TENSOR DECOMPOSITIONS WITH BLOCK-TOEPLITZ STRUCTURE AND APPLICATIONS IN SIGNAL PROCESSING

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ABSTRACT

Tensor decompositions with Toeplitz or block-Toeplitz structure are common in signal processing. For instance, they show up in blind system identification and deconvolution. We illustrate that by simultaneously taking the tensor nature and the block-Toeplitz structure of the problem into account new uniqueness results and numerical methods for computing a tensor decomposition with block-Toeplitz structure can be obtained.

Index Terms— tensor, block tensor decomposition, block-Toeplitz matrix, blind signal separation, blind system identification.

1. INTRODUCTION

Many problems in signal processing can be formulated as tensor decomposition problems involving Toeplitz or block-Toeplitz structured matrix factors. Such problems appear in blind system identification [8], [18], [19] and in convolutive blind signal separation [2], [4], [11]. Moreover, they also appear in biomedical engineering [10]. However, the block-Toeplitz structure has been more or less ignored in the existing literature. Consequently no identifiability results or computational methods tailored for the problems have been suggested so far.

This motivated us to develop identifiability conditions and numerical methods for tensor decompositions with block-Toeplitz structures. In this short communication we will focus on convolutive blind signal separation [4], [11]. However, the results can for instance also be used in blind system identification problems [8], [18], [19].

The paper is organized as follows. The rest of the introduction will present our notation followed by a quick review of the Canonical Polyadic Decomposition (CPD), $(L, L, 1)$- and $(\cdot, L, P)$-block tensor decompositions. Section 2 reviews the CPD based blind receiver for oversampled signals. Next, in sections 3 and 4 we illustrate how block-Toeplitz structured tensors appear in convolutive blind signal separation and what benefits we can expect by exploiting the structure. In section 5 some numerical experiments are reported while section 6 concludes the paper.

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1.1. Notations

Vectors, matrices and tensors are denoted by lower case boldface, upper case boldface and upper case calligraphic letters, respectively.

Let Blockdiag $(\mathbf{X}_1, \ldots, \mathbf{X}_R)$ denote a block-diagonal matrix holding the matrices $\{\mathbf{X}_r\}$ on its block-diagonal.

The outer product of N vectors $\mathbf{a}^{(n)} \in \mathbb{C}^{I_n}$ is denoted by $\circ$ such that $\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}$ satisfies

$$\left(\mathbf{a}^{(1)} \circ \mathbf{a}^{(2)} \circ \cdots \circ \mathbf{a}^{(N)}\right)_{i_1, i_2, \ldots, i_N} = a^{(1)}_{i_1} a^{(2)}_{i_2} \cdots a^{(N)}_{i_N}.$$

Given a tensor $\mathcal{T} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ and a matrix $\mathbf{B} \in \mathbb{C}^{J_1 \times J_2}$. Let $I_k = J_2$ for some $k \in [1, N]$, then the mode-k product of $\mathcal{T}$ and $\mathbf{B}$ is equal to $\mathcal{Y} = \mathcal{A} \underset{k}{\bullet} \mathbf{B} \in \mathbb{C}^{I_1 \times \cdots \times I_{k-1} \times J_1 \times I_{k+1} \times \cdots \times I_N}$, defined by

$$\mathcal{Y}_{i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_n} = \left(\mathcal{T} \underset{k}{\bullet} \mathbf{B}\right)_{i_1, \ldots, i_{k-1}, j_k, i_{k+1}, \ldots, i_n} = \sum_{i_k=1}^{I_k} \mathcal{T}_{i_1, \ldots, i_k, \ldots, i_m} \mathbf{B}_{j_k, i_k}.$$

1.2. Canonical Polyadic Decomposition

A third order rank-1 tensor $\mathcal{X} \in \mathbb{C}^{I \times J \times K}$ is defined as the outer product of non-zero vectors $\mathbf{a} \in \mathbb{C}^I$,$\mathbf{b} \in \mathbb{C}^J$ and $\mathbf{c} \in \mathbb{C}^K$ such that $\mathcal{X}_{ijk} = a_i b_j c_k$. The rank of a tensor $\mathcal{X}$ is equal to the minimal number of rank-1 tensors that yield $\mathcal{X}$ in a linear combination. Assume that the rank of $\mathcal{X}$ is R, then it can be written as

$$\mathcal{X} = \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r, \tag{1}$$

where $\mathbf{a}_r \in \mathbb{C}^I$, $\mathbf{b}_r \in \mathbb{C}^J$ and $\mathbf{c}_r \in \mathbb{C}^K$. This decomposition will be referred to as the CPD of $\mathcal{X}$.[1] It can be shown that

[1] The CPD is also known as the PARAFAC decomposition [7] or the CANDECOMP [1].

under mild conditions the CPD of a tensor is essentially unique [9], [3], i.e., it is unique up to the inherent permutation and scaling ambiguites of the decomposition.

1.3. $(L, L, 1)$-Block Tensor Decomposition

We say that a tensor $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$ is a $(L, L, 1)$-BTD if it can be decomposed as [5]:

$$\mathcal{T} = \sum_{r=1}^{R} \left(\mathbf{A}^{(r)} \mathbf{B}^{(r)T} \right) \circ \mathbf{c}^{(r)}, \tag{2}$$

where

$$\begin{aligned} \mathbf{A}^{(r)} &\in \mathbb{C}^{I \times L} \\ \mathbf{B}^{(r)} &\in \mathbb{C}^{J \times L} \\ \mathbf{c}^{(r)} &\in \mathbb{C}^{K}. \end{aligned}$$

We say that the $(L, L, 1)$-BTD of $\mathcal{T}$ in (2) is essentially unique if all triplets $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}\right)$ satisfying relation (2) are related via

$$\begin{aligned} \widehat{\mathbf{A}} &= \mathbf{A}\text{Blockdiag}\left(\alpha_1 \mathbf{F}_1, \ldots, \alpha_R \mathbf{F}_R\right) \\ \widehat{\mathbf{B}} &= \mathbf{B}\text{Blockdiag}\left(\beta_1 \mathbf{F}_1^{-1}, \ldots, \beta_R \mathbf{F}_R^{-1}\right) \\ \widehat{\mathbf{c}}^{(r)} &= \gamma_r \mathbf{c}^{(r)} \end{aligned}$$

up to a permutation of the $(L, L, 1)$-terms, where $\mathbf{F}_r \in \mathbb{C}^{L \times L}$ are nonsingular matrices and $\alpha_r, \beta_r, \gamma_r \in \mathbb{C}$ are scalars satisfying $\alpha_r \beta_r \gamma_r = 1$, $\forall r \in [1, R]$. Essential uniqueness conditions for the $(L, L, 1)$-BTD can be found in [5], [12], [6].

1.4. $(\cdot, L, P)$-Block Tensor Decomposition

We say that a tensor $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$ is a $(\cdot, L, P)$-BTD if it can written as [5]:

$$\mathcal{T} = \sum_{r=1}^{R} \mathcal{H}^{(r)} \underset{2}{\bullet} \mathbf{A}^{(r)} \underset{3}{\bullet} \mathbf{B}^{(r)}, \tag{3}$$

where

$$\begin{aligned} \mathcal{H}^{(r)} &\in \mathbb{C}^{I \times L \times P} \\ \mathbf{A}^{(r)} &\in \mathbb{C}^{J \times L} \\ \mathbf{B}^{(r)} &\in \mathbb{C}^{K \times P}. \end{aligned}$$

We say that the $(\cdot, L, P)$-BTD of $\mathcal{T}$ in (3) is essentially unique if all triplets $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \{\widehat{\mathcal{H}}^{(r)}\}\right)$ satisfying relation (3) are related via

$$\begin{aligned} \widehat{\mathbf{A}} &= \mathbf{A}\text{Blockdiag}\left(\mathbf{F}_1, \ldots, \mathbf{F}_R\right) \\ \widehat{\mathbf{B}} &= \mathbf{B}\text{Blockdiag}\left(\mathbf{G}_1, \ldots, \mathbf{G}_R\right) \\ \widehat{\mathcal{H}}^{(r)} &= \mathcal{H}^{(r)} \underset{2}{\bullet} \mathbf{F}_r^{-1} \underset{3}{\bullet} \mathbf{G}_r^{-1} \end{aligned}$$

up to a permutation of the $(\cdot, L, P)$-terms, where $\mathbf{F}_r \in \mathbb{C}^{L \times L}$ and $\mathbf{G}_r \in \mathbb{C}^{P \times P}$ are nonsingular matrices. Essential uniqueness conditions for the $(\cdot, L, P)$-BTD can be found in [5].

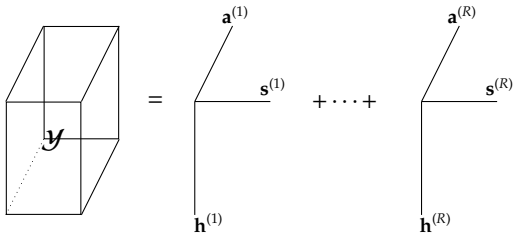


Fig. 1. A visual representation of the CPD of the observation tensor $\mathcal{Y}$ of the form (4) for the case of flat fading channels.

2. BLIND SEPARATION OF OVERSAMPLED SIGNALS

The connection between tensor decompositions and blind separation of oversampled signals, such as synchronous DS-CDMA was established in [13]. It was done by formulating the separation as a CPD problem. More precisely, consider a DS-CDMA uplink system equipped with I antennas and R users transmitting synchronously over a flat fading channel. It was explained that the output of the ith receive antenna at the jth chip period and kth symbol period is given by

$$y_{jki} = \sum_{r=1}^{R} h_{jr} s_{kr} a_{ir},$$

where h_{jr}, s_{kr} and a_{ir} denote the channel gain coefficient associated with user r at the jth chip period, the transmitted symbol from user r at the kth symbol period and the antenna gain response between user r and the ith antenna. By storing the coefficients into vectors as follows $\left(\mathbf{h}^{(r)}\right)_j = h_{jr}$, $\left(\mathbf{s}^{(r)}\right)_k = s_{kr}$ and $\left(\mathbf{a}^{(r)}\right)_i = a_{ir}$, we get the CPD

$$\mathbb{C}^{J \times K \times I} \ni \mathcal{Y} = \sum_{r=1}^{R} \mathbf{h}_r \circ \mathbf{s}_r \circ \mathbf{a}_r, \tag{4}$$

see [13] for further details. A visual representation of the decomposition (4) is given in figure 1. Due to the uniqueness properties associated with the CPD it is possible to blindly recover and separate the transmitted symbol sequences $\{\mathbf{s}_r\}$ based on only the observed data $\mathcal{Y}$. However, in practice the channel may suffer from InterSymbol Interference (ISI) and multipath behaviour. Such properties cannot be handled by the quite restrictive CPD modeling approach.

3. BLIND SEPARATION OF OVERSAMPLED SIGNALS WITH INTERSYMBOL INTERFERENCE

As mentioned earlier, the CPD model is in many practical problems too restrictive. A partial attempt to overcome the ISI problem for the case where multipath propagations take place in the far field was done in [14]. In [4] the problem was considered as a $(L, L, 1)$-BTD problem, as briefly explained next.

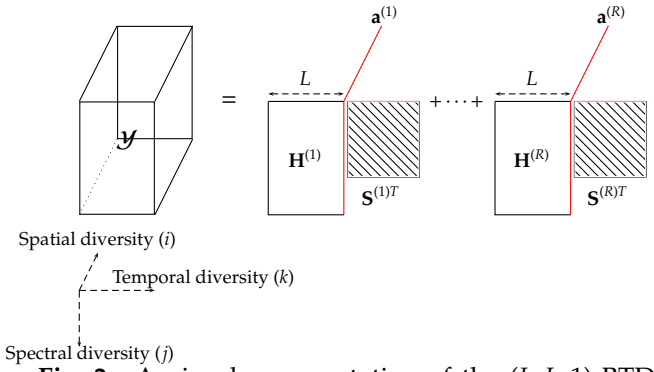


Fig. 2. A visual representation of the $(L, L, 1)$-BTD of the observation tensor $\mathcal{Y}$ of the form (6) for the case of channels with ISI. Note that the symbol matrices $\mathbf{S}^{(r)}$ are Toeplitz structured. The red lines indicate the CPD model that would occur if no ISI was present.

Let the transmitted signal from user r at the kth symbol period be $s_k^{(r)}$. Moreover, let $h_{j+(l-1)J}^{(r)}$ be the result of convolving the channel impulse response and spreading sequence of the rth user, where the coefficient corresponds to the jth chip and the kth symbol period. The signal received by the ith antenna is

$$y_{jki} = \sum_{r=1}^{R} a_{ir} \sum_{l=0}^{L-1} h_{j+(l-1)J}^{(r)} s_{k-l}^{(r)}, \tag{5}$$

where a_{ir} is the response of the ith antenna to the signal of the rth user. It can be shown that if we store the observed data elements y_{jki} into a tensor $\mathcal{Y} \in \mathbb{C}^{J \times K \times I}$, then it admits the factorization

$$\mathcal{Y} = \sum_{r=1}^{R} \left(\mathbf{H}^{(r)} \mathbf{S}^{(r)T} \right) \circ \mathbf{a}^{(r)}, \tag{6}$$

where $\mathbf{H}^{(r)} \in \mathbb{C}^{J \times L}$ contains the coefficients of the convolution of the channel impulse response and spreading sequence associated with user r, $\mathbf{S}^{(r)} \in \mathbb{C}^{K \times L}$ contains the symbol sequences transmitted by user r and $\mathbf{a}^{(r)} \in \mathbb{C}^{I}$ contains the antenna array response coefficients associated with user r, see [4] for further details. Due to the ISI the symbol matrices $\mathbf{S}^{(r)}$ are Toeplitz structured. Hence, by comparing (6) with (2) it is clear that (6) is a block-Toeplitz structured $(L, L, 1)$-BTD. A visual representation of the decomposition (6) can be seen in figure 2.

In [4] the authors relied on more general identifiability results for $(L, L, 1)$-BTDs that do not take the block-Toeplitz structure of the problem into account. We have shown in [15], [16] that by exploiting the block-Toeplitz structure of the problem more relaxed identifiability conditions can be obtained. For instance, it was shown in [16] that a $(L, L, 1)$-BTD with a block-Toeplitz matrix factor is generically essentially unique if

$$\begin{cases} J \geq L \\ K \geq R(L+1)+1 \\ IJ \geq RL \\ JL(JL-1)I(I-1) \geq 2R(R-1) \end{cases}$$

As an example, when the number of available symbol periods is $K = 100$, the spreading code length is fixed to $J = 4$ and the filter order is $L = 2$, then the number of users R we expect to be able to separate as function of number of receive antennas I used can be seen in table 1 for the cases when only the $(L, L, 1)$-BTD structure is exploited and when both the $(L, L, 1)$-BTD structure and block-Toeplitz structure is taken into account. By inspecting the table it is clear that by simultaneously exploiting structures, better identifiability conditions are obtained.

$L = 2,\quad I$	2	3	4	5
$(L, L, 1)$-BTD	2	3	4	5
$(L, L, 1)$-BTD+block-Toeplitz	4	6	8	10

Table 1. Value for R for which the $(L, L, 1)$-BTD and $(L, L, 1)$-BTD+block-Toeplitz conditions are expected to guarantee essential uniqueness of a block-Toeplitz structured $(L, L, 1)$-BTD with $J = 4$, $K = 100$, $L = 2$ and I is varying. (Sufficient condition for generic data.)

Not only can one obtain improved identifiablity results by exploiting the block-Toeplitz structure of the problem, but it also turns out that the problem of computing the $(L, L, 1)$-BTD (5) leads to a Simultaneous Matrix Diagonalization (SMD) problem, which numerically can be solved efficiently, the algorithm is derived in [16]. In section 5 we will show simulation results.

4. BLIND SEPARATION OF OVERSAMPLED SIGNALS WITH INTERSYMBOL INTERFERENCE AND MULTIPLE DIRECTIONS OF ARRIVAL

In [11] the authors generalized the model in the previous section to the case of ISI and multiple directions of arrival, i.e., the multipath is not limited to the far field of the array. As an example we assume that there are R DS-CDMA modulated signals where each user r is transmitting over P discrete paths and the receiver is equipped with I receive antennas. Let the transmitted signal from user r at the kth symbol period be $s_k^{(r)}$. Furthermore, let $h_{j+(l-1)J}^{(r,p)}$ be the convolution of the channel impulse response and spreading sequence of the pth path of the rth user corresponding to the jth chip. The signal received by the ith antenna can be written as

$$y_{jki} = \sum_{r=1}^{R} \sum_{p=1}^{P} a_{ip}^{(r)} \sum_{l=0}^{L-1} h_{j+(l-1)J}^{(r,p)} s_{k-l}^{(r)}, \tag{7}$$

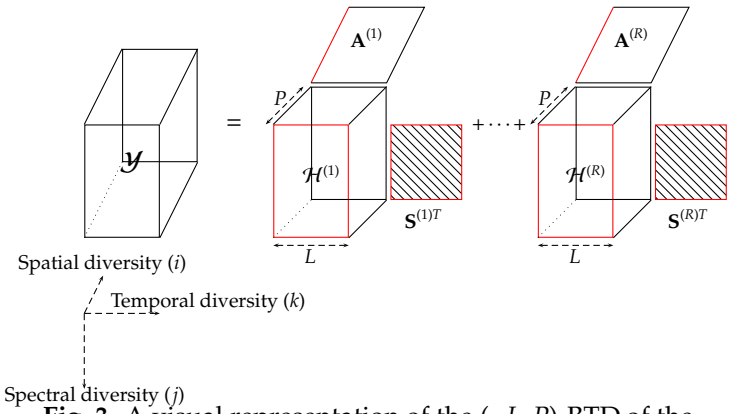


Fig. 3. A visual representation of the $(\cdot, L, P)$-BTD of the observation tensor $\mathcal{Y}$ of the form (8) for the case of channels with ISI and multiple directions of arrival. Note that the symbol matrices $\mathbf{S}^{(r)}$ are Toeplitz structured. The red lines indicate the block-Toeplitz structured $(L, L, 1)$-BTD model that would occur if the multipath occured only in the far field.

where $a_{ip}^{(r)}$ is the response of the ith antenna to the signal of the pth path of user r, see [11] for further details. Stack the channel coefficients $h_{j+(l-1)J}^{(r,p)}$ into the tensors $\mathcal{H}^{(r)} \in \mathbb{C}^{J \times L_r \times P_r}$, the transmitted symbols $s_{k-l}^{(r)}$ into the Toeplitz matrices $\mathbf{S}^{(r)} \in \mathbb{C}^{K \times L}$ and the antenna response coefficients $a_{ip}^{(r)}$ into the matrices $\mathbf{A}^{(r)} \in \mathbb{C}^{I \times P}$, then it can be shown that the observed data (7) can now be stored in the tensor $\mathcal{Y} \in \mathbb{C}^{J \times K \times I}$ admitting the factorization

$$\mathcal{Y} \quad = \quad \sum_{r=1}^{R} \mathcal{H}^{(r)} \bullet_2 \mathbf{S}^{(r)} \bullet_3 \mathbf{A}^{(r)}. \tag{8}$$

Since the matrices $\mathbf{S}^{(r)}$ are Toeplitz structured, (8) is actually a block-Toeplitz structured $(\cdot, L, P)$-BTD. A visual representation of the decomposition (8) is given in figure 3.

In [11] the authors relied on more general identifiability results for $(\cdot, L, P)$-BTDs that do not take the block-Toeplitz structure of the problem into account. We have shown in [15], [17] that by exploiting the block-Toeplitz structure of the problem more relaxed identifiability conditions can be obtained. As an example, when the number of available symbol periods is $K = 100$, the spreading code length is fixed to $J = 4$, the filter order is $L = 2$ and the number of paths associated with each user is $P = 2$, then the number of users R we expect to be able to separate as function of number of receive antennas I used can be seen in table 2 for the cases when only the $(\cdot, L, P)$-BTD structure is exploited and when both the $(\cdot, L, P)$-BTD structure and block-Toeplitz structure is taken into account. Again, by inspecting the table it is clear that by simultaneously the structures better identifiability conditions are obtained.

$P = 2, L = 2, \quad I$	2	3	4	5
$(\cdot, L, P)$-BTD	1	1	2	2
$(\cdot, L, P)$-BTD+block-Toeplitz	4	6	8	10

Table 2. Value for R for which the $(\cdot, L, P)$-BTD and $(\cdot, L, P)$-BTD+block-Toeplitz conditions are expected to guarantee essential uniqueness of a block-Toeplitz structured $(\cdot, L, P)$-BTD with $J = 4$, $K = 100$, $L = 2$, $P = 2$ and I is varying. (Sufficient condition for generic data.)

It turns out that by exploiting the block-Toeplitz structure, the problem of computing the $(\cdot, L, P)$-BTD (8) also leads to a SMD problem, see [17] for further details. In the following section we will also illustrate this by a numerical example.

5. NUMERICAL EXPERIMENTS

Let us compare the SMD method with a randomly initialized Alternating Least Squares (ALS) method. The SMD method in which the block-Toeplitz structure is taken into account is denoted by SMD-Toeplitz. A randomly intialized ALS method in which the block-Toeplitz structure is taken into account is denoted by ALS-Toeplitz. The SMD method be followed by a few ALS fine-tuning steps, is denoted by SMD-ALS-Toeplitz.

The entries of the matrices $\mathbf{H}^{(r)}$, tensors $\mathcal{H}^{(r)}$, antenna response vectors $\mathbf{a}^{(r)}$ and matrices $\mathbf{A}^{(r)}$ are randomly drawn from a Gaussian distribution with zero mean and unit variance while the entries of the symbol matrices $\mathbf{S}^{(r)}$ are randomly drawn from a QPSK constellation. The transmitted data are perturbed by additive Gaussian noise with with zero mean and unit variance.

Let us first consider the case in section 3. The model parameters are $I = 3$, $J = 3$, $K = 100$, $L = 2$ and $R = 3$. The mean SER values over 100 trials while SNR is varying from 0 to 20 dB SNR can be seen in figure 1. We first notice that the computationally efficient SMD-Toeplitz method performs about same as the more costly ALS-Toeplitz method. For this case the simulation indicated that the SMD-Toeplitz method is about ten times faster than the ALS-Toeplitz method. This suggests to use it as a mean to initialize the ALS-Toeplitz method. Indeed, we observe that the SMD-ALS-Toeplitz performs better than the ALS-Toeplitz method. Note that by using several random initalizations, then the ALS-Toeplitz method would likely yield the same performance as the SMD-ALS-Toeplitz method, but at a significantly increased computational cost.

Second,we consider the case in section 4. The model parameters are $I = 3$, $J = 6$, $K = 100$, $L = 2$, $P = 2$ and $R = 3$. The mean SER values over 100 trials while SNR is varying from -10 to 10 dB SNR can be seen in figure 2. Again, we notice that the SMD-ALS-Toeplitz yields

the best performance.

6. CONCLUSION

Many tensor decompositions problems in signals involves block-Toeplitz structures. In this short communication we illustrated its use in blind separation of oversampled signals such as DS-CDMA with ISI and multiple directions of arrival. We illustrated that by simultaneously taking the tensor and block-Toeplitz structure of the problem into account it is possible to obtain better identifiability results and computationally efficient numerical methods. Further details will be presented in [15],[16], [17].

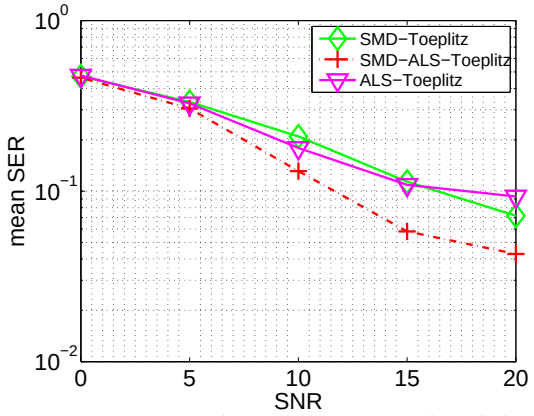


Fig. 4. Mean SER values over 100 trials while SNR is varying from 0 to 20 dB for the $(L, L, 1)$-BTD simulation case.

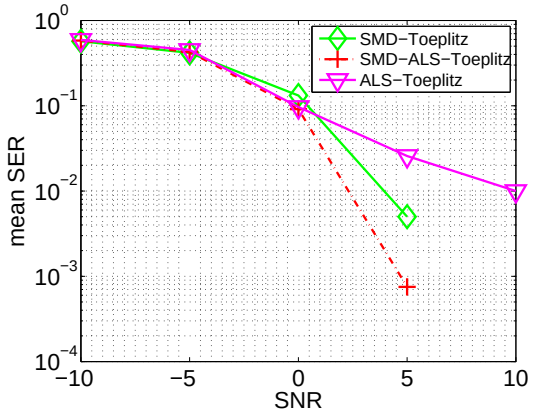


Fig. 5. Mean SER values over 100 trials while SNR is varying from −10 to 10 dB for the $(\cdot, L, P)$-BTD simulation case.

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