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The Hilbert Distance between LTI Passive Systems

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***Abstract*— We study the Hilbert metric on the space of continuous-time linear time-invariant (LTI) passive systems. It is shown that the Hilbert metric between two LTI passive systems is efficiently computable in terms of suitable $\mathcal{H}_\infty$-norms of the spectral factors of the two transfer functions. The Hilbert metric has the interpretation of a phase distance between the two systems. It also enjoys favorable invariance properties under positive scalings and congruence transformations, which confers the physical interpretation of a distortion measure between the supplied energies.**

I. Introduction

Passive systems are ubiquitous in the physical world. Passivity is a physical property of systems originating from circuit network theory [1] and has been recognized as one of the cornerstones of control theory [2]. A key characteristic of passivity is that it is a *phase* property. A phase-shift of any sinusoidal input to a single-input single-output (SISO) linear time-invariant (LTI) passive system is at most 90 degrees; i.e., the Bode phase plot is contained in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

In this paper, we yearn for an answer to the following question for multi-input multi-output (MIMO) LTI systems:

Given two passive systems P_1 and P_2, what is a "good" distance function $d(\cdot,\cdot)$ between P_1 and P_2?

A first motivation stems from model reduction studies [3] under passivity constraints. In such a scenario, P_1 models a physical passive system and P_2 is a reduced-order approximated passive model to be identified with some error bound with respect to a suitable distance. A popular choice of the error distance is the $\mathcal{H}_\infty$-norm of the error system. The question of how to guarantee the passivity of the approximated model has been central to, e.g., positive real balancing methods [3, Sec. 7.5].

A second motivation is to capture possible passive system uncertainties in robust control [4]. Given a nominal system P_1, we treat P_2 as a perturbed version of P_1 and thereby a distance function $d(P_1, P_2)$ can mathematically quantify how large the perturbation is. Historically, the *gap metric*, introduced by Zames and El-Sakkary [5] in 1980 and further developed in [6] [7], [8], [9], has been the preferred candidate to study such approximation problems. The gap metric $d_{\mathrm{G}}(\cdot,\cdot)$ is a *chordal distance* function between two subspaces, i.e., two systems' input-output trajectories. Its computable formula [6] boils down to a two-block $\mathcal{H}_\infty$ problem via the use of coprime factorizations. However, is the gap metric suitable for comparing only passive systems? To answer this, let us first consider three passive systems:

$$P_1(s) = \frac{s+2}{s+1},\ P_2(s) = \frac{0.5(s+2)}{s+1} \text{ and } P_3(s) = \frac{2(s+2)}{s+1}$$

whose only difference lies in various positive gain scalings. Once we acknowledge passivity to be a phase characteristic, a reasonable distance function between the systems is expected to be zero, since the systems have the exactly same phase plots. The gap metric between them is however nonzero. As an extreme case, $d_{\mathrm{G}}(P_1, \alpha P_1) \to \frac{\sqrt{2}}{2}$ as $\alpha \to \infty$, monotonically nondecreasing in $\alpha > 0$.

One could argue that a "good" distance function $d(\cdot,\cdot)$ for the approximation problem in the space of passive systems should be a *phase distance* endowed with three desirable properties. First, it should be invariant under arbitrary positive scalings (gain invariance):

$$d(\alpha P_1, \beta P_2) = d(P_1, P_2)$$

for all $\alpha, \beta > 0$, and under congruence transformations (coordinate/dynamic-gain invariance):

$$d(T^* P_1 T, T^* P_2 T) = d(P_1, P_2)$$

for some invertible T. Second, it should be mathematically tractable with acceptable computational efficiency. Third, it should have a physical interpretation.

Our paper shows that the *Hilbert metric* defined on the space of LTI passive systems enjoys all properties mentioned earlier. The Hilbert metric, first introduced by Hilbert in 1895 [10], is a classical distance function between two rays in a closed cone [11], that is, a *conal* and *projective distance*. This metric has been extensively utilized in contraction analysis [12], [13], [14], [15], [16]. Recently, the Hilbert metric was introduced in the cone of discrete-time rational spectral densities [17] with a computable formula involving $\mathcal{H}_\infty$-norms and spectral factorizations. This paper is inspired by [17] and by observation that the space of continuous-time LTI passive systems forms a closed convex cone. In short, we develop a computable formula of the proposed Hilbert metric and further bridge a link between the Hilbert metric and Bode phase responses for SISO minimum-phase systems. We also show that the congruence invariance property of the Hilbert metric has an energy-based interpretation in multi-port electric circuits.

The remainder of this paper is organized as follows. In Section II, preliminaries on the Hilbert metric in cones and passivity for LTI systems are introduced. Section III presents

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our main result, a formula of the Hilbert metric on the cone of LTI passive systems. A connection between the Hilbert metric and classical system phase is established. In Section IV, an energy-based interpretation of the congruence invariance of the Hilbert metric is provided using multi-port circuits. Section V draws a comparison between the Hilbert metric and gap metric and Section VI concludes this paper.

II. Notation and Preliminaries

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be the field of real or complex numbers, and $\mathbb{F}^n$ be the linear space of n-dimensional vectors over $\mathbb{F}$. Denote $\mathbb{C}_+$ and $\bar{\mathbb{C}}_+$ as the open and closed complex right half-plane, respectively. The transpose and conjugate transpose of matrices are denoted by $(\cdot)^\top$ and $(\cdot)^*$, respectively. Let $(\cdot)^{-*} := [(\cdot)^*]^{-1}$. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$, respectively. The argument of $z \in \mathbb{C}$ is denoted by $\angle z$ and if $z = 0$, then $\angle z = \emptyset$.

For a complex matrix $A \in \mathbb{C}^{n\times n}$, the largest (resp. smallest) singular value of A is denoted by $\sigma_{\max}(A)$ (resp. $\sigma_{\min}(A)$). The Hermitian part of A is denoted as $\mathrm{He}[A] := \frac{1}{2}(A + A^*)$. The set of Hermitian matrices is denoted by $\mathcal{S}^n := \{A \in \mathbb{C}^{n\times n} \mid A = A^*\}$. For $A \in \mathcal{S}^n$, $A \geq 0$ (resp. $A > 0$) denotes that A is positive semi-definite (resp. positive definite). Their corresponding sets are denoted by

$$\mathcal{S}^n_+ := \{A \in \mathcal{S}^n \mid A \geq 0\} \text{ and } \mathcal{S}^n_{++} := \{A \in \mathcal{S}^n \mid A > 0\}.$$

The largest (resp. smallest) eigenvalue of $A \in \mathcal{S}^n$ is denoted by $\lambda_{\max}(A)$ (resp. $\lambda_{\min}(A)$). Any matrix $A \in \mathcal{S}^n_+$ (resp. $A \in \mathcal{S}^n_{++}$) has a unique square root $A^{\frac{1}{2}} \in \mathcal{S}^n_+$ (resp. $A^{\frac{1}{2}} \in \mathcal{S}^n_{++}$), that is, $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$.

A. The Hilbert Metric

Let $\mathcal{K}$ be a closed, solid, pointed and convex cone in a Banach space $\mathcal{B}$; namely, $\mathcal{K}$ is a closed subset of $\mathcal{B}$ with the following properties: (i) $\mathrm{int}(\mathcal{K})$, the interior of $\mathcal{K}$, is nonempty, (ii) $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$, (iii) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$ and (iv) $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$. Such a cone $\mathcal{K}$ induces a partial ordering $\preceq_{\mathcal{K}}$ on $\mathcal{B}$ by

$$x \preceq_{\mathcal{K}} y \;\Leftrightarrow\; y - x \in \mathcal{K}.$$

Moreover, denote by $x \prec_{\mathcal{K}} y \Leftrightarrow y - x \in \mathrm{int}(\mathcal{K})$. For $x, y \in \mathcal{K}$, y is said to dominate x if there exists $\beta > 0$ such that $x \preceq_{\mathcal{K}} \beta y$. If y dominates x, and x dominates y, we say $x \sim_{\mathcal{K}} y$ which defines an equivalence relation on $\mathcal{K}$. The corresponding equivalence classes are often called parts or components of $\mathcal{K}$.

Let $\mathcal{K}^+ := \mathcal{K} \setminus \{0\}$. The classical Hilbert metric [11] can be defined on the equivalence classes in $\mathcal{K}^+$ as below.

Definition 1: The *Hilbert metric* $d_{\mathrm{H}} : \mathcal{K}^+ \times \mathcal{K}^+ \to [0, \infty]$ between two elements $x, y \in \mathcal{K}^+$ is defined by

$$d_{\mathrm{H}}(x, y) := \log\left(\frac{M(x, y)}{m(x, y)}\right)$$

if $x \sim_{\mathcal{K}} y$, where the ratio in $\log(\cdot)$ is defined by

$$M(x, y) := \inf\{\beta \in \mathbb{R} \mid x \preceq_{\mathcal{K}} \beta y\}$$
$$m(x, y) := \sup\{\alpha \in \mathbb{R} \mid \alpha y \preceq_{\mathcal{K}} x\},$$

with the convention that $\inf\{\emptyset\} = \infty$ and $\sup\{\emptyset\} = 0$ so that $d_{\mathrm{H}}(x, y) = \infty$ whenever $M(x, y) = \infty$ or $m(x, y) = 0$.

The Hilbert metric on $\mathcal{K}^+$ is a *projective* metric, that is,

$$d_{\mathrm{H}}(\alpha x, \beta y) = d_{\mathrm{H}}(x, y)\ \forall \alpha, \beta > 0.$$

The Hilbert metric measures a distance between the two rays $\{\alpha x \mid \alpha > 0, x \in \mathcal{K}^+\}$ and $\{\beta y \mid \beta > 0, y \in \mathcal{K}^+\}$ in the cone. Clearly, it holds that $d_{\mathrm{H}}(x, y) = 0$ if and only if $y = \alpha x$ for some $\alpha > 0$. This makes the interior of $\mathcal{K}$ equipped with $d_{\mathrm{H}}(\cdot, \cdot)$ a pseudo-metric space as follows.

Lemma 1 ([11, Th. 2.1]): The ordered pair $(\mathrm{int}(\mathcal{K}), d_{\mathrm{H}})$ is a pseudo-metric space.

An illustrative example of $\mathcal{K}$ is the set of positive semi-definite matrices $\mathcal{S}^n_+$ with its interior $\mathcal{S}^n_{++}$ contained in the Banach space $\mathcal{S}^n$. The partial ordering $\preceq_{\mathcal{S}_+}$ is expressed by $\leq$. For $A, B \in \mathcal{S}^n_{++}$, it holds that

$$\begin{aligned} M(A, B) &= \inf\{\beta \mid A \leq \beta B\} \\ &= \inf\{\beta \mid B^{-\frac{1}{2}}AB^{-\frac{1}{2}} \leq \beta I\} \\ &= \lambda_{\max}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) = \lambda_{\max}(AB^{-1}), \\ m(A, B) &= \sup\{\alpha \mid \alpha B \leq A\} \\ &= \sup\{\alpha \mid \alpha I \leq B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\} \\ &= \lambda_{\min}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) = \lambda_{\min}(AB^{-1}). \end{aligned}$$

The Hilbert metric between A and B can be expressed by

$$d_{\mathrm{H}}(A, B) = \log\left(\frac{\lambda_{\max}(AB^{-1})}{\lambda_{\min}(AB^{-1})}\right)$$

which involves the extreme eigenvalues only. Other interesting examples arise in polyhedral cones and Lorentz cones which can be found in [18, Ch. 2].

B. Passive Systems and Spectral Factorizations

Let $\mathcal{RL}^{n\times n}_\infty$ be the space of real-rational proper matrix-valued functions $j\mathbb{R} \to \mathbb{C}^{n\times n}$ that are bounded on $j\mathbb{R}$ with $\mathcal{L}_\infty$-norm defined by $\|P\|_{\mathcal{L}_\infty} := \operatorname{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}(P(j\omega))$. Let $\mathcal{RH}^{n\times n}_\infty$ be a closed subspace of $\mathcal{RL}^{n\times n}_\infty$ with functions that are analytic in $\mathbb{C}_+$ and bounded on $j\mathbb{R}$ with $\mathcal{H}_\infty$-norm defined by

$$\|P\|_{\mathcal{H}_\infty} := \sup_{\mathrm{Re}[s]>0} \sigma_{\max}(P(s)) = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}(P(j\omega)).$$

An $n \times n$ continuous-time MIMO LTI system P is said to be stable if $P \in \mathcal{RH}^{n\times n}_\infty$; i.e., P has no poles in $\bar{\mathbb{C}}_+$. Moreover, $P \in \mathcal{RH}^{n\times n}_\infty$ is said to be minimum-phase if P has no transmission zeros in $\bar{\mathbb{C}}_+$.

Define the following set of para-Hermitian functions:

$$\mathcal{F}^n := \big\{H \in \mathcal{RL}^{n\times n}_\infty \mid H^*(s) = H(s), H(s) \text{ has full normal rank, } H(j\omega) \geq 0\ \forall \omega \in [-\infty, \infty]\big\}, \quad (1)$$

where $H^*(s) := H(-s)^\top$. Any function $H \in \mathcal{F}^n$ has a *spectral factorization* [19], [20] defined by

$$H(s) = W^*(s)W(s), \quad (2)$$

where $W \in \mathcal{RH}^{n\times n}_\infty$ satisfying $W^{-1} \in \mathcal{RH}^{n\times n}_\infty$ is minimum-phase. Then W is called a *spectral factor* of H

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which is uniquely determined up to pre-multiplication of some unitary matrix $U \in \mathbb{C}^{n\times n}$, i.e., $\bar{W}(s) = UW(s)$.

For LTI systems, the concept of passivity is equivalent to that of positive realness [1, Sec. 2.7]. For the sake of technical brevity, in this paper we restrict ourselves to the following class of stable passive systems.

Definition 2: A system $P \in \mathcal{RH}_\infty^{n\times n}$ with full normal rank is said to be *passive* (a.k.a. *positive real*) if

$$P(j\omega) + P^*(j\omega) \geq 0 \tag{3}$$

for all $\omega \in [-\infty, \infty]$. Moreover, P is said to be *strictly passive* (a.k.a. *strongly positive real*) if

$$P(j\omega) + P^*(j\omega) > 0$$

for all $\omega \in [-\infty, \infty]$.

Denote by $\mathcal{P}_+^n$ (resp. $\mathcal{P}_{++}^n$) the set of passive (reps. strictly passive) LTI systems per Definition 2. It is well known that passivity should be regarded as a *phase property*. For a SISO LTI system P, condition (3) is equivalent to that $\angle P(j\omega) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $\omega \in [-\infty, \infty]$. For more general systems, such as MIMO nonlinear systems, an analog phase characterization $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of passivity has been established in the recent works [21], [22], [23].

III. The Hilbert Distance between Passive Systems

This section establishes the Hilbert metric on the space of LTI passive systems as our main result. It is shown that the Hilbert metric between two passive systems can be computed by using the theories of spectral factorization and $\mathcal{H}_\infty$ control. Moreover, the Hilbert metric is interpreted as a phase distance for SISO minimum-phase passive systems.

The key observation is to acknowledge that the space of LTI passive systems $\mathcal{P}_+^n$ forms a closed and convex cone with the interior $\mathcal{P}_{++}^n$. To examine that $\mathcal{P}_+^n$ is a convex cone, note that for $P_1, P_2 \in \mathcal{P}_+^n$, it holds that $(\alpha P_1 + \beta P_2) \in \mathcal{P}_+^n$ for all $\alpha, \beta \geq 0$ since by Definition 2, we have

$$\begin{aligned}&(\alpha P_1(j\omega) + \beta P_2(j\omega)) + (\alpha P_1(j\omega) + \beta P_2(j\omega))^*\\ &\quad\geq \alpha\left(P_1(j\omega) + P_1^*(j\omega)\right) + \beta\left(P_2(j\omega) + P_2^*(j\omega)\right) \geq 0\end{aligned}$$

for all $\omega \in [-\infty, \infty]$. The partial ordering induced by $\mathcal{P}_+^n$ can be represented as follows.

Proposition 1: For $P_1, P_2 \in \mathcal{P}_+^n$, it holds that $P_1 \preceq_{\mathcal{P}_+^n} P_2$ if and only if

$$\mathrm{He}[P_1(j\omega)] \leq \mathrm{He}[P_2(j\omega)]$$

for all $\omega \in [-\infty, \infty]$.

Proof: By definition, we have $P_1 \preceq_{\mathcal{P}_+^n} P_2$ if and only if $P_2 - P_1 \in \mathcal{P}_+^n$; namely, $P_2 - P_1$ is positive real and by (3) we have

$$P_2(j\omega) - P_1(j\omega) + (P_2(j\omega) - P_1(j\omega))^* \geq 0$$

for all $\omega \in [-\infty, \infty]$. This shows that $\mathrm{He}[P_2(j\omega)] \geq \mathrm{He}[P_1(j\omega)]$ for all $\omega \in [-\infty, \infty]$. ∎

Consider a passive system $P \in \mathcal{P}_+^n$. Since $P \in \mathcal{RH}_\infty^{n\times n}$, notice that the Hermitian part of P belongs to $\mathcal{RL}_\infty^{n\times n}$ and is an element of the set $\mathcal{F}^n$ in (1), thereby admitting a spectral factorization according to (2). Specifically, we have

$$\mathrm{He}[P(s)] = \frac{1}{2}\left(P(s) + P^*(s)\right) = W^*(s)W(s), \tag{4}$$

where $W, W^{-1} \in \mathcal{RH}_\infty^{n\times n}$ is a corresponding spectral factor of $\mathrm{He}[P(s)]$. Based on spectral factorizations, we establish the Hilbert metric on $\mathcal{P}_+^n \setminus \{0\}$ as elaborated below.

Theorem 1 (The Hilbert distance): Given two passive systems $P_1, P_2 \in \mathcal{P}_+^n \setminus \{0\}$, let $W_1, W_2 \in \mathcal{RH}_\infty^{n\times n}$ be their corresponding spectral factors defined in (4). The Hilbert metric between P_1 and P_2 can be computed as

$$d_{\mathrm{H}}(P_1, P_2) = \log\left(\left\|W_1W_2^{-1}\right\|_{\mathcal{H}_\infty}^2 \left\|W_2W_1^{-1}\right\|_{\mathcal{H}_\infty}^2\right) \tag{5}$$

if $P_1 \sim_{\mathcal{P}_+^n} P_2$. Otherwise, $d_{\mathrm{H}}(P_1, P_2) = \infty$.

Proof: By hypothesis, it follows from (4) that

$$\begin{aligned}\mathrm{He}[P_1(s)] &= W_1^*(s)W_1(s),\\ \mathrm{He}[P_2(s)] &= W_2^*(s)W_2(s),\end{aligned} \tag{6}$$

where the spectral factors $W_1, W_1^{-1}, W_2, W_2^{-1} \in \mathcal{RH}_\infty^{n\times n}$. For the notational brevity, we adopt $\forall\omega$ as a shorthand of $\forall\omega \in [-\infty, \infty]$. Following from Definition 1, Proposition 1 and (6), for two systems $P_1, P_2 \in \mathcal{P}_+^n \setminus \{0\}$ such that $P_1 \sim_{\mathcal{P}_+^n} P_2$, we have that

$$\begin{aligned}&M(P_1, P_2)\\ &= \inf\left\{\beta \mid \mathrm{He}[P_1(j\omega)] \leq \beta \mathrm{He}[P_2(j\omega)]\ \forall\omega\right\}\\ &= \inf\left\{\beta \mid W_1^*(j\omega)W_1(j\omega) \leq \beta W_2^*(j\omega)W_2(j\omega)\ \forall\omega\right\}\\ &= \inf\left\{\beta \mid W_2^{-*}(j\omega)W_1^*(j\omega)W_1(j\omega)W_2^{-1}(j\omega) \leq \beta I\ \forall\omega\right\}\\ &= \inf\left\{\beta \mid \left[\sigma_{\max}\left(W_1(j\omega)W_2^{-1}(j\omega)\right)\right]^2 \leq \beta I\ \forall\omega\right\}\\ &= \left\|W_1W_2^{-1}\right\|_{\mathcal{H}_\infty}^2,\end{aligned}$$

where $W_1W_2^{-1}$ belongs to $\mathcal{RH}_\infty^{n\times n}$ as $W_1, W_2^{-1} \in \mathcal{RH}_\infty^{n\times n}$. In a similar fashion, we obtain that

$$\begin{aligned}&m(P_1, P_2)\\ &= \sup\left\{\alpha \mid \alpha \mathrm{He}[P_2(j\omega)] \leq \mathrm{He}[P_1(j\omega)]\ \forall\omega\right\}\\ &= \sup\left\{\alpha \mid \alpha W_2^*(j\omega)W_2(j\omega) \leq W_1^*(j\omega)W_1(j\omega)\ \forall\omega\right\}\\ &= \sup\left\{\alpha \mid \alpha I \leq W_2^{-*}(j\omega)W_1^*(j\omega)W_1(j\omega)W_2^{-1}(j\omega)\ \forall\omega\right\}\\ &= \sup\left\{\alpha \mid \alpha I \leq \left[\sigma_{\min}\left(W_1(j\omega)W_2(j\omega)^{-1}\right)\right]^2\ \forall\omega\right\}\\ &= \frac{1}{\left\|(W_1W_2^{-1})^{-1}\right\|_{\mathcal{H}_\infty}^2} = \frac{1}{\left\|W_2W_1^{-1}\right\|_{\mathcal{H}_\infty}^2},\end{aligned}$$

where $W_2W_1^{-1}$ belongs to $\mathcal{RH}_\infty^{n\times n}$. This yields the formula of the Hilbert metric $d_{\mathrm{H}}(P_1, P_2)$ given by

$$\log\left(\frac{M(P_1, P_2)}{m(P_1, P_2)}\right) = \log\left(\left\|W_1W_2^{-1}\right\|_{\mathcal{H}_\infty}^2 \left\|W_2W_1^{-1}\right\|_{\mathcal{H}_\infty}^2\right)$$

which completes the proof. ∎

Theorem 1 indicates that computation of the Hilbert metric $d_{\mathrm{H}}(\cdot,\cdot)$ on $\mathcal{P}_+^n \setminus \{0\}$ only requires the spectral factors and $\mathcal{H}_\infty$-norms in (5), both of which can be efficiently computed by mature state-space methods. For example, see [20, Th. 1] for solutions to spectral factorizations and [4, Sec. 4.7] for

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algorithms to $\mathcal{H}_\infty$-norms. It is noteworthy that $d_{\rm H}(P_1,P_2)$ in (5) is finite for all strictly passive systems $P_1,P_2\in\mathcal{P}^n_{++}$, namely, on the interior $\mathrm{int}(\mathcal{P}^n_+)$, according to Lemma 1.

A discrete-time counterpart of Theorem 1 was studied in [17, Th. 1] which proposed the Hilbert metric as a distance between discrete-time bounded rational spectral densities. A specialization of Theorem 1 involving static matrices deserves to be discussed. Denote by $\mathcal{A}^n_+ \coloneqq \{A\in\mathbb{C}^{n\times n} \mid A+A^*\geq 0\}$ the set of matrices termed accretive matrices. For any $A,B\in\mathcal{A}^n_+\setminus\{0\}$, formula (5) reduces to

$$d_{\rm H}(A,B)=\log\left(\frac{\lambda_{\max}\left((\mathrm{He}[B])^{-\frac{1}{2}}\mathrm{He}[A](\mathrm{He}[B])^{-\frac{1}{2}}\right)}{\lambda_{\min}\left((\mathrm{He}[B])^{-\frac{1}{2}}\mathrm{He}[A](\mathrm{He}[B])^{-\frac{1}{2}}\right)}\right). \tag{7}$$

The Hilbert metric (7) has a significant difference from the metric studied in [24, Th. 4.4] for strictly accretive matrices through symmetric polar decompositions. For instance, the later metric is unitarily invariant, while the Hilbert metric is congruence-invariant which will be elaborated in Section IV.

We provide an illustrative example of computation of the Hilbert metric between two SISO passive systems.

Example 1: Consider two passive systems

$$P_1(s)=\frac{\alpha(s+0.5)}{s+1}\text{ and }P_2(s)=\frac{\beta(s+2)}{s+1},$$

where $\alpha,\beta>0$. The corresponding spectral factors of $\mathrm{He}[P_1(s)]$ and $\mathrm{He}[P_2(s)]$ can be easily constructed:

$$W_1(s)=\frac{\sqrt{\alpha}(s+\sqrt{2}/2)}{s+1}\text{ and }W_2(s)=\frac{\sqrt{\beta}(s+\sqrt{2})}{s+1}.$$

The Hilbert metric between P_1 and P_2 is then given by

$$\begin{aligned}d_{\rm H}(P_1,P_2)&=\log\left(\frac{\alpha}{\beta}\left\|\frac{s+\sqrt{2}/2}{s+\sqrt{2}}\right\|^2_{\mathcal{H}_\infty}\frac{\beta}{\alpha}\left\|\frac{s+\sqrt{2}}{s+\sqrt{2}/2}\right\|^2_{\mathcal{H}_\infty}\right)\\&=\log(4)\ \forall\alpha,\beta>0.\end{aligned}$$

Example 1 manifests an intrinsic property of $d_{\rm H}(P_1,P_2)$ in (5), that is, the invariance under positive gain scalings:

$$d_{\rm H}(\alpha P_1,\beta P_2)=d_{\rm H}(P_1,P_2)\ \forall\alpha,\beta>0.$$

The gain information of passive systems seems to be irrelevant as suggested in the property above. This suggests the interpretation of the Hilbert metric as a phase measure.

A. The Hilbert Metric as a Phase Distance

The following result follows from Definition 1 and Proposition 1. The proof is omitted for brevity.

Proposition 2: For $P_1,P_2\in\mathcal{P}^n_+\setminus\{0\}$, it holds that $d_{\rm H}(P_1,P_2)=0$ if and only if

$$\mathrm{He}[P_1(j\omega)]=\alpha\mathrm{He}[P_2(j\omega)]$$

for some $\alpha>0$ and all $\omega\in[-\infty,\infty]$.

The statement above shows that two frequency responses have zero distance separation if they share the same Hermitian part up to some positive scaling. Interestingly, the latter is a frequency-wise phase property for SISO systems.

Proposition 3: For $P_1\in\mathcal{P}^1_+\setminus\{0\}$ and $P_2\in\mathcal{P}^1_+\setminus\{0\}$ that are minimum-phase systems, it holds that $d_{\rm H}(P_1,P_2)=0$ if and only if

$$\angle P_1(j\omega)=\angle P_2(j\omega)$$

for all $\omega\in[-\infty,\infty]$.

Proof: According to Proposition 2, it is equivalent to proving the statement that $\mathrm{Re}[P_1(j\omega)]=\alpha\mathrm{Re}[P_2(j\omega)]$ for some $\alpha>0$ and all $\omega\in[-\infty,\infty]$ if and only if $\angle P_1(j\omega)=\angle P_2(j\omega)$ for all $\omega\in[-\infty,\infty]$.

We first show the sufficiency. Since $\angle P_1(j\omega)=\angle P_2(j\omega)$ for all $\omega\in[-\infty,\infty]$ and P_k is stable and minimum-phase, it follows from the Bode gain/phase relation that $|P_k(j\omega)|$ can be uniquely constructed from $\angle P_k(j\omega)$ alone up to positive scaling for $k=1,2$. Specifically, for any frequency $\omega_0\in[-\infty,\infty]$, we have the relation [25, p. 321]:

$$\begin{aligned}\log\frac{|P_k(j\omega_0)|}{|P_k(j0)|}=&\\-\frac{\omega_0}{\pi}\int_{-\infty}^{\infty}\frac{d\left[\left(\angle P_k(j\omega)\right)/\omega\right]}{d\nu}\log\coth\frac{|\nu|}{2}\,d\nu&\end{aligned}$$

for $k=1,2$, where $\nu\coloneqq\log(\omega/\omega_0)$. It then follows that

$$\log\frac{|P_1(j\omega_0)|}{|P_1(j0)|}=\log\frac{|P_2(j\omega_0)|}{|P_2(j0)|}$$

for all $\omega_0\in[-\infty,\infty]$ which is equivalent to that

$$|P_1(j\omega)|=\frac{|P_1(j0)|}{|P_2(j0)|}|P_2(j\omega)|$$

for all $\omega\in[-\infty,\infty]$. Let $\alpha\coloneqq\frac{|P_1(j0)|}{|P_2(j0)|}>0$. we have

$$\begin{aligned}\mathrm{Re}[P_1(j\omega)]&=|P_1(j\omega)|\cos\angle P_1(j\omega)\\&=\alpha\,|P_2(j\omega)|\cos\angle P_2(j\omega)=\alpha\mathrm{Re}[P_2(j\omega)]\end{aligned}$$

for all $\omega\in[-\infty,\infty]$.

We next show the necessity using properties of analytic functions. For $k=1,2$, since a minimum-phase $P_k(s)$ is causal, it is shown in [26, p. 344] that for the frequency response $P_k(j\omega)$, its imaginary part $\mathrm{Im}[P_k(j\omega)]$ is uniquely determined by its real part $\mathrm{Re}[P_k(j\omega)]$. Concretely, for every $\omega_0\in[-\infty,\infty]$, we have

$$\begin{aligned}\mathrm{Im}[P_k(j\omega_0)]=&\\\frac{1}{2\pi}\mathrm{p.v.}\int_{-\infty}^{\infty}\mathrm{Re}[P_k(j\omega)]\cot\left(\frac{\omega-\omega_0}{2}\right)d\omega,&\end{aligned}\tag{8}$$

for $k=1,2$, where the integral is taken under Cauchy principal values (p.v.). By the hypothesis of $\mathrm{Re}[P_1(jw)]=\alpha\mathrm{Re}[P_2(j\omega)]$ for $\alpha>0$ and all $\omega\in[-\infty,\infty]$, it follows from formula (8) that

$$\mathrm{Im}[P_1(j\omega)]=\alpha\mathrm{Im}[P_2(j\omega)]$$

for $\alpha>0$ and all $\omega\in[-\infty,\infty]$. This implies that $P_1(j\omega)=\alpha P_2(j\omega)$ for $\alpha>0$ and all $\omega\in[-\infty,\infty]$. Thus, $\angle P_1(j\omega)=\angle P_2(j\omega)$ for all $\omega\in[-\infty,\infty]$. ∎

For SISO minimum-phase passive systems, Proposition 3 shows that a zero Hilbert distance requires exactly the same Bode phase plots. The Hilbert distance can thus be counted

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as a quantitative characterization of the *frequency-wise phase difference* between passive systems. It is worth noting that the case of same Nyquist plots may still have nonzero Hilbert distances. For example, consider the following three passive systems:

$$P_1(s)=\frac{s+1}{s+0.5},\ P_2(s)=\frac{s+2}{s+1}\text{ and }P_3(s)=\frac{s+4}{s+2}.$$

One can check that the three systems have the same Nyquist plots, while their Bode phase responses are completely different. Therefore, the Hilbert distance between the three systems is still nonzero.

A generalization of Proposition 3 to MIMO passive systems is currently under investigation.

IV. An Energy Distortion Interpretation from Multi-port Circuits

The Hilbert metric is invariant under congruence transformations as detailed below.

Proposition 4: For two passive systems $P_1,P_2\in\mathcal{P}_+^n\setminus\{0\}$ and any minimum-phase filter $T\in\mathcal{RH}_\infty^{n\times n}$ such that $T^{-1}\in\mathcal{RH}_\infty^{n\times n}$, it holds that

$$d_{\mathrm{H}}(P_1,P_2)=d_{\mathrm{H}}(T^*P_1T,T^*P_2T). \tag{9}$$

Proof: For $k=1,2$, denote $\tilde{P}_k\coloneqq T^*P_kT$. Note that

$$\begin{aligned}\frac{1}{2}\left(\tilde{P}_k(s)+\tilde{P}_k^*(s)\right)&=T^*(s)\frac{P_k(s)+P_k^*(s)}{2}T(s)\\&=T^*(s)W_k^*(s)W_k(s)T(s)\\&=\tilde{W}_k^*(s)\tilde{W}_k(s),\end{aligned}$$

where the second equality utilizes the spectral factorization in (4) and $W_kT=:\tilde{W}_k\in\mathcal{RH}_\infty^{n\times n}$ for $k=1,2$. Since $T^{-1}\in\mathcal{RH}_\infty^{n\times n}$ and so is W_k^{-1}, then $\tilde{W}_k(s)$ is a minimum-phase spectral factor of $\mathrm{He}[\tilde{P}_k(s)]$ as $\tilde{W}_k^{-1}\in\mathcal{RH}_\infty^{n\times n}$ for $k=1,2$. Moreover, it holds that

$$\begin{aligned}\tilde{W}_1\tilde{W}_2^{-1}&=W_1T\left(W_2T\right)^{-1}=W_1W_2^{-1}\\\tilde{W}_2\tilde{W}_1^{-1}&=W_2T\left(W_1T\right)^{-1}=W_2W_1^{-1}.\end{aligned}$$

This implies that

$$\begin{aligned}&\left\|\tilde{W}_1\tilde{W}_2^{-1}\right\|_{\mathcal{H}_\infty}^2\left\|\tilde{W}_2\tilde{W}_1^{-1}\right\|_{\mathcal{H}_\infty}^2\\&\qquad=\left\|W_1W_2^{-1}\right\|_{\mathcal{H}_\infty}^2\left\|W_2W_1^{-1}\right\|_{\mathcal{H}_\infty}^2.\end{aligned}$$

The proof is then completed by invoking Theorem 1. ∎

The remainder of this section is dedicated to a physical interpretation of the congruence invariance property (9) of the Hilbert metric. We next interpret (9) using n-port electric circuits from a perspective of supplied energy distortion.

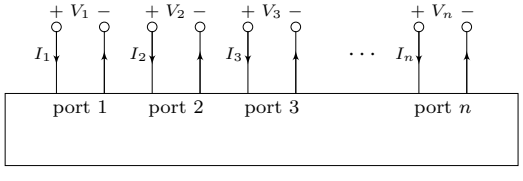


Fig. 1. A passive n-port circuit.

An n-port circuit illustrated by Fig. 1 adopts a port-based external description of connections of electrical components such as resistors, capacitors and inductors. The external behaviors of an n-port circuit are completely characterized by its port-voltage/current variables. A stable LTI n-port circuit can be identified as a causal operator $\boldsymbol{P}:\mathcal{L}_2^n\to\mathcal{L}_2^n$ with transfer function $P\in\mathcal{P}_+^n$, where $\mathcal{L}_2^n$ denotes the following set of energy-bounded $\mathbb{R}^n$-valued signals:

$$\mathcal{L}_2^n\coloneqq\left\{u\colon\mathbb{R}_+\to\mathbb{R}^n\ \middle|\ \langle u\,,u\rangle\coloneqq\int_0^\infty u(t)^\top u(t)\,dt<\infty\right\}.$$

For a circuit $\boldsymbol{P}$ in Fig. 1, we choose the port-voltage variable $v=\begin{bmatrix}V_1&V_2&\cdots&V_n\end{bmatrix}^\top$ as the input and the port-current variable $i=\begin{bmatrix}I_1&I_2&\cdots&I_n\end{bmatrix}^\top=\boldsymbol{P}v$ as the output. By the well-known Plancherel's theorem, the *supplied energy* of $\boldsymbol{P}$ can be expressed both in the time domain and frequency domain:

$$\begin{aligned}E&\coloneqq\langle v\,,\boldsymbol{P}v\rangle=\int_0^\infty v(t)^\top(\boldsymbol{P}v)(t)\,dt\\&=\langle\hat{v}\,,\widehat{\boldsymbol{P}v}\rangle=\frac{1}{2\pi}\int_{-\infty}^\infty\hat{v}(j\omega)^*P(j\omega)\hat{v}(j\omega)\,d\omega,\end{aligned}$$

where $\hat{v}$ denotes the Fourier transform of v.

Consider two n-port passive circuits $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ and view $\boldsymbol{P}_2$ as a distorted version of $\boldsymbol{P}_1$. The Hilbert distance $d_{\mathrm{H}}(\boldsymbol{P}_1,\boldsymbol{P}_2)$ may be interpreted as a distortion measure between the energy supplies of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. A good measure of distortion should be invariant to the representation of the signals. Namely, if we filter the voltages with any stable LTI filter, the distortion between the energy supplies should not be affected by the linear transformation.

The scenario can be formulated by the invariance under congruence transformations stated in Proposition 4. Specifically, we choose an LTI filter $\boldsymbol{T}:\mathcal{L}_2^n\to\mathcal{L}_2^n$ such that $T,T^{-1}\in\mathcal{RH}_\infty^{n\times n}$. For $k=1,2$, the new supplied energy of $\boldsymbol{P}_k$ with the filtered port-voltage input $\boldsymbol{T}v_k$ and filtered port-current output $\boldsymbol{P}_k\boldsymbol{T}v_k$ is given by

$$\begin{aligned}\tilde{E}_k&=\langle\widehat{\boldsymbol{T}v_k}\,,\widehat{\boldsymbol{P}_k\boldsymbol{T}v_k}\rangle=\langle T\hat{v}_k\,,PT\hat{v}_k\rangle\\&=\frac{1}{2\pi}\int_{-\infty}^\infty\hat{v}_k(j\omega)^*T(j\omega)^*P_k(j\omega)T(j\omega)\hat{v}_k(j\omega)\,d\omega.\end{aligned}$$

Then $\tilde{E}_k$ corresponds to the supplied energy transferred to a filtered circuit defined through the transformation $\tilde{P}_k=T^*P_kT$ in the frequency domain for $k=1,2$. By (9), we have $d_{\mathrm{H}}(P_1,P_2)=d_{\mathrm{H}}(\tilde{P}_1,\tilde{P}_2)$. Here is a toy example. When $n=2$ and $T=\mathrm{diag}(10^{-3},10^3)$ is static, the units of measurements for the voltage and current in $\mathrm{port}\ 1$ become $[\mathrm{kV}]$ and $[\mathrm{kA}]$ and in $\mathrm{port}\ 2$ $[\mathrm{mV}]$ and $[\mathrm{mA}]$. In this case, $\tilde{E}_k$ is consistent with E_k in quantity when performing calculations correctly for $k=1,2$.

V. Comparison to the Gap Metric

This section draws a brief comparison between the proposed Hilbert metric and the celebrated gap metric.

The gap metric was introduced in [5] for the purpose of measuring a difference between possibly unstable LTI

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systems. Denote by $\mathcal{R}^{n\times n}$ the set of $n\times n$ real-rational proper matrix-valued functions. The normalized right coprime factorization of a system $P \in \mathcal{R}^{n\times n}$ is given by

$$P(s) = N(s)D^{-1}(s),$$

where $N, D \in \mathcal{RH}_\infty^{n\times n}$ and D has a proper inverse such that $D^*(s)D(s) + N^*(s)N(s) = I$. The graph symbol of P is represented by a closed subspace:

$$\mathcal{G}(P) = \begin{bmatrix} D \\ N \end{bmatrix} \mathcal{H}_2^n \subset \mathcal{H}_2^n \times \mathcal{H}_2^n,$$

where $\mathcal{H}_2^n$ is an isomorphism of $\mathcal{L}_2^n$ via the Laplace transform. In [6, Prop. 1], a computable formula of the gap metric $d_{\mathrm{G}} : \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times n} \to [0, 1]$ was provided:

$$d_{\mathrm{G}}(P_1, P_2) = \max \left\{ \inf_{Q\in\mathcal{H}_\infty^{n\times n}} \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} Q \right\|_{\mathcal{H}_\infty}, \inf_{Q\in\mathcal{H}_\infty^{n\times n}} \left\| \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} - \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} Q \right\|_{\mathcal{H}_\infty} \right\}. \quad (10)$$

In what follows, we lay out the gap metric $d_{\mathrm{G}}(\cdot,\cdot)$ in (10) and the Hilbert metric $d_{\mathrm{H}}(\cdot,\cdot)$ in (5) side by side and pinpoint their major differences.

(i) Space: $d_{\mathrm{G}}(\cdot,\cdot)$ as a general distance is defined on the space of arbitrary systems, while $d_{\mathrm{H}}(\cdot,\cdot)$ as a phase distance is only targeted at the space of passive systems.

(ii) Computation: formula (10) for $d_{\mathrm{G}}(\cdot,\cdot)$ employs two coprime factorizations, while formula (5) for $d_{\mathrm{H}}(\cdot,\cdot)$ two spectral factorizations.

(iii) Geometry: $d_{\mathrm{G}}(\cdot,\cdot)$ is known as a *chordal* distance function (e.g., see [27, Fig. 12.4]) between two closed subspaces $\mathcal{G}(P_1)$ and $\mathcal{G}(P_2)$, while $d_{\mathrm{H}}(\cdot,\cdot)$ is a conal distance between two rays in the closed cone of passive systems.

(iv) Invariance: $d_{\mathrm{G}}(\cdot,\cdot)$ is unitarily invariant in the sense that

$$d_{\mathrm{G}}(UP_1V, UP_2V) = d_{\mathrm{G}}(P_1, P_2)$$

for any unitary matrices $U, V \in \mathbb{C}^{n\times n}$, while $d_{\mathrm{H}}(\cdot,\cdot)$ is invariant under congruence transformations (9).

VI. Conclusion

In this paper, we developed the Hilbert metric on the cone of MIMO LTI passive systems. We showed that the distance can be computed from suitable spectral factorizations and $\mathcal{H}_\infty$ norms. Furthermore, it was shown for SISO minimum-phase systems that the Hilbert metric is a phase distance. A brief comparison was made between the Hilbert metric and the gap metric.

The Hilbert metric provides a fresh perspective on approximation problems and robustness analysis in the space of physical LTI passive models. We note that the Hilbert distance can be naturally extended to the cone of other phase systems like negative imaginary systems [28] and sectorial systems [21], [29].

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