QPALM: Augmented Lagrangian method for Quadratic Programs

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1 Introduction

Applications of model predictive control (MPC) arise more and more in industry, because of its capacity to optimize a desired objective over a prediction horizon while explicitly taking into account constraints. The very core of MPC is to solve an optimization problem at every time instant, and then apply the first control input of the calculated sequence of inputs. Low sampling times result in faster responses to model-plant mismatch and disturbances in the environment. Moreover, solvers often run on embedded hardware with memory limitations. Therefore, substantial research is being performed into fast and efficient optimization solvers.

Augmented Lagrangian methods are known to be an efficient framework for dealing with constrained optimization [1]. This abstract presents QPALM, an augmented Lagrangian method with exact line-search to solve quadratic programs (QP). In this case, the exact line-search reduces to finding the zero of a monotone piecewise affine function, which can be solved efficiently. We calculate update directions using the limited memory quasi-Newton method L-BGFS, resulting in an algorithm free of matrix inverses.

2 Methodology

We consider the following quadratic program

$$\begin{aligned} \textbf{minimize} \quad & \tfrac{1}{2}x^\top Qx + q^\top x, \\ \textbf{subject to} \quad & Ax \in C \end{aligned}$$

with $Q \in \mathbb{R}^{n\times n}$ a positive definite matrix, $q \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $C = \{z : \underline{b} \le z \le \overline{b}\}$ and $\underline{b}_i$, $\overline{b}_i$ are m- dimensional vectors whose components satisfy $-\infty \le \underline{b}_i \le \overline{b}_i \le \infty$, $i = 1,\dots,m$. Equality constraints can be incorporated in this formulation as well, by taking $\underline{b}_i = \overline{b}_i$. By introducing a slack vector $z = Ax$, we apply the augmented Lagrangian method where the equality constraints are relaxed. The inner subproblem is a minimization of the augmented Lagrangian, which after elimination of z can be expressed as

$$\textbf{minimize} \quad \varphi_\Sigma(x) = f(x) + \tfrac{1}{2}\operatorname{dist}^2_{\Sigma,C}(Ax + \Sigma^{-1}y^k),$$

with Σ a diagonal matrix with penalty parameters, and $\operatorname{dist}_{\Sigma,C}(z) = \min\{\|w - z\|_\Sigma \mid w \in C\}$. Termination requires the primal and dual residuals to be below tolerances which are composed of an absolute and a relative contribution

$$\begin{aligned} \varepsilon_{\mathrm{p}}(x,z,\varepsilon_{\mathrm{a}},\varepsilon_{\mathrm{r}}) &= \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{r}}\max\{\|Ax\|_\infty, \|z\|_\infty\} \\ \varepsilon_{\mathrm{d}}(x,y,\varepsilon_{\mathrm{a}},\varepsilon_{\mathrm{r}}) &= \varepsilon_{\mathrm{a}} + \varepsilon_{\mathrm{r}}\max\{\|Qx\|_\infty, \|A^\top y\|_\infty, \|q\|_\infty\} \end{aligned}$$

where ε_{a}, ε_{r} are small positive numbers representing absolute and positive tolerance, respectively. The algorithm is outlined in alg. 1. The proposed method will be benchmarked against state-of-the-art QP solvers, and is expected to perform favorably against these solvers.

Algorithm 1 QPALM

Require: x^0, y^0, ε_{a}, ε_{r}
  Initialize: $\rho, \vartheta \in (0,1)$, $\Delta > 0$, $\hat{\varepsilon}_{\mathrm{a}} > 0$, $\hat{\varepsilon}_{\mathrm{r}} \in (0,1]$
$\Sigma = \operatorname{diag}(\sigma_1,\dots,\sigma_m)$, where $\sigma_i > 0$
$r_p = Ax^0 - \min(\overline{b}, \max(Ax^0 + \Sigma^{-1}y^0, \underline{b}))$
for $k = 0,1,\dots$ **do**
  $z^k = \min(\overline{b}, \max(Ax^k + \Sigma^{-1}y^k, \underline{b}))$
  $\hat{y}^k = y^k + \Sigma(Ax^k - z^k)$
  $\nabla\varphi_\Sigma(x^k) = \nabla f(x^k) + A^\top\hat{y}^k$
  if $\begin{cases} \|\nabla\varphi_\Sigma(x^k)\|_\infty \le \varepsilon_{\mathrm{d}}(x^k,\hat{y}^k,\varepsilon_{\mathrm{a}},\varepsilon_{\mathrm{r}}) \\ \|Ax^k - z^k\|_\infty \le \varepsilon_{\mathrm{p}}(x^k,z^k,\varepsilon_{\mathrm{a}},\varepsilon_{\mathrm{r}}) \end{cases}$ **then**
    Exit with solution x^k.
  else if $\|\nabla\varphi_\Sigma(x^k)\|_\infty \le \varepsilon_{\mathrm{d}}(x^k,\hat{y}^k,\hat{\varepsilon}_{\mathrm{a}},\hat{\varepsilon}_{\mathrm{r}})$ **then**
    $y^{k+1} = \hat{y}^k$
    $\hat{\varepsilon}_{\mathrm{a}} \leftarrow \max\{\rho\hat{\varepsilon}_{\mathrm{a}}, \varepsilon_{\mathrm{a}}\}$, $\hat{\varepsilon}_{\mathrm{r}} \leftarrow \max\{\rho\hat{\varepsilon}_{\mathrm{r}}, \varepsilon_{\mathrm{r}}\}$
    $\sigma_i \leftarrow \begin{cases} \Delta\sigma_i, & |A_ix^k - z_i^k| > \vartheta|r_{\mathrm{p},i}| \\ \sigma_i, & \text{otherwise} \end{cases}$
    $r_{\mathrm{p}} \leftarrow Ax^k - z^k$
  else
    Compute d^k via L-BFGS
    Determine τ_k by exact line-search on φ_Σ
    $x^{k+1} = x^k + \tau_k d^k$, $y^{k+1} = y^k$
  end if
end for

References

[1] D. P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*. Athena Scientific, 1996.

Acknowledgement This work benefits from KU Leuven-BOF PFV/10/002 Centre of Excellence: Optimization in Engineering (OPTEC), from project G0C4515N of the Research Foundation - Flanders (FWO - Flanders), from Flanders Make ICON: Avoidance of collisions and obstacles in narrow lanes, and from the KU Leuven Research project C14/15/067: B-spline based certificates of positivity with applications in engineering. FWO-FNRS under EOS Project no 30468160 (SeLMA), FWO projects: G086318N; G086518N