Preprints of the 7th IFAC Symposium on Robust Control Design
The International Federation of Automatic Control
Aalborg, Denmark, June 20-22, 2012

Optimization of Stable Periodic Attractors for Nonlinear Dynamic Systems

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Abstract: In this paper we discuss numerical strategies to find periodic orbits of nonlinear dynamic system. These orbits shall be open-loop stable and robust with respect to bounded disturbances. For this aim, we review and extend existing techniques from the field of reachability analysis and ellipsoidal calculus to compute robust positive invariant tubes for nonlinear dynamic systems. We suggest a conservative approximation strategy for robust stability optimization providing guarantees on the region of attraction. The technique is tested by applying it to an inverted spring pendulum for which robust and open-loop stable orbits exist.

1. INTRODUCTION

The question of the stability and existence of periodic orbits of dynamic systems has attracted many researchers during this and the last century. Starting with the original work of Lyapunov [1907] many contributions have been published. For example, at the end of the 20th century, Matthieu and Hill have analyzed an interesting class of differential equations, the Matthieu-Hill differential equations, for which it can be proven that non-trivial open-loop stable periodic orbits exist (cf. Wolf [2010]) and which can be seen as an important prototype class of problems for which nontrivial open-loop stable orbits can be observed.

In general, it is extremely difficult to analyze the periodic orbits of a nonlinear dynamic system. For example Hilbert's 16th problem (published in 1900) is asking for the number and configuration of the periodic limit cycles of a general polynomial vector field in the plane. In fact, this problem is up to now still unsolved (cf. Llibre [2008]) and must be considered as one of the hardest problems ever posed in mathematics. This illustrates how difficult the analysis of such periodic cycles can be – and here we talk about a dynamic system with two differential states only. On the other hand, in practical applications, we have often at least a rough idea or physical intuition of when and where periodic cycles can be expected. Here, we can think of periodically driven spring-damper systems, periodic thermodynamic Carnot processes, bicycles, humanoid and walking robots, controllable kites, many periodically operating power generating devices, etc. Thus the question how to find and optimize the stability of periodic orbits numerically is highly relevant and, of course, this question has also been addressed by many authors.

Starting with the work of Kalman [1963] and Bittanti et al. [1991] periodic Lyapunov and Riccati equations became an important field of research for analyzing the stability of linear periodic systems. Some of the existing modern robust stability optimization techniques are based on the optimization of the so called pseudo-spectral abscissa. In this context we refer to the work of Burke et al. [2003, 2006] as well as to the work of Trefethen and Embree [2005]. In these approaches non-smooth (but derivative based) optimization algorithms are developed. Similar approaches have been proposed in Vanbiervliet et al. [2009] and Diehl et al. [2009], where a smoothed version of the spectral abscissa is optimized such that existing derivative based, local optimal control techniques can be employed. For interesting applications of open-loop stability optimization we refer to Mombaur [2001] and Lü et al. [2006].

In this paper, we are interested in both the robustness and the stability of periodic orbits. Here, we consider a nonlinear dynamic system which depends on an uncertain but bounded time-varying input such that the state of the system is only known to be in a certain reachability tube (cf. Bertsekas and Rhodes [1971], Rakovic and Kouramas [2006], Rakovic and Fiacchini [2008]). In this context, we employ existing concepts from the field of reachability analysis (Aubin [1991], J. Lygeros and Sastry [1999], I.M. Mitchell and Tomlin [2005]) and ellipsoidal calculus (Kurzhanski and Varaiya [2002]). We propose a technique to compute and optimize an ellipsoidal outer tube containing the reachable points in state space. For periodic systems it is sometimes possible to find reachability periodic tubes. For the case that we find such a periodic tube it is - under some mild additional assumption - possible to show the robust stability of the dynamic system, as for example discussed by Blanchini and Miani [2008].

The contribution of this paper is that we show how to use the existing ellipsoidal analysis techniques to solve robust optimal control and stability optimization problems with state of the art optimal control software. For this aim, we start in Section 2 with the mathematical problem formulation while Section 3 introduces ellipsoidal outer approximation strategies for the reachable tube in an uncertain dynamic system. An approximate solution strategy for robust and stable optimal control is presented in Section 4. Finally, we demonstrate in Section 5 how the presented techniques can be used to find and optimize open-loop stable orbits of an inverted spring pendulum.

Notation: We denote with $\Pi(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$. The set $\mathbb{S}^n_x \subset \mathbb{R}^{n\times n}$ denotes the set of symmetric and positive semi-definite matrices in $\mathbb{R}^{n\times n}$.

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2. PROBLEM FORMULATION

In this paper we are interested in uncertain dynamic system of the form

$$\forall \tau \in \mathbb{R}: \quad \dot{x}(\tau) \;=\; f(u(\tau), x(\tau), w(\tau))\,, \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ the control input, and $w(t) \in W \subseteq \mathbb{R}^{n_w}$ an uncertain function which affects the dynamics. Here, we assume that the right-hand side function f is uniformly Lipschitz continuous in x, while the uncertainty set $W \subseteq \mathbb{R}^{n_w}$ is compact and given.

Definition 1. A set-valued function $X : [t_1, t_2] \to \Pi(\mathbb{R}^{n_x})$ is called a robust positive invariant tube on the interval $[t_1, t_2]$, if the inclusion

$$X(t') \;\supseteq\; \left\{ x(t') \in \mathbb{R}^{n_x} \;\middle|\; \begin{array}{l} \exists x(\cdot), w(\cdot): \\ x(t) \in X(t) \\ \dot{x}(\tau) \;=\; f(u(\tau), x(\tau), w(\tau)) \\ w(\tau) \in W \quad \text{for all } \tau \in [t, t'] \end{array} \right\}$$

is satisfied for all $t, t' \in [t_1, t_2]$ with $t' \geq t$.

In the following, we say that the function X satisfies an inclusion of the form

$$\forall \tau \in [0, T_\mathrm{e}]: \quad X(\tau^+) \;\supseteq\; F(u(\tau), X(\tau), W)\,, \quad X(0) = X_0\,,$$

if and only if it is a robust positive invariant tube on the interval $[0, T_\mathrm{e}]$. Note that this is only a formal notation which is motivated by the fact that a forward propagation "F" of the reachable set $X(\tau)$ by an infinitesimal time step $\tau^+ - \tau$ depends on the current set $X(\tau)$, the current control input $u(\tau)$, and the current uncertainty bound W only [1].

The periodic robust optimization problems of our interest can now be written as

$$\begin{aligned} \min_{u(\cdot), T_\mathrm{e}, X(\cdot)} \quad & J[u(\cdot), T_\mathrm{e}, X(\cdot)] \\ \text{s.t.} \quad & X(\tau^+) \;\supseteq\; F(u(\tau), X(\tau), W) \\ & X(0) \;\supseteq\; X(T_\mathrm{e}) \\ & 0 \;\geq\; H(\tau, u(\tau), X(\tau), W) \;\text{ f.a. }\; \tau \in [0, T_\mathrm{e}]\,. \end{aligned} \tag{2}$$

Here, J denotes an objective functional while the function H can be used to formulate robust path constraints. In this paper, we assume that H is the robust counterpart function of a given path constraint function h which is defined component-wise (with $i \in \{1, \ldots, n_H\}$) as

$$H_i(\tau, u(\tau), X(\tau), W(\tau)) \;:=\; \sup_{\substack{x \in X(\tau) \\ w \in W}} h_i(\tau, u(\tau), x, w)\,.$$

The interpretation of this constraint is the following: if we have $H_i(\tau, u(\tau), X(\tau), W(\tau)) \leq 0$, then we know that the constraint of the form $h_i(\tau, u(\tau), x, w) \leq 0$ must be satisfied for all feasible choices of the uncertainty function w. In other words, we have a way to impose robust satisfaction of path constraints. As we allow an explicit time dependence of h, robust point constraints can be formulated with the above technique, too.

An interesting point of the above formulation is the periodic boundary inclusion $X(0) \supseteq X(T_\mathrm{e})$ —a constraint which is in a similar fashion used to define invariant sets in control (Blanchini and Miani [2008]). If this inclusion is satisfied, we can continue the control input u periodically such that there exists a continuation of our robust positive invariant tube which satisfies $X(t + T_\mathrm{e}) \subseteq X(t)$ for all $t \in \mathbb{R}$. In other words, if we start the dynamic system with an initial value $x_0 \in X(0)$ the state of the system will remain inside the bounded tube X for all times $t \geq 0$ and for all feasible uncertainty realizations w satisfying $w(\tau) \in W$.

[1] Alternatively, our notation for uncertainty propagation in dynamic systems could also be replaced by a differential inclusion (cf. Deimling [1992]).

3. ELLIPSOIDAL OUTER APPROXIMATION STRATEGIES

The major difficulty with optimal control problems of the form (2) is that the optimization variable X is a set valued function. This is in contrast to standard nonlinear optimal control problems where all optimization variables are vector valued functions or parameters. Thus, in order to apply numerical techniques, we have to find a way to store set valued functions in a computer. One way to do this is to not search among all set valued functions but try to find at least an optimized ellipsoidal tube which is a feasible point of the problem (2). Here, the advantage of ellipsoids

$$\mathscr{E}(Q, q) \;:=\; \left\{ \; q + Q^{\frac{1}{2}} v \;\middle|\; \exists v \in \mathbb{R}^n: \;\; v^T v \;\leq\; 1 \right\}.$$

is that they can be characterized by their center $q \in \mathbb{R}^{n_x}$ and a symmetric and positive semi-definite matrix $Q \in \mathbb{S}^{n_x}_+$. Most of the existing ellipsoidal outer approximation strategies are in one or the other way based on the following Lemma:

Lemma 1. Let $\sum_{i=1}^N \mathscr{E}(Q_i, q_i)$ be the Minkowski sum of N given ellipsoids with $Q_i \succeq 0$. If we choose any positive multipliers $\lambda_1, \ldots, \lambda_N$ with $\sum_{i=1}^N \lambda_i = 1$ then we have

$$\sum_{i=1}^N \mathscr{E}(Q_i, q_i) \;\subseteq\; \mathscr{E}\left(\sum_{i=1}^N \frac{1}{\lambda_i} Q_i\,, \sum_{i=1}^N q_i \right).$$

In other words, any positive vector $\lambda \in \mathbb{R}^N_{++}$ yields an ellipsoidal outer approximation of the considered Minkowski sum.

A proof of this Lemma can, e.g., be found in the work of Ben-Tal et al. [2009] or Kurzhanski and Varaiya [2002].

For uncertainty affine dynamic systems it is well-known how to compute ellipsoidal outer approximations of the reachable sets. The corresponding techniques have mainly been developed by Kurzhanski and Filippova [1993], Kurzhanski and Varaiya [2002] which basically transfer the above Lemma to an infinite sum of ellipsoids. For example, if we consider a linear dynamic system of the form

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \quad \text{with} \quad x(0) \in X_0 := \mathscr{E}(Q_0, 0)$$

and $W := \{w \in \mathbb{R}^{n_w} \mid w^\top w \leq 1\}$, we can compute a robust positive invariant tube of the form $X(t) = \mathscr{E}(Q(t), 0)$ choosing any positive function $\kappa > 0$ and propagating a Lyapunov differential equation of the form

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)^\top + \kappa(t)Q(t) + \frac{1}{\kappa(t)} B(t)B(t)^\top\,.$$

which can be started with $Q(0) = Q_0$. A proof of this statement can, e.g., be found in Kurzhanski and Varaiya [2002] or the references therein.

For general nonlinear dynamic systems, the computation of ellipsoidal robust positive invariant tubes is more difficult than for uncertainty affine dynamics. The main strategy which we propose to deal with this issue is to carefully overestimate the influence of the nonlinear terms in the dynamic system. For this

aim, we define a reference function q to be the solution of the dynamic system for no uncertainty, i.e., for $w = 0$, such that

$$\forall \tau \in \mathbb{R}: \quad \dot{q}(\tau) = f(u(\tau), q(\tau), 0) \quad \text{with} \quad q(0) = q_0 .$$

Now, we decompose the dynamic system into a linear and a nonlinear part:

$$\dot{x}(\tau) = d(\tau) + A(\tau)(x(\tau) - q(\tau)) + B(\tau)w(\tau) + f_{\mathrm{n}}(u(\tau), q(\tau), x(\tau), w(\tau)) .$$

Here, $A := \partial_x f$ and $B := \partial_w f$ are partial derivatives of the right hand side function f with respect to x and w while the function f_{n} collects all nonlinear terms such that the dynamic equations coincide with the original system. A practical way to overestimate the nonlinear terms is to construct parametric bounds on the components f_{n}^i which may depend on the current uncertainty set. In the following, we assume that we have functions l_i such that

$$\left| f_{\mathrm{n}}^i(u, q, x, w) \right| \leq l_i(u, q, Q)$$

for all $x \in \mathscr{E}(Q, q)$, for all $w \in W$, and for all $Q \in S_+^{n_x}$. Now, the idea is to transfer the existing techniques from the field of ellipsoidal calculus by regarding the nonlinearities as additional locally bounded uncertainties. For this aim, we add the contributions of all the unknown terms in one matrix valued function

$$\Omega(t, u, \kappa, q, Q) := \frac{B(t)B(t)^\top}{\kappa_0} + \operatorname{diag}\left(\frac{l_1(u, q, Q)^2}{\kappa_1}, \ldots, \frac{l_{n_x}(u, q, Q)^2}{\kappa_{n_x}} \right) .$$

Now, if we choose any positive function $\kappa(t) \in \mathbb{R}_{++}^{n_x+1}$ and propagate the differential equation

$$\begin{aligned} \dot{Q}(t) &= \Phi(t, u(t), \kappa(t), q(t), Q(t)) \\ &:= A(t)Q(t) + Q(t)A(t)^\top + \sum_{i=0}^{n_x} \kappa_i(t) Q(t) \\ &\quad + \Omega(t, u(t), \kappa(t), q(t), Q(t)) \end{aligned}$$

with $Q(0) = Q_0$ being a given symmetric and positive semi-definite matrix, then $X(t) := \mathscr{E}(Q(t), q(t))$ is a robust positive invariant tube for the nonlinear dynamic system

$$\dot{x}(t) = f(u(t), x(t), w(t)) \quad \text{with} \quad x(0) \in \mathscr{E}(Q_0, q_0) \tag{3}$$

and $W := \{w \in \mathbb{R}^{n_w} \mid w^\top w \leq 1\}$. Note that a proof of this statement follows immediately if we combine Lemma 1 with the analysis for uncertainty affine dynamic system by Kurzhanski and Varaiya [2002]. This is due to the fact that the uncertain terms in the sum

$$w(t) + \sum_{i=1}^{n_x} e_i f_{\mathrm{n}}^i(u(t), q(t), x(t), w(t))$$

are at each time t known to be bounded in a sum of $n_x + 1$ ellipsoids such that the above ellipsoidal outer approximation strategy can be applied. For the details of this argumentation we refer to Houska [2011]. Finally, we note that for the nonlinear case, the differential equation for Q is not necessarily a Lyapunov equation anymore, as the function Ω is nonlinear in Q. Thus, in the nonlinear case, we have in general neither a guarantee for the existence of solutions of this differential equation nor a guarantee on how accurate the ellipsoidal outer approximation is. However, we shall see later that the above approach works well in practical examples where the uncertainties are not too large. This is due to fact that we can still optimize the function κ which parameterizes our ellipsoidal outer approximation.

4. CONSERVATIVE ROBUST STABILITY OPTIMIZATION FOR DYNAMIC SYSTEMS

Using the ellipsoidal techniques from the previous section for the construction of robust positive invariant tubes, we find a strategy to solve the original robust optimization problem (2) in a conservative approximation. Here, we suggest to solve a problem of the form

$$\min_{u(\cdot), \kappa(\tau), T_{\mathrm{e}}, q(\cdot), Q(\cdot)} J[u(\cdot), T_{\mathrm{e}}, \mathscr{E}(q(\cdot), Q(\cdot))]$$
$$\text{s.t.} \quad \begin{cases} \forall \tau \in [0, T_{\mathrm{e}}]: \\ \dot{q}(\tau) = f(u(\tau), q(\tau), 0) \\ \dot{Q}(\tau) = \Phi(t, u(t), \kappa(t), q(t), Q(t)) \\ q(0) = q(T_{\mathrm{e}}) \\ Q(0) = Q(T_{\mathrm{e}}) \\ \kappa(\tau) > 0 \\ Q(\tau) \succeq 0 \\ 0 \geq H(\tau, u(\tau), \mathscr{E}(q(\tau), Q(\tau)), W) . \end{cases} \tag{4}$$

Note that this optimization problem is similar to the original problem (2) but the difference is that the set valued function X has been replaced by the parameterized ellipsoidal tube $\mathscr{E}(q(\cdot), Q(\cdot))$ which is for all inputs $\kappa > 0$ robust positive invariant, as shown in the previous section. In some cases it is possible to transform the problem (4) further such that we end up with a standard optimal control problem. For example, if the original path constraint is a simple affine constraint of the form

$$h_i(\tau, u(\tau), x, w) = C_i(t)x(t) + D_i(t)u(t) + e_i(t) \leq 0 ,$$

we can find an explicit expression for the associated robust path constraint:

$$H_i(\tau, u(\tau), \mathscr{E}(q(\tau), Q(\tau)), W) = C_i(t)q(t) + D_i(t)u(t) + e_i(t) + \sqrt{C_i(t)P(t)C_i(t)^\top} .$$

Now, we know that every feasible solution of the auxiliary problem (4) corresponds to a feasible point of the original set-valued optimal control problem (2). This is due to the fact that the propagation of Q has been constructed in such a way that the ellipsoidal tube $\mathscr{E}(q(\cdot), Q(\cdot))$ is always robust positive invariant with respect to the given nonlinear system and with respect to the uncertainty set $W := \{w \in \mathbb{R}^{n_w} \mid w^\top w \leq 1\}$.

Finally, we note that if we can find a solution of problem (4) then the tube $\mathscr{E}(q(\cdot), Q(\cdot))$ can be continued periodically, i.e., we know that once we start the system with an initial value inside this tube, the state of the dynamic system will remain inside this tube for all times and for all feasible disturbances w. Additionally, we can make a statement about the stability and region of attraction of the nominal periodic orbit q:

Lemma 2. Let us assume that the solution of the linear Lyapunov equation

$$\dot{\Delta}(\tau) = A(\tau)\Delta(\tau) + \Delta(\tau)A(\tau)^\top + B(t)B(t)^\top , \quad \Delta(0) = 0$$

has a solution Δ with $\Delta(T_{\mathrm{e}}) \succ 0$, i.e., we assume that the system (A, B) is reachable. If the nonlinearity estimates satisfy $l_i(u, q, \alpha Q_1) \leq \sqrt{\alpha} l_i(u, q, Q_2)$ for all $Q_1, Q_2 \in \mathbb{S}_+^{n_x}$ with $Q_1 \preceq Q_2$ and all $\alpha \in [0, 1]$ and if (u, q, Q) corresponds to a periodic solution of the problem (2), then the differential equation

$$\dot{y}(t) = f(u(t), y(t), 0) \quad \text{with} \quad y(0) = y_0$$

is asymptotically attracted by the orbit $q(t)$ for all $y_0 \in \mathscr{E}(q(0),Q(0))$.

Proof: The main idea of the proof is to analyze a differential equation of the form

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^\top + \sum_{i=1}^{n_x} \kappa_i(t)P(t) + \operatorname{diag}\left(\frac{l_1(u(t),q(t),P(t))^2}{\kappa_1(t)}, \ldots, \frac{l_{n_x}(u(t),q(t),P(t))^2}{\kappa_{n_x}(t)} \right)$$

for all $t \in [0,\infty)$ and $P(0) = Q(0)$. As the differential equation $\dot{y}(t) = f(u(t),y(t),0)$ is not affected by uncertainties, we must have $y(t) \in \mathscr{E}(q(t),P(t))$. Thus, our aim is to show that we have $\overline{\lim_t} \|P(t)\| \to 0$ as this implies $\overline{\lim_t} \|y(t)-q(t)\| \to 0$.

Now the idea is to compute the difference $\bar{\Delta}(t) := Q(t) - P(t)$ between the two matrix valued functions Q and P via a differential equation of the form

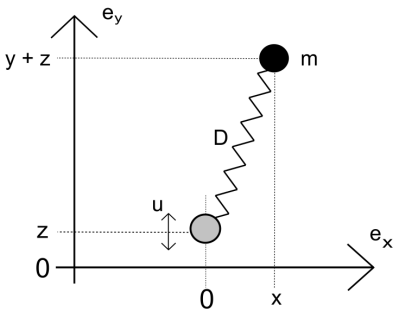
$$\frac{\partial}{\partial t}\bar{\Delta}(\tau) = A(\tau)\bar{\Delta}(\tau) + \bar{\Delta}(\tau)A(\tau)^\top + \sum_{i=1}^{n_x} \kappa_i(t)\bar{\Delta}(t) + \frac{B(t)B(t)^\top}{\kappa_0}, \quad \bar{\Delta}(0) = 0$$

which satisfies $\bar{\Delta}(t) \geq \frac{1}{\kappa_0}\Delta(t)$ for all $t \in [0,T_\mathrm{e}]$. Thus, we can use our reachability assumption to conclude that we have $\bar{\Delta}(T_\mathrm{e}) \succ 0$. As we have $l_i(u,q,P) \leq l_i(u,q,Q)$ whenever $P \preceq Q$ we must have $P(T) \prec P(0)$. In other words, there exists an $\alpha \in [0,1)$ with $P(T) \preceq \alpha P(0)$. Finally, periodic continuation yields $P(nT) \preceq \alpha^n P(0)$ as we can use our scaling assumption on the functions l_i. The latter statement implies $\overline{\lim_t} \|P(t)\| \to 0$ which is equivalent to the statement of the Lemma. □

Summarizing the properties of formulation (2) we have first shown that any solution of this auxiliary problem yields a feasible but possibly sub-optimal solution of the original set-valued robust optimal control problem (2). And second, if the additional requirements from Lemma 2 are satisfied, we can even show the nominal asymptotic open-loop stability of the nonlinear dynamic system providing an explicit guarantee on the region of attraction.

5. OPEN-LOOP STABLE ORBITS OF AN INVERTED SPRING PENDULUM

In this section, we illustrate how the techniques from the previous section can be used to find open loop stable orbits for periodic systems in practice. In order to derive a simple but nonlinear model for an inverted spring pendulum in the 2-dimensional Euclidean space $\mathbb{R}^2$, we first introduce the mass m, which is attached at one end of a spring with given relaxed length l and spring constant D. The other end of the spring is mounted at a point which can move along the vertical axis. We assume that this mounting point has at time t the coordinate $(0,z(t))^T$, while we can control the associated acceleration $u(t) := \ddot{z}(t)$. The velocity of the mounting point will be denoted by $v_z(t) := \dot{z}(t)$. Moreover, we assume that the position of the mass point is given by $(x(t),z(t)+y(t))^T$, i.e., (x,y) is the relative position coordinate of the mass with respect to the moving oscillatory base. Figure 1 shows a sketch of this construction as well as the numerical values for the given physical constants.



Param.	Value
L	$1\,\mathrm{m}$
m	$0.1\,\mathrm{kg}$
D	$700\,\frac{\mathrm{N}}{\mathrm{m}}$
g	$9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$
b	$5\,\frac{1}{\mathrm{s}}$
$\overline{w}$	$0.03\,\frac{\mathrm{N}}{\mathrm{kg}}$
$\overline{u}$	$200\,\frac{\mathrm{N}}{\mathrm{kg}}$
$\overline{v}_z$	$3.2\,\frac{\mathrm{m}}{\mathrm{s}}$

Fig. 1. A sketch of the inverted spring pendulum showing the choice of coordinates as well as all given numerical values for the parameters which are needed within the problem formulation.

Summarizing the states of our dynamic system within one vector $\xi := (x,y,v_x,v_y,z,v_z)$, the stacked equations of motion $\dot{\xi}(t) = f(\xi(t),u(t),w(t))$ can be derived by an application of Newton's law:

$$f(\xi,u,w) = \begin{pmatrix} v_x \\ v_y \\ -\frac{Dx}{m}\left(1 - \frac{l}{\sqrt{x^2+y^2}}\right) - bv_x + w \\ -g + u - \frac{Dy}{m}\left(1 - \frac{l}{\sqrt{x^2+y^2}}\right) - bv_y \\ v_z \\ u \end{pmatrix}.$$

In this context, the function w is assumed to be an uncertain force acting at the mass point in horizontal direction. The associated uncertainty set is in our example assumed to be a simple interval of the form $W(\tau) := [-\overline{w},\overline{w}]$.

Our aim is to operate the spring pendulum in an open-loop stable periodic orbit with period time $T_\mathrm{e} \in \mathbb{R}_{++}$ at its "inverted" position. Our objective is to minimize a generalized Lagrange term of the form

$$J[u(\cdot),T_\mathrm{e},X(\cdot)] := \int_0^{T_\mathrm{e}} \max_{\xi \in X(\tau)} \frac{\left(e_x^T \xi\right)^2}{T_\mathrm{e}}\,\mathrm{d}\tau$$

with $e_x^T := (1,0,\ldots,0)^T \in \mathbb{R}^6$. The constraint function H is in our example given by

$$H(\tau,u(\tau),X(\tau)) := \begin{pmatrix} u(\tau) - \overline{u} \\ -u(\tau) + \overline{u} \\ \max\limits_{\xi \in X(\tau)} e_{v_z}^T \xi - \overline{v}_z \\ \min\limits_{\xi \in X(\tau)} -e_{v_z}^T \xi + \overline{v}_z \end{pmatrix}$$

with $e_{v_z}^T := (0,\ldots,0,1)^T \in \mathbb{R}^6$. The values for these bounds are all given in Figure 1. Note that the period time $T_\mathrm{e} > 0$ is a free optimization variable, too.

In order to construct a conservative but tractable formulation for this optimal control problem, we need to find a suitable nonlinearity estimate. For this aim, we first observe that only the third and fourth component of the right-hand side function include nonlinear terms. In order to over-estimate the influence of these terms, we use the nonlinearity estimates

$$l_3(q,Q) := \frac{Dl}{m}\frac{\sqrt{Q_{11}Q_{22}}}{q_2(q_2-\sqrt{Q_{22}})} + \frac{1}{2}\frac{Dl}{m}\frac{(Q_{11})^{\frac{3}{2}}}{q_2(q_2-\sqrt{Q_{22}})^2} \quad (5)$$

$$l_4(q,Q) := \frac{Dl}{m}\frac{Q_{11}}{(q_2-\sqrt{Q_{22}})^2}\,. \quad (6)$$

Note that these nonlinearity estimates satisfy the requirements from Lemma 2. For example the estimate (6) can be proven by noting that the nonlinear term in the fourth component of the differential equation satisfies

$$\left|\frac{q_2+\Delta y}{\sqrt{(\Delta x)^2+(q_2+\Delta y)^2}} - 1\right| \leq \frac{(\Delta x)^2}{|q_2+\Delta y|\sqrt{(\Delta x)^2+(q_2+\Delta y)^2}}$$
$$\leq \frac{(\Delta x)^2}{(q_2+\Delta y)^2} \leq \frac{Q_{11}}{(q_2-\sqrt{Q_{22}})^2}$$

using $|\Delta x| \leq \sqrt{Q_{11}}$ and $|\Delta y| \leq \sqrt{Q_{22}}$ as well as $q_{33} = 0$. The nonlinearity estimate for the third component can be derived analogously. Now, we can setup the robust periodic optimal control problem of the form (2) solving it approximately based on the formulation (4). Here, we mention that the Lagrange term can be evaluated as

$$J[u(\cdot), T_\mathrm{e}, \mathscr{E}(q(\cdot),Q(\cdot))] = \int_0^{T_\mathrm{e}} \frac{Q_{1,1}(\tau)}{T_\mathrm{e}}\,\mathrm{d}\tau\,.$$

Note that the problem of the form (4) requires in this example 6 differential states to implement the dynamics of the nominal path $q : [0,T_\mathrm{e}] \to \mathbb{R}^6$ as well as 36 differential states for the associated nonlinear differential equation for $Q : [0,T_\mathrm{e}] \to \mathbb{R}^{6\times 6}$. Using symmetry and the fact that the states z and v_z are not affected by the uncertainties a differential equation with 16 states can be implemented. Collecting the control inputs, we need one primal control input u, which denotes the acceleration of the oscillatory base, and 3 dual control inputs $\kappa \in \mathbb{R}^3$ to optimize the estimate of the influence the uncertainty itself and the nonlinear terms, respectively. Thus, we need 4 control inputs in total.

Remark 1. Note that the existence of open-loop stable periodic orbits of the inverted spring pendulum is well-known in the literature. As early as in 1908 Stephenson has predicted this phenomenon (Stephenson [1908]). For a more recent article we refer to the work of Arinstein and Gitterman Arinstein and Gitterman [2008], where the open-loop stable orbits of an inverted spring pendulum are theoretically analyzed with an approximation technique using Mathieu's differential equation (Wolf [2010]). In the current paper, we have used this existing approximate analysis to find a good initial guess for the optimal control algorithm. In addition, we refer to the work of Kabamba, Meerkov, and Poh Kabamba et al. [1998] on stability and robustness in vibrational control, where similar periodically operated dynamic systems are discussed from a control perspective.

A locally optimal and robustly open-loop stable periodic orbit is visualized in Figure 2. This orbit has been found by solving the above problem formulation numerically using `ACADO Toolkit` (Houska [2011]). The optimal value for the cycle duration is in this example $T_\mathrm{e} \approx 79\,\mathrm{ms}$. Note that for a sinusoidal driving force at the oscillatory base the resonance frequency of the spring would be

$$\omega_\mathrm{r} = \omega_0\sqrt{1-\frac{1}{2}\left(\frac{b}{\omega_0}\right)^2} \quad \text{with} \quad \omega_0 := \sqrt{\frac{D}{m}}\,.$$

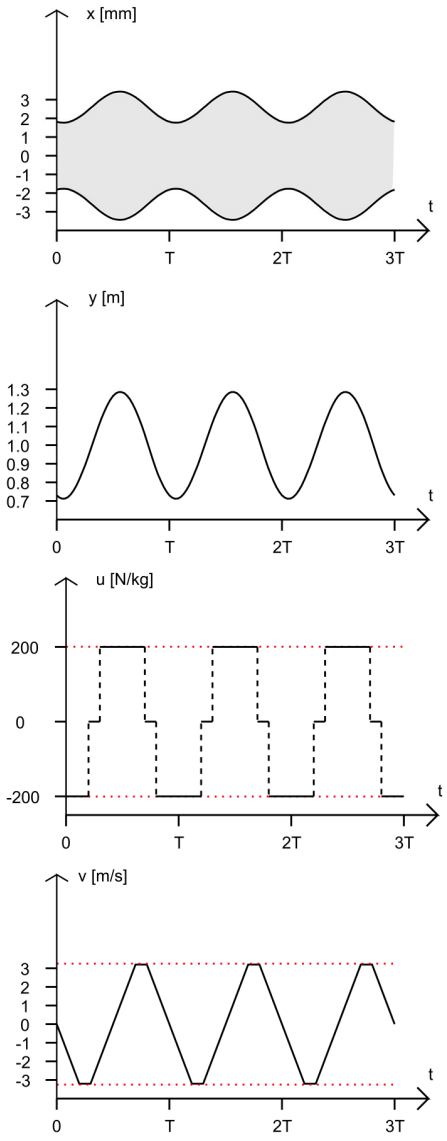


Fig. 2. The upper part of the figure shows a projection of the optimized periodic robust positive invariant tube $\mathscr{E}(q(\cdot),Q(\cdot))$ onto the $t-x$-plane. Here, the grey shaded area represents the region in which the horizontal displacement $x(t)$ of the mass point against the vertical axis can be guaranteed to be. The second figure shows the optimal central path of the y-coordinate of the mass point. The third figure shows the optimal control input over three periods while the associated vertical velocity profile of the mounting point is shown in the lower part.

If we use the parameters from Figure 1, we find a corresponding resonance cycle duration of

$$T_\mathrm{r} = \frac{2\pi}{\omega_\mathrm{r}} \approx 75\,\mathrm{ms}\,,$$

at which a maximum nominal oscillation amplitude of the spring in y-direction can be expected. Note that the value for T_r is close to our optimal result for T_e. The interpretation of this effect is that we need a significant amplitude of the spring oscillation in order to obtain the maximum stability – and, if there would be no oscillation of the spring, the pendulum cannot possibly be stable at its inverted position. Thus, from a physical point of view, it is clear that we have

to choose a driving frequency which is close to resonance. On the other hand, if the system is exactly at resonance, it might be more sensitive with respect to disturbances. In this sense the numerical result for the time T_e is in agreement with our physical expectation.

6. CONCLUSIONS

In this paper, we have discussed a strategy to approximate periodic robust optimal control problems with a standard optimal control problem. Here, the main idea was to construct parameterized ellipsoidal robust positive invariant tubes such that we can over-estimate the reachable tube of the uncertain dynamic system. For the case that we succeed in finding periodic robust invariant tubes, Lemma 2 has shown that – under some mild additional assumptions – the asymptotic stability of the nominal system can be proven. Here, we have provided a computationally available guarantee on the region of attraction of the nonlinear dynamic system to a periodic orbit. Finally, we have illustrated the technique by robustly optimizing a periodic operation of an inverted pendulum. In this example, the technique was successful and yielded a tube in which the system is guaranteed to be stable and robust with respect to the bounded disturbances.

ACKNOWLEDGEMENTS

The research was supported by the Research Council KUL via GOA/11/05 Ambiorics, GOA/10/09 MaNet, CoE EF/05/006 Optimization in Engineering (OPTEC) en PFV/10/002 (OPTEC), IOF-SCORES4CHEM and PhD/postdoc/fellow grants, the Flemish Government via FWO (PhD/postdoc grants, projects G0226.06, G0321.06, G.0302.07, G.0320.08, G.0558.08, G.0557.08, G.0588.09, G.0377.09, research communities ICCoS, ANMMM, MLDM) and via IWT (PhD Grants, Eureka-Flite+, SBO LeCoPro, SBO Climaqs, SBO POM, O&O-Dsquare), the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO, Dynamical systems, control and optimization, 2007-2011), the IBBT, the EU (ERNSI; FP7-HD-MPC (INFSO-ICT-223854), COST intelliCIS, FP7-EMBOCON (ICT-248940), FP7-SADCO (MC ITN-264735), ERC HIGHWIND (259 166)), the Contract Research (AMINAL), the Helmholtz Gemeinschaft via viCERP and the ACCM.

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