$

since

$$\begin{aligned}
d^*_{(j,n-j)}(r) &\le d^*_{(j,n-j)}(0) = mj \le m(i-1) \\
&= d^*_{(i,n-i)}\big(a(i-1)\big) \le d^*_{(i,n-i)}(r)
\end{aligned}$$

for $0 \le r \le a(i-1)$. In addition, for $2 \le i \le i^*$,

$$d^*_{(i,n-i)}(r) \le d^*_{(i-1,n-i+1)}(r) \quad \text{for } a(i-1) \le r \le n \tag{38}$$

which is proved as follows. By Lemma 1, $d^*(1+s_2(i-1); m-i+1, n-i+1) = m(i-1)$ but $d^*(1; m, i) < m(i-1)$, and thus $1 + s_2(i-1) > a(i-1)$ and $d^*_{(i,n-i)}(r) \le d^*_{(i-1,n-i+1)}(r) = m(i-1)$ for $a(i-1) \le r \le 1 + s_2(i-1)$. For any r such that $1 + s_2(i-1) < r \le n$, $d^*_{(i-1,n-i+1)}(r) = d^*(1+t_2; m-i+1, n-i+1)$ for some $t_2 > s_2(i-1)$. $d^*(1+t_2; m-i+1, n-i+1) = d^*(t_2; m-i, n-i)$ but $d^*(r-t_2; m, i) < d^*(r-1-t_2; m, i-1)$. Hence $d^*_{(i,n-i)}(r) \le d^*_{(i-1,n-i+1)}(r)$ for $1 + s_2(i-1) < r \le n$.

From (37), (38), and Proposition 4, for each $1 \le i \le i^*$, we have for $a(i) \le r \le a(i-1)$

$$d^*_{(i,n-i)}(r) \ge d^*_{(j,n-j)}(r) \text{ for all } 1 \le j \le n. \tag{39}$$

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