Investigation of a Polynomial Matrix Generalised EVD for Multi-Channel Wiener Filtering

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***Abstract*—State-of-the-art narrowband noise cancellation techniques utilise the generalised eigenvalue decomposition (GEVD) for multi-channel Wiener filtering, which can be applied to independent frequency bins in order to achieve broadband processing. Here we investigate the extension of the GEVD to broadband, polynomial matrices, akin to strategies that have already been developed by McWhirter *et. al* on the polynomial matrix eigenvalue decomposition (PEVD). In our approach we extend the Cholesky method for calculating the scalar GEVD to polynomial matrices. In this paper we outline our Cholesky-like approach, which utilises recently developed techniques for polynomial matrix spectral factorisation and polynomial matrix inversion.**

I. Introduction

In a multi-microphone scenario with M sensors, the generalised eigenvalue decomposition (GEVD) has been utilised to enable a low rank approximation of a multichannel Wiener filter (MWF) [3], [8]. This MWF is meant to separate a desired signal from $K-1$ interfering signals, with $K < M$. The MWF $\hat{\mathbf{W}}$ used in [3] is calculated as

$$\hat{\mathbf{W}} = \mathbf{R}_{yy}^{-1}\mathbf{R}_{ss}\hat{\mathbf{E}}_d \,, \tag{1}$$

where $\mathbf{R}_{yy}$ is the signal-plus-noise covariance matrix, $\mathbf{R}_{ss}$ is the estimated signal-only covariance matrix, and $\hat{\mathbf{E}}_d$ is used to select the signal of interest.

In scenarios such as the audio processing examples in [3] and [8], the signal-plus-noise covariance matrix is easily calculated from the received signals, but the signal-only covariance matrix cannot be obtained in this way. Instead, the signal-only covariance matrix must be estimated from the signal-plus-noise and noise-only covariance matrices; during this estimation process, the rank of the matrix is artificially increased. To avoid using matrices with an artificially high rank, the estimated covariance matrix is replaced by a low-rank approximation [3]. Methods using both the conventional eigenvalue decomposition (EVD) and GEVD have been explored in [8] with results indicating the GEVD based low-rank approximation performs better as it effectively selects the modes with the highest SNR.

In brief, the GEVD jointly diagonalises two separate covariance matrices using a common set of generalised eigenvectors. Since the audio application area is inherently broadband, while both the covariance matrix and the GEVD factorisation are defined for the narrowband case, a broadband extension of the GEVD is required. In [3], this is addressed by a frequency domain approach which solves independent narrowband problems in DFT bins. In order to exploit the spectral coherence of e.g. audio signals, in this paper we aim to find a broadband signal processing approach based on polynomial space-time covariance matrices.

For a broadband array vector $\mathbf{x}[n]$, where explicit lags τ need to be taken into account, the space-time covariance matrix is defined as $\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\}$ where $\mathcal{E}\{\cdot\}$ is the expectation operator. Its z-transform $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ\, \mathbf{R}[\tau]$ forms a parahermitian polynomial matrix [9], for which an extension from the EVD to a polynomial EVD (PEVD) exists [5]. In this contribution, we extend the PEVD to a polynomial version of the GEVD (PGEVD) in order to directly address the broadband MWF problem. Wiener filters have previously been formulated using polynomial matrix techniques [6] but initially could not be solved due to an absence of tools. We here complement the polynomial Wiener filter solution in [12] by a PGEVD approach.

Below, Sec. II reviews the (narrowband) GEVD, in particular the Cholesky approach on which our extension will be based. Sec. III introduces polynomial matrices and the PEVD. Inversion of polynomial matrices is covered in Sec. IV. Sec. V outlines the extension of the Cholesky approach to polynomial matrices, with results presented in Sec. VI and conclusions drawn in Sec. VII.

II. Generalised Eigenvalue Decomposition

The GEVD solves the problem of $\mathbf{R}_1\mathbf{v} = d\mathbf{R}_2\mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$, where $\mathbf{v}$ is an eigenvector of the pencil $\mathbf{R}_1 - d\mathbf{R}_2$ [2]. In matrix form we can write this as

$$\mathbf{R}_1\mathbf{V} = \mathbf{R}_2\mathbf{V}\mathbf{D} \,, \tag{2}$$

where the diagonal matrix $\mathbf{D}$ contains the generalised eigenvalues and $\mathbf{V}$ holds the corresponding generalised eigenvectors. Applying the generalised eigenvectors to $\mathbf{R}_1$ and $\mathbf{R}_2$

results in

$$\mathbf{V}^{\mathrm{H}}\mathbf{R}_1\mathbf{V} = \mathbf{D}\ , \tag{3}$$

$$\mathbf{V}^{\mathrm{H}}\mathbf{R}_2\mathbf{V} = \mathbf{I}\ , \tag{4}$$

where $\mathbf{I}$ is the identity. Rearranging (2), or (3) and (4) to

$$\mathbf{R}_2^{-1}\mathbf{R}_1 = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} \quad , \tag{5}$$

it is evident that $\mathbf{R}_2^{-1}\mathbf{R}_1$ is no longer Hermitian and therefore $\mathbf{V}$ is not guaranteed to be unitary, even if — as in the case of MWF [3], [8] — $\mathbf{R}_1$ and $\mathbf{R}_2$ are Hermitian covariance matrices.

Among a number of options to calculate the GEVD or joint diagonalisation of Hermitian matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ is a Cholesky-based approach [2]. In a first step, a Cholesky decomposition is used to obtain

$$\mathbf{R}_2 = \mathbf{L}\mathbf{L}^{\mathrm{H}}\ , \tag{6}$$

with $\mathbf{L}$ lower left triangular. Next we form an intermediate Hermitian matrix $\mathbf{C}$ using $\mathbf{R}_1$ and $\mathbf{L}^{-1}$,

$$\mathbf{C} = \mathbf{L}^{-1}\mathbf{R}_1\mathbf{L}^{-\mathrm{H}}\ . \tag{7}$$

Finally, an EVD is performed on the intermediate Hermitian matrix,

$$\mathbf{C} = \mathbf{Y}\mathbf{D}\mathbf{Y}^{\mathrm{H}}\ , \tag{8}$$

such that $\mathbf{D}$ contains the generalised eigenvalues and the generalised eigenvectors are the columns of $\mathbf{V} = \mathbf{L}^{-\mathrm{H}}\mathbf{Y}$.

In both [3] and [8] the GEVD is used to obtain a low-rank approximation of the signal-only covariance matrix, $\mathbf{R}_{ss}$. Using the GEVD for low-rank approximation, $\mathbf{R}_1$ and $\mathbf{R}_2$ would be the signal-plus-noise and noise-only covariance matrices respectively. $\mathbf{R}_1$ is estimated from the array input data during a signal-plus-noise period, $\mathbf{R}_2$ can either be estimated from a noise-only data segment or come from a-priori knowledge. Various methods for low-rank approximation of $\mathbf{R}_{ss}$ are based around $\mathbf{R}_{ss} = \mathbf{R}_1 - \mathbf{R}_2$, and the GEVD based approach has been shown to give the most reliable estimation of $\mathbf{R}_{ss}$ [8]. Rearranging (3) and (4) we get

$$\mathbf{R}_1 = \mathbf{V}^{-\mathrm{H}}\mathbf{D}\mathbf{V}^{-1} \quad , \tag{9}$$

$$\mathbf{R}_2 = \mathbf{V}^{-\mathrm{H}}\mathbf{V}^{-1} \quad , \tag{10}$$

which can be used in the estimation of $\mathbf{R}_{ss}$ to get

$$\mathbf{R}_{ss} = \mathbf{V}^{-\mathrm{H}}\mathbf{D}\mathbf{V}^{-1} - \mathbf{V}^{-\mathrm{H}}\mathbf{V}^{-1} \tag{11}$$

$$= \mathbf{V}^{-\mathrm{H}}(\mathbf{D} - \mathbf{I})\mathbf{V}^{-1}\ . \tag{12}$$

If the generalised eigenvalues are ordered, a low-rank estimation can be achieved by setting the lower diagonal elements of $(\mathbf{D} - \mathbf{I})$ in (12) to zero.

III. Polynomial Matrices and the PEVD

The methods introduced in the previous section are used in narrowband scenarios where the delays can be synthesised using a phase shift term. To deal with broadband signals, explicit delays need to be modelled. In the z-domain this raises the need for polynomials, which sparked the development of polynomial methods [5]. The cross-spectral density matrix $\boldsymbol{R}(z)$, from Sec. I, is parahermitian i.e. $\boldsymbol{R}(z) = \tilde{\boldsymbol{R}}(z)$ where $\tilde{\cdot}$ represents the parahermitian transpose that consists of a Hermitian transpose and time reversal, i.e. $\tilde{\boldsymbol{R}}(z) = \boldsymbol{R}^{\mathrm{H}}(z^{-1})$. The parahermitian transpose can be seen as the polynomial extension of the scalar Hermitian transpose.

The polynomial EVD (PEVD) has been proposed in [5] to extend the EVD from narrowband to broadband scenarios; here,

$$\boldsymbol{R}(z) \approx \tilde{\boldsymbol{Q}}(z)\boldsymbol{\Gamma}(z)\boldsymbol{Q}(z) \quad , \tag{13}$$

where $\boldsymbol{\Gamma}(z)$ is a diagonal matrix containing the polynomial eigenvalues and the paraunitary matrix $\boldsymbol{Q}(z)$ (i.e. $\boldsymbol{Q}(z)\tilde{\boldsymbol{Q}}(z) = \mathbf{I}$) holds the polynomial eigenvectors.

Two main approaches exist for iteratively calculating the PEVD; those based on the second order sequential best rotation (SBR2) algorithm [5], and those based on sequential matrix diagonalisation (SMD) algorithm [7]. Both the approaches have had various extensions e.g. coding gain optimisation [14], or multiple shift methods to transfer more energy [15], [16] etc. In general the SMD methods transfer a greater amount of energy at each iteration, but are more costly per iteration than the SBR2 based methods.

Both SBR2 and SMD based algorithms follow the same three general steps during one iteration:

1) search the parahermitian matrix to find high energy regions;
2) time shift the high energy region onto the lag zero slice of the parahermitian matrix; and
3) transfer energy from the lag zero slice onto the diagonal.

After I iterations we obtain an approximately diagonal polynomial matrix $\boldsymbol{\Gamma}(z)$. The operations used to time shift and transfer the energy onto the diagonal of the lag zero slice are combined from each iteration to produce the paraunitary matrix $\boldsymbol{Q}(z)$. The number of iterations can either be set in advance or determined at run time using an appropriate stopping criterion, such as the amount of energy to be transferred during the current iteration.

With each iteration of the PEVD algorithm the polynomial matrices $\boldsymbol{\Gamma}(z)$ and $\boldsymbol{Q}(z)$ will generally increase in order. The increasing order can be curtailed using an appropriate truncation technique for either parahermitian [5] or paraunitary matrices [1]. For the results in this paper we have utilised the standard SMD approach introduced in [7]. Fig. 1 shows an example of (a) the input parahermitian matrix $\boldsymbol{R}_2(z)$ and (b) the diagonalised $\boldsymbol{\Gamma}_2(z)$ after 500 algorithm iterations.

IV. Polynomial Matrix Inverse

In our approach we will use spectral factorisation [10], [11] of the polynomial eigenvalues to take the place of the Cholesky

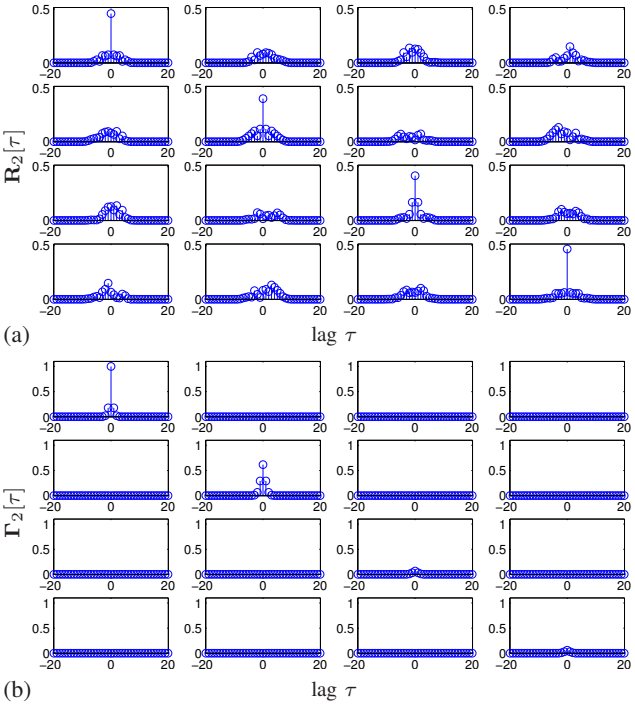


Fig. 1. Space-time covariance matrix (a) $\mathbf{R}_2[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{R}_2(z)$ and (b) $\mathbf{\Gamma}_2[\tau] \circ\!\!-\!\!\bullet\, \mathbf{\Gamma}_2(z)$ showing only lags $|\tau| \leq 20$.

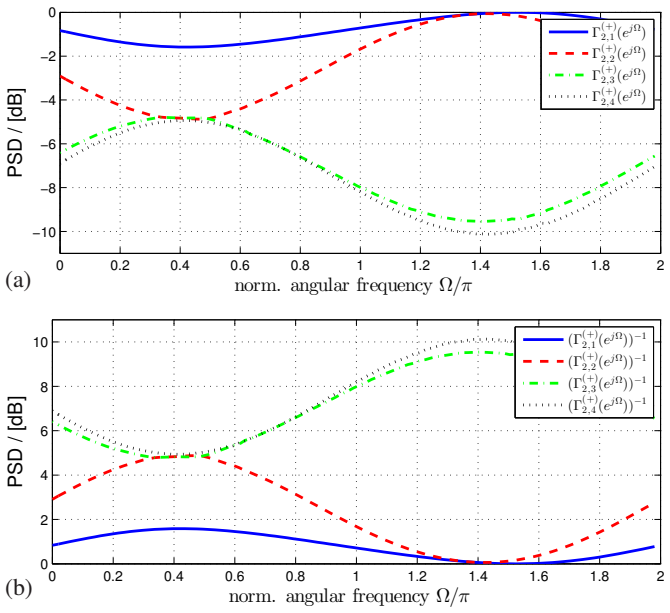


Fig. 2. Power spectral density (PSD) of the (a) minimum phase components, $\mathbf{\Gamma}_2^{(+)}(z)$ and (b) inverted minimum phase components $(\mathbf{\Gamma}_2^{(+)}(z))^{-1}$.

decomposition. The spectral factorisation is carried out on each of the individual polynomial eigenvalues as in [10], [11] using a method such as *sfact()* from [13]. After spectral factorisation the PEVD equation from (13) becomes

$$\boldsymbol{R}_2(z) \approx \tilde{\boldsymbol{Q}}_2(z)\mathbf{\Gamma}_2^{(+)}(z)\mathbf{\Gamma}_2^{(-)}(z)\boldsymbol{Q}_2(z) \quad , \tag{14}$$

where $\mathbf{\Gamma}_2^{(+)}(z)$ and $\mathbf{\Gamma}_2^{(-)}(z) = \tilde{\mathbf{\Gamma}}_2^{(+)}(z)$ are the minimum and maximum phase components of $\mathbf{\Gamma}_2(z)$ respectively.

Like the scalar method outlined above we can construct $\boldsymbol{L}(z) = \tilde{\boldsymbol{Q}}_2(z)\mathbf{\Gamma}_2^{(+)}(z)$ and so $\boldsymbol{R}_2(z) = \boldsymbol{L}(z)\tilde{\boldsymbol{L}}(z)$. It is possible to calculate the inverse of $\boldsymbol{L}(z)$ by inverting both $\tilde{\boldsymbol{Q}}_2(z)$ and $\mathbf{\Gamma}_2^{(+)}(z)$ independently. The inverse of a paraunitary matrix is simply its parahermitian transpose, i.e. $\boldsymbol{Q}_2(z)\tilde{\boldsymbol{Q}}_2(z) = \mathbf{I}$. The inverse of $\mathbf{\Gamma}_2^{(+)}(z) \bullet\!\!-\!\!\circ\, \mathbf{\Gamma}_2^{(+)}[\tau]$ is calculated in the frequency domain by taking the FFT of each of the diagonal entries, $\mathbf{\Gamma}_{2,i}^{(+)}[\tau] \;\forall\; i = 1 \ldots M$, where $\mathbf{\Gamma}_{2,i}^{(+)}[\tau]$ is the ith diagonal entry.

To illustrate the inversion of $\mathbf{\Gamma}_2^{(+)}(z)$ we can look at the power spectral densities of $\mathbf{\Gamma}_2^{(+)}(z)$ in Fig. 2(a), and its inverse $(\mathbf{\Gamma}_2^{(+)}(z))^{-1}$ in Fig. 2(b). Typically the PEVD methods from Sec. III will resolve the eigenvalue with the highest power $(\mathbf{\Gamma}_{2,1}^{(+)}(z))$ better than the eigenvalue with the lowest power $(\mathbf{\Gamma}_{2,4}^{(+)}(z))$; this has important consequences for the inversion as small errors in $\mathbf{\Gamma}_{2,4}^{(+)}(z)$ can get significantly amplified in $(\mathbf{\Gamma}_{2,4}^{(+)}(z))^{-1}$.

V. GEVD Extended to Polynomial Matrices

Extending the GEVD in (2) to polynomial, parahermitian, matrices $\boldsymbol{R}_i(z)$, $i = \{1, 2\}$, leads to a joint diagonalisation problem akin to (3) and (4)

$$\tilde{\boldsymbol{V}}(z)\boldsymbol{R}_1(z)\boldsymbol{V}(z) = \boldsymbol{D}(z) \quad , \tag{15}$$

$$\tilde{\boldsymbol{V}}(z)\boldsymbol{R}_2(z)\boldsymbol{V}(z) = \mathbf{I} \quad . \tag{16}$$

Here $\boldsymbol{D}(z)$ contains the polynomial generalised eigenvectors. The factorisations in (16) can be shown to exist if the PEVDs of $\boldsymbol{R}_i(z)$, $i = \{1, 2\}$, exist [4], and if $\boldsymbol{R}_2(z)$ is invertible. Due to non-uniqueness of the PEVD [1], the generalised polynomial eigenvectors in $\boldsymbol{V}(z)$ can at the very least be arbitrarily delayed w.r.t. each other, leading to a variability in their polynomial degree.

For the computation of $\boldsymbol{V}(z)$ and $\boldsymbol{D}(z)$, a two-step Cholesky-style approach can be performed. The first step starts by calculating the PEVD

$$\boldsymbol{R}_2(z) = \tilde{\boldsymbol{Q}}_2(z)\mathbf{\Gamma}_2(z)\boldsymbol{Q}_2(z) \quad , \tag{17}$$

followed by the spectral factorisation outlined in Sec. IV. Now the factor $\boldsymbol{L}(z) = \tilde{\boldsymbol{Q}}_2(z)\mathbf{\Gamma}_2^{(+)}(z)$ is not lower left triangular but easily inverted, such that $\boldsymbol{L}^{-1}(z) = (\mathbf{\Gamma}_2^{(+)}(z))^{-1}\boldsymbol{Q}_2(z)$, using the procedure described in Sec. IV.

For the second step, we start by constructing the intermediate (parahermitian) matrix

$$\boldsymbol{C}(z) = \boldsymbol{L}^{-1}(z)\boldsymbol{R}_1(z)\tilde{\boldsymbol{L}}^{-1}(z) \quad . \tag{18}$$

Next, using the PEVD, we decompose the intermediate matrix into

$$\boldsymbol{C}(z) = \tilde{\boldsymbol{Q}}(z)\boldsymbol{D}(z)\boldsymbol{Q}(z) \quad , \tag{19}$$

where $\boldsymbol{D}(z)$ contains the polynomial generalised eigenvalues and the polynomial generalised eigenvectors are calculated as $\boldsymbol{V}(z) = \tilde{\boldsymbol{L}}^{-1}(z)\boldsymbol{Q}(z)$, noting a possible order reduction due to ambiguity [1]. The steps used to compute the polynomial

generalised eigenvalue decomposition are summarised in Algorithm 1, for the general case where $\boldsymbol{R}_1(z) \in \mathbb{C}^{M\times M}$ and $\boldsymbol{R}_2(z) \in \mathbb{C}^{M\times M}$ i.e. they both have a matrix dimension of M.

Algorithm 1 Algorithm for Polynomial GEVD

Input: parahermitian matrices $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$
Output: polynomial generalised eigenvectors $\boldsymbol{V}(z)$

Calculate PEVD of $\boldsymbol{R}_2(z)$
$[\boldsymbol{Q}_2(z), \boldsymbol{\Gamma}_2(z)] =$ *SMD*($\boldsymbol{R}_2(z)$,*iter*)
Spectral factorisation:
for $i = 1$ to M **do**
  $\boldsymbol{\Gamma}_{2,i}^{(+)}(z) =$ *sfact*($\boldsymbol{\Gamma}_{2,i}(z)$)
end for
Transform z-domain to time domain
$\boldsymbol{\Gamma}_2^{(+)}[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{\Gamma}_2^{(+)}(z)$
Inversion:
for $i = 1$ to M **do**
  $(\boldsymbol{\Gamma}_{2,i}^{(+)}[\tau])^{-1} =$ *IFFT*($1/($*FFT*($\boldsymbol{\Gamma}_{2,i}^{(+)}[\tau]$)))
end for
Transform time domain to z-domain
$(\boldsymbol{\Gamma}_2^{(+)}(z))^{-1} \bullet\!\!-\!\!\circ\, (\boldsymbol{\Gamma}_2^{(+)}[\tau])^{-1}$
$\boldsymbol{L}^{-1}(z) = (\boldsymbol{\Gamma}_2^{(+)}(z))^{-1}\boldsymbol{Q}_2(z)$
Construct intermediate matrix:
$\boldsymbol{C}(z) = \boldsymbol{L}^{-1}(z)\boldsymbol{R}_1(z)\tilde{\boldsymbol{L}}^{-1}(z)$
Calculate PEVD of $\boldsymbol{C}(z)$
$[\boldsymbol{Q}(z), \boldsymbol{D}(z)] =$ *SMD*($\boldsymbol{C}(z)$,*iter*)
Form generalised eigenvectors:
$\boldsymbol{V}(z) = \tilde{\boldsymbol{L}}^{-1}(z)\boldsymbol{Q}(z)$
return $\boldsymbol{V}(z)$

VI. Results

A. Performance Metrics

Diagonalisation. Since the goal of the GEVD is to minimise off-diagonal energy of both $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$, a suitable normalised metric similar to [7] is

$$E_{n,\text{norm}}^{(i)} = \frac{\sum_\tau \sum_{k=1}^{M} \|\hat{\mathbf{s}}_{n,k}^{(i)}[\tau]\|_2^2}{\sum_\tau \|\mathbf{R}_n[\tau]\|_\text{F}^2} \tag{20}$$

where $\hat{\mathbf{s}}_{n,k}^{(i)}[\tau]$ is the modified k-th column vector containing all but the on-diagonal elements of $\mathbf{S}_n^{(i)}[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{S}_n^{(i)}(z)$ for $n = 1, 2$. The partially diagonalised $\boldsymbol{S}_n^{(i)}(z)$ is calculated as

$$\boldsymbol{S}_1^{(i)}(z) = \tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_1(z)\boldsymbol{V}^{(i)}(z) \quad , \tag{21}$$

$$\boldsymbol{S}_2^{(i)}(z) = \tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_2(z)\boldsymbol{V}^{(i)}(z) \quad , \tag{22}$$

where $\boldsymbol{V}^{(i)}(z)$ is the generalised eigenvectors calculated after i iterations in each of the PEVD steps of Sec. V.

Identity Error. Ideally the generalised eigenvectors in $\boldsymbol{V}(z)$ should turn $\boldsymbol{R}_2(z)$ into an $M \times M$ identity matrix; however, in reality $\boldsymbol{S}_2^{(i)}(z)$ only approximates the identity. To quantify the closeness of $\boldsymbol{S}_2^{(i)}(z)$ to the identity we define

$$\boldsymbol{E}^{(i)}(z) = \mathbf{I}_{M\times M} - \boldsymbol{S}_2^{(i)}(z) \, . \tag{23}$$

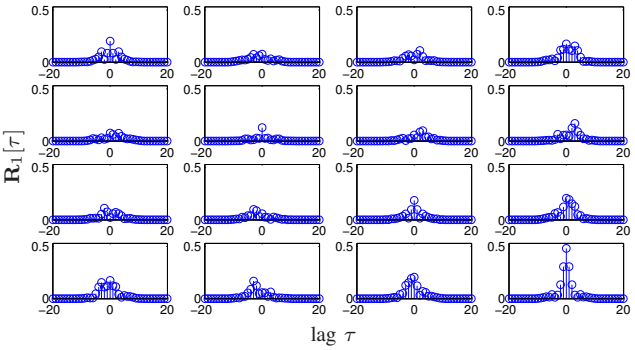


Fig. 3. Space-time covariance matrix $\mathbf{R}_1[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{R}_1(z)$ for lags $|\tau| \leq 20$.

With $\mathbf{E}^{(i)}[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{E}^{(i)}(z)$ the identity error is given by the distance metric

$$\xi^{(i)} = \sum_\tau \|\mathbf{E}^{(i)}[\tau]\|_F^2 \, . \tag{24}$$

B. Simulation Scenario

As an example, $\boldsymbol{R}_1(z) \in \mathbb{C}^{4\times4}$ and $\boldsymbol{R}_2(z) \in \mathbb{C}^{4\times4}$ are generated from the source model in [7]. The order of both $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$ from the source model is 89 and the source model is set up such that $\boldsymbol{R}_1(z)$ is rank 1, with the contributing source having a dynamic range of approximately 10 dB. $\boldsymbol{R}_2(z)$ is full rank and its sources have a dynamic range of approximately 20 dB. The SMD algorithm in [7] is used to calculate the PEVD and is run for 500 iterations with the performance metrics from Sec. VI-A recorded after every 10 iterations.

C. Joint Diagonalisation

The example PGEVD uses the two parahermitian matrices $\boldsymbol{R}_1(z)$ and $\boldsymbol{R}_2(z)$, shown in Fig. 3 and Fig. 1 (a) respectively. Following the algorithm outlined in Sec. V, with each of the PEVD steps having 500 iterations, we obtain the jointly diagonalised systems in Fig. 4, closely approximating (15) and (16).

D. Algorithm Convergence

Although the PGEVD as such is not iterative, we are able to vary the number of PEVD iterations and show how performance metrics improve with more iterations. First we look at the reduction in off-diagonal energy for both $\tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_1(z)\boldsymbol{V}^{(i)}(z)$ and $\tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_2(z)\boldsymbol{V}^{(i)}(z)$ in Fig. 5. The convergence of $\boldsymbol{S}_1^{(i)}(z)$ is heavily influenced by the initial PEVD of $\boldsymbol{S}_2^{(i)}(z)$, and its effect can beneficial or detrimental on the second PEVD stage as the number of iterations increase. If the number of iterations for the first PEVD were fixed the curve for $\boldsymbol{R}_1(z)$ would be smooth; varying the number of iterations, as we do here, causes the fluctuations in the curve for $\boldsymbol{R}_1(z)$.

In addition we can examine how close the decomposition $\tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_2(z)\boldsymbol{V}^{(i)}(z)$ is to the identity (delayed by an appropriate amount) at each iteration. Fig. 6 shows how

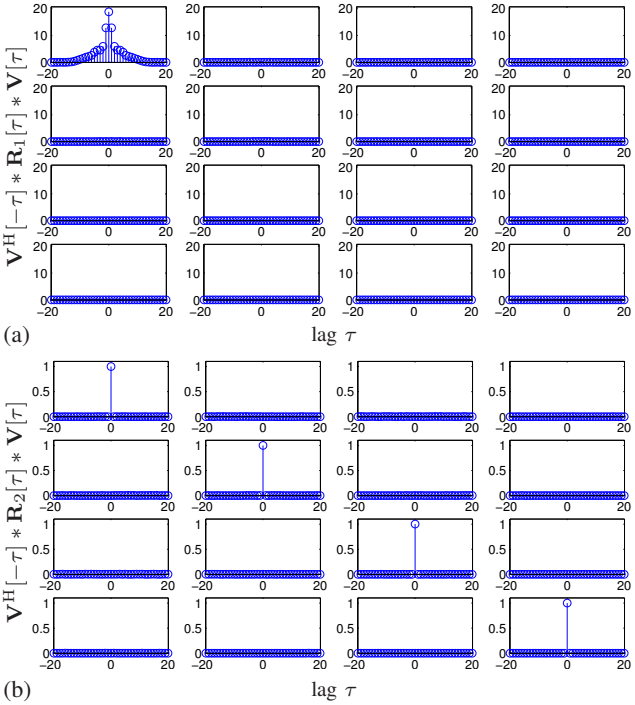


Fig. 4. Results of joint diagonalisation for (a) $\mathbf{S}_1^{(500)}[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{S}_1^{(500)}(z)$ and (b) $\mathbf{S}_2^{(500)}[\tau] \circ\!\!-\!\!\bullet\, \boldsymbol{S}_2^{(500)}(z)$, for lags $|\tau| \leq 20$.

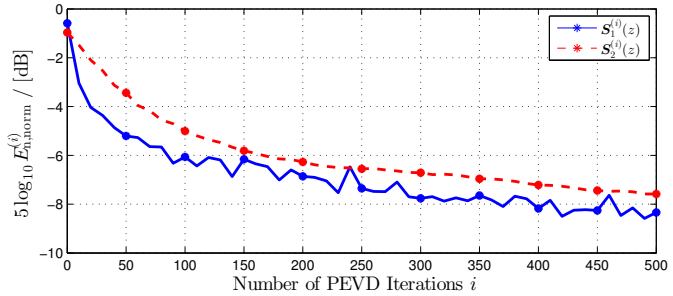


Fig. 5. Reduction in off-diagonal energy for $\boldsymbol{S}_1^{(i)}(z)$ and $\boldsymbol{S}_2^{(i)}(z)$ as the number of PEVD iterations, i, increase.

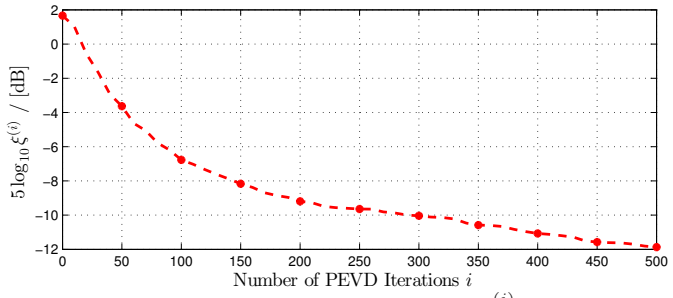


Fig. 6. Difference between the identity matrix, $\mathbf{I}$, and $\boldsymbol{S}_2^{(i)}(z)$ as the number of PEVD iterations, i, increase.

$\tilde{\boldsymbol{V}}^{(i)}(z)\boldsymbol{R}_2(z)\boldsymbol{V}^{(i)}(z)$ converges towards to the identity as the number of PEVD iterations, i, increase.

VII. Conclusion

To extend the GEVD to broadband scenarios we have proposed a polynomial version. Since existing polynomial matrix factorisations are based on the paraunitary property, which the generalised polynomial eigenvectors do not fulfil, an indirect Cholesky-style approach has been suggested, which involves two PEVDs and the inversion of a parahermitian matrix. The effectiveness of this approach has been demonstrated using an example.

Although not covered in this paper, the numerical robustness of the polynomial matrix inversion can be enhanced by regularisation. Based on the proposed approach, it is therefore now possible to extend problems such as the multichannel Wiener filter, which elegantly rely on the GEVD, to the broadband case by utilising its polynomial extension.

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