

Current and Future Persectives on the Block Macaulay Matrix

Numerical Analysis Seminar at FMF Ljubljana

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December 10, 2025



Current and Future Persectives on the Block Macaulay Matrix

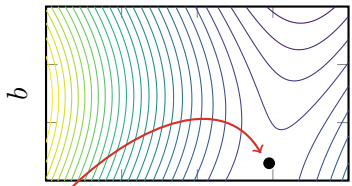
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STADIUS
Center for Dynamical Systems, Signal Processing and Data Analytics

The talk presents joint efforts with and insights from Bart De Moor, Bor Plestenjak, and Matías Bender.

Introducing the block Macaulay matrix

optimization problem

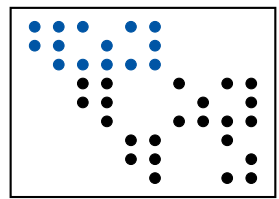


tackle optimization problem

multiparameter eigenvalue reformulation

$$\mathcal{A}\underbrace{(a,b)}_{\lambda}\underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_{z} = \mathbf{0}$$

leads to the construction of a structured block Macaulay matrix



use numerical algorithms

achieve global optimality

critical points

cost	a	b
0.8681	0.4751	0.4598
1.9231	0.2294	−0.2528



Outline

1 | Multiparameter Eigenvalue Problems
2 | Current Block Macaulay Algorithms
  a | Shift-Invariance in the (Right) Null Space
  b | Complementary Approch in the Column Space
3 | Paths for Future Research
  a | Similarities with the Kronecker Algorithm
  b | Bihomogeneous Structure and Lifting Approach
  c | Other problem structures



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What is a multiparameter eigenvalue problem?

A **two-parameter eigenvalue problem** (of degree d) is given by

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda_1 + \boldsymbol{A}_{01}\lambda_2 + \cdots + \boldsymbol{A}_{0d}\lambda_2^d\right)\boldsymbol{z} = \boldsymbol{0}.$$

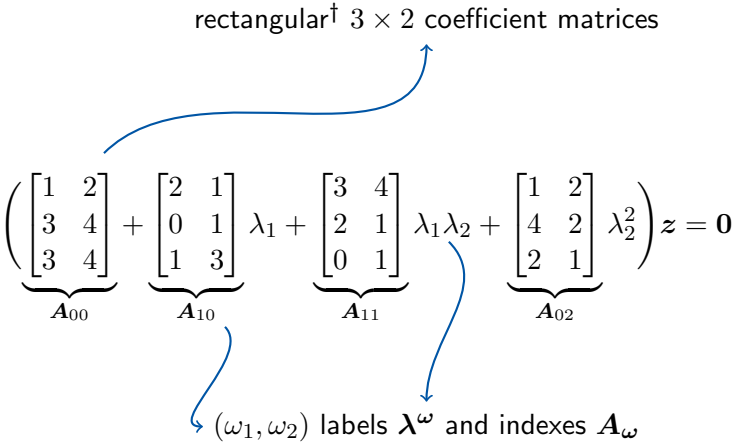
It has rectangular coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} = \boldsymbol{A}_{(\omega_1 \cdots \omega_m)} \in \mathbb{C}^{k \times l}$ with $k \geq l + m - 1$.

The **multiparameter eigenvalue problem (MEP)** $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \boldsymbol{0}$ consists in finding all m-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}_0^l$, so that (with $\|\boldsymbol{z}\|_2 = 1$)

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}} \boldsymbol{z} = \boldsymbol{0}.$$

Vermeersch and De Moor (2022); Vermeersch (2023); Vermeersch and De Moor (2025)

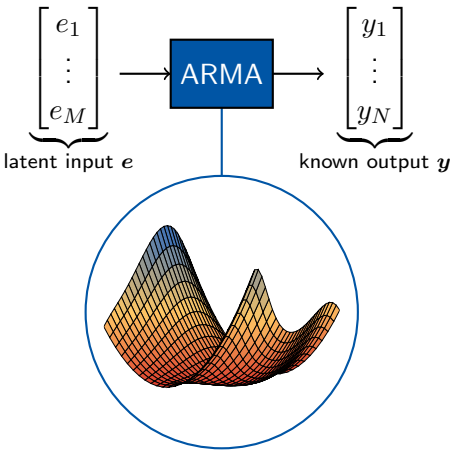
Example of a two-parameter eigenvalue problem



†We consider only rectangular problems in this presentation.

System identification as a potential application

first-order ARMA$(1,1)$ model with N data points ($\boldsymbol{y}$) and two parameters (α and γ)



$$\min_{\alpha,\gamma,\boldsymbol{e}} e_1^2 + \cdots + e_N^2$$

subject to $y_k + \alpha y_{k-1} = e_k + \gamma e_{k-1}$

$$\downarrow$$

$$p_1(\alpha,\gamma,\boldsymbol{e}) = \cdots = p_s(\alpha,\gamma,\boldsymbol{e}) = 0$$

or

$$\left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\alpha + \boldsymbol{A}_{01}\gamma + \boldsymbol{A}_{02}\gamma^2\right)\boldsymbol{z} = \boldsymbol{0}$$

$$\downarrow$$

parameters α and γ

Vermeersch and De Moor (2019)

Multiparameter eigentuple-finding algorithms

Two types of numerical solution algorithms:

optimization-based algorithms

- nonlinear optimization algorithms
- fast, but find only one solution per run
- potential convergence issues and depend on initial guess
- gradient descent, Gauss–Newton, Newton–Raphson

matrix-based algorithms

- structured matrices
- good approximations of all solutions
- slower than optimization-based algorithms
- randomized sketching, Kronecker matrices, **block Macaulay matrices**

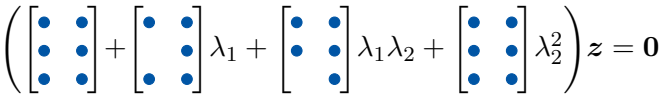
Vermeersch (2023); Vermeersch and De Moor (2025); Hochstenbach et al. (2024) and references therein



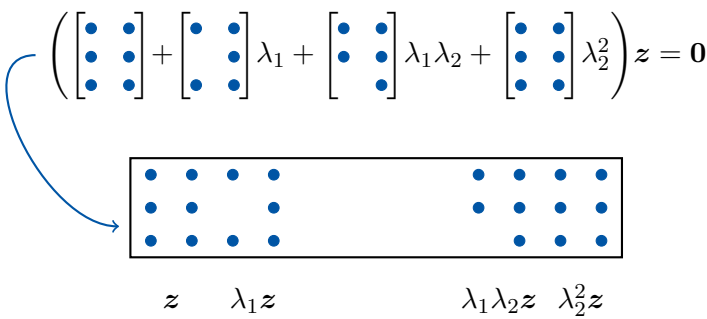
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Constructing the block Macaulay matrix

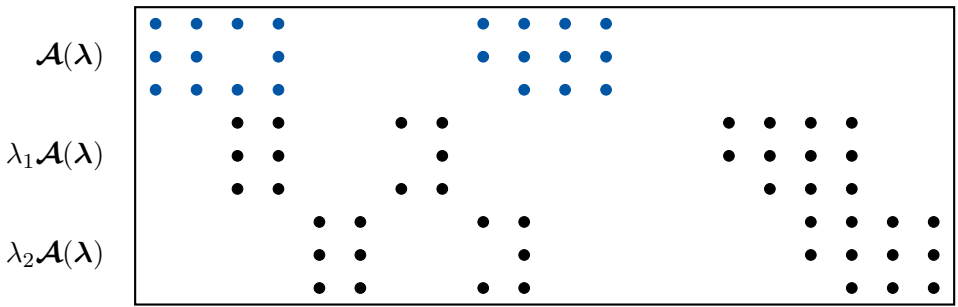


Constructing the block Macaulay matrix



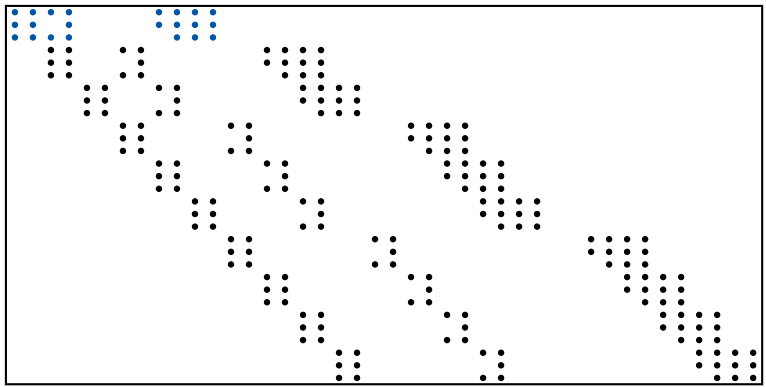
Constructing the block Macaulay matrix

$$\left(\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\\ \bullet & \bullet\end{bmatrix}+\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\lambda_1+\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\lambda_1\lambda_2+\begin{bmatrix}\bullet & \bullet\\ \bullet & \bullet\\ \bullet & \bullet\end{bmatrix}\lambda_2^2\right)\boldsymbol{z}=\mathbf{0}$$



Constructing the block Macaulay matrix

$M_d =$





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Exploiting the structure of the (right) null space

Every solution yields exactly one vector in the multivariate block Vandermonde basis matrix ($d \geq d^*$):

$$\boldsymbol{V}_d = \left[\begin{array}{cc} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1 \boldsymbol{z}|_{(1)} & \lambda_1 \boldsymbol{z}|_{(2)} \\ \lambda_2 \boldsymbol{z}|_{(1)} & \lambda_2 \boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2 \boldsymbol{z}|_{(1)} & \lambda_1^2 \boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2 \boldsymbol{z}|_{(1)} & \lambda_1\lambda_2 \boldsymbol{z}|_{(2)} \\ \lambda_2^2 \boldsymbol{z}|_{(1)} & \lambda_2^2 \boldsymbol{z}|_{(2)} \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \\ \\ \downarrow \\ \boldsymbol{v}_d|_{(1)} \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \\ \\ \downarrow \\ \boldsymbol{v}_d|_{(2)} \end{array}$$

Monomial multiplicative property

one affine solution

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}}_{\boldsymbol{S}_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1 \boldsymbol{z}|_{(1)} \\ \lambda_2 \boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2 \boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2 \boldsymbol{z}|_{(1)} \\ \lambda_2^2 \boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}} \lambda_1|_{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots \end{bmatrix}}_{\boldsymbol{S}_{\lambda_1}} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1 \boldsymbol{z}|_{(1)} \\ \lambda_2 \boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2 \boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2 \boldsymbol{z}|_{(1)} \\ \lambda_2^2 \boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}}$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

one affine solution

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} \\ \lambda_2\boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}} \lambda_1|_{(1)} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} \\ \lambda_2\boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}}$$

select the linearly independent rows of $\boldsymbol{V}_d$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

multiple affine solutions

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d}$$

$$\begin{bmatrix} \lambda_1|_{(1)} & 0 \\ 0 & \lambda_1|_{(2)} \end{bmatrix}$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

multiple affine solutions

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d}$$

standard eigenvalue problem!

$$(\boldsymbol{S}_1\boldsymbol{V}_d)\,\boldsymbol{D}_{\lambda_1} = (\boldsymbol{S}_{\lambda_1}\boldsymbol{V}_d)$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

- Solution vectors $\boldsymbol{v}_d$ in $\boldsymbol{V}_d$ not known in advanced:

numerical basis matrix $\boldsymbol{Z}_d$ of the right null space ($\boldsymbol{V}_d = \boldsymbol{Z}_d \boldsymbol{T}$)

$$\Downarrow$$

$$(\boldsymbol{S}_1 \boldsymbol{Z}_d)\, \boldsymbol{T} \boldsymbol{D}_{\lambda_1} = (\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d)\, \boldsymbol{T}$$

$$\boldsymbol{T} \boldsymbol{D}_{\lambda_1} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d)$$

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

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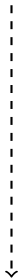
- Possible to **shift with any polynomial** in the variables:

$$\boldsymbol{T} \boldsymbol{D}_{\lambda_2} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{\lambda_2} \boldsymbol{Z}_d)\,,$$

which leads to the same $\boldsymbol{T}$

Null space based solution approach

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0}$$



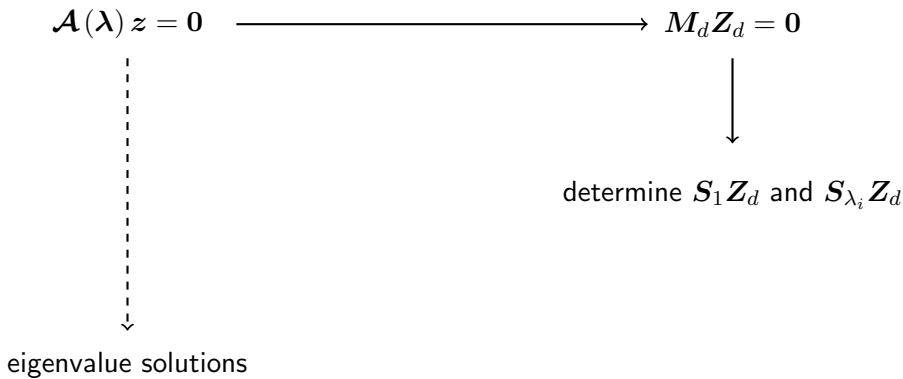
eigenvalue solutions

Null space based solution approach



Vermeersch and De Moor (2022)

Null space based solution approach



Vermeersch and De Moor (2022)

Null space based solution approach

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \longrightarrow \boldsymbol{M}_d \boldsymbol{Z}_d = \boldsymbol{0}$$

$$\downarrow$$

determine $\boldsymbol{S}_1 \boldsymbol{Z}_d$ and $\boldsymbol{S}_{\lambda_i} \boldsymbol{Z}_d$

$$\downarrow$$

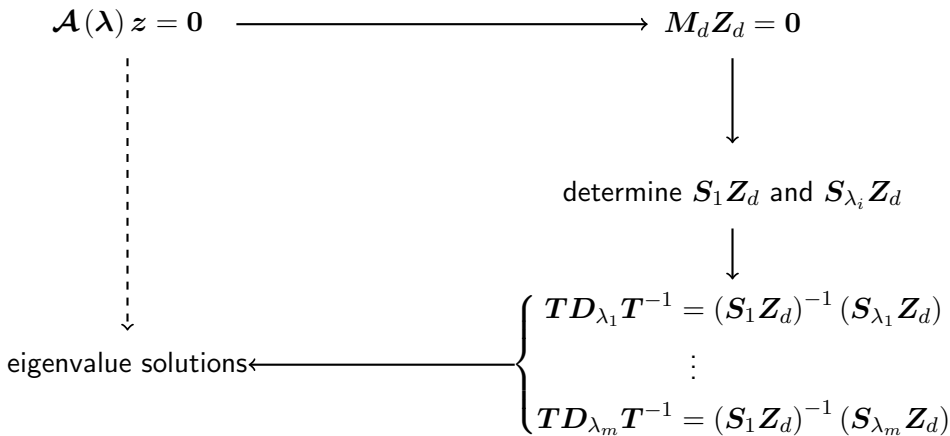
$$\begin{cases} \boldsymbol{T}\boldsymbol{D}_{\lambda_1}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_1}\boldsymbol{Z}_d) \\ \vdots \\ \boldsymbol{T}\boldsymbol{D}_{\lambda_m}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_m}\boldsymbol{Z}_d) \end{cases}$$

$$\vdots$$

eigenvalue solutions

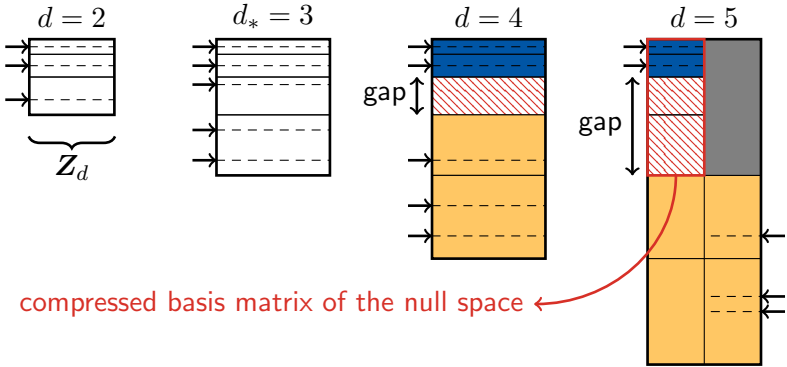
Vermeersch and De Moor (2022)

Null space based solution approach



Vermeersch and De Moor (2022)

Removing solutions at infinity



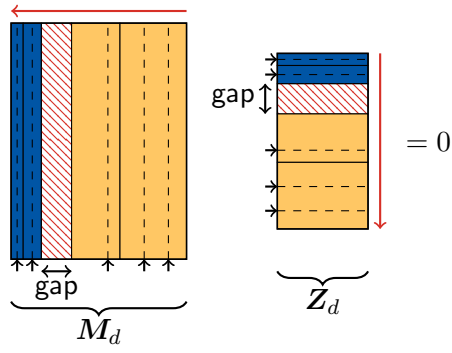
A column compression deflates the solutions at infinity from the eigenvalue problems



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What about the column space?



The column space and (right) null space of the Macaulay matrix are complementary subspaces: linearly independent rows in the (right) null space are related to linearly dependent columns in the column space

Reordered block Macaulay matrix

$$\underbrace{\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{bmatrix}}_{\boldsymbol{N}_d} \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix}}_{\boldsymbol{Z}_d} = \boldsymbol{0}$$

$\boldsymbol{A}$: corresponds to $\boldsymbol{S}_1 \boldsymbol{Z}_d$

$\boldsymbol{B}$: contains rows of $\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d$ not in $\boldsymbol{A}$

$\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$: remaining rows of $\boldsymbol{Z}_d$

The multiplicative property

$$(\boldsymbol{S}_1 \boldsymbol{Z}_d)\, \boldsymbol{T} \boldsymbol{D}_{\lambda_1} = (\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d)\, \boldsymbol{T}$$

can be rewritten as

$$\boldsymbol{A} \boldsymbol{D}_{\lambda_1} \boldsymbol{A}^{-1} = \widetilde{\boldsymbol{S}}_{\lambda_1} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{B} \boldsymbol{A}^{-1} \end{bmatrix}.$$

Vermeersch and De Moor (2022)

Reordered block Macaulay matrix

reordered block Macaulay matrix

$$\underbrace{\begin{bmatrix} \boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{M}_3 & \boldsymbol{M}_4 \end{bmatrix}}_{\boldsymbol{N}_d} \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{D} & \boldsymbol{E} \end{bmatrix}}_{\boldsymbol{Z}_d} = \boldsymbol{0}$$

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Vermeersch and De Moor (2022)

Backward QR decomposition

The backward QR decomposition **eliminates** the dependency on **the (right) null space**

$$\begin{bmatrix} \boldsymbol{R}_{14} & \boldsymbol{R}_{13} & \boldsymbol{R}_{12} & \boldsymbol{R}_{11} \\ \boldsymbol{R}_{24} & \boldsymbol{R}_{23} & \boldsymbol{R}_{22} & \mathbf{0} \\ \boldsymbol{R}_{34} & \boldsymbol{R}_{33} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{R}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \mathbf{0} \\ \boldsymbol{B} & \mathbf{0} \\ \boldsymbol{C} & \mathbf{0} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} = \mathbf{0}.$$

(Vermeersch and De Moor, 2021)

Backward QR decomposition

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$$\begin{bmatrix} \boldsymbol{R}_{14} & \boldsymbol{R}_{13} & \boldsymbol{R}_{12} & \boldsymbol{R}_{11} \\ \boldsymbol{R}_{24} & \boldsymbol{R}_{23} & \boldsymbol{R}_{22} & \mathbf{0} \\ \boldsymbol{R}_{34} & \boldsymbol{R}_{33} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{R}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \mathbf{0} \\ \boldsymbol{B} & \mathbf{0} \\ \boldsymbol{C} & \mathbf{0} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} = \mathbf{0}.$$

$$\boldsymbol{B}\boldsymbol{A}^{-1} = -\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34}$$

(Vermeersch and De Moor, 2021)

Backward QR decomposition

The backward QR decomposition **eliminates** the dependency on **the (right) null space**

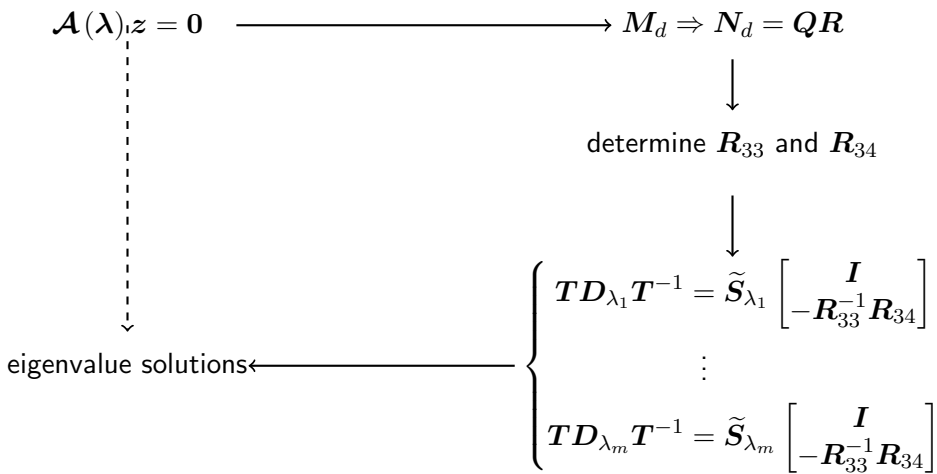
$$\begin{bmatrix} \boldsymbol{R}_{14} & \boldsymbol{R}_{13} & \boldsymbol{R}_{12} & \boldsymbol{R}_{11} \\ \boldsymbol{R}_{24} & \boldsymbol{R}_{23} & \boldsymbol{R}_{22} & \boldsymbol{0} \\ \boldsymbol{R}_{34} & \boldsymbol{R}_{33} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{R}_{44} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix} = \boldsymbol{0}.$$

$$\boldsymbol{B}\boldsymbol{A}^{-1} = -\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34}$$

$$\boldsymbol{A}\boldsymbol{D}_{\lambda_1}\boldsymbol{A}^{-1} = \widetilde{\boldsymbol{S}}_{\lambda_1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{R}_{33}^{-1}\boldsymbol{R}_{34} \end{bmatrix},$$

(Vermeersch and De Moor, 2021)

Column space based solution approach



(Vermeersch and De Moor, 2022)



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Identifying some open problems

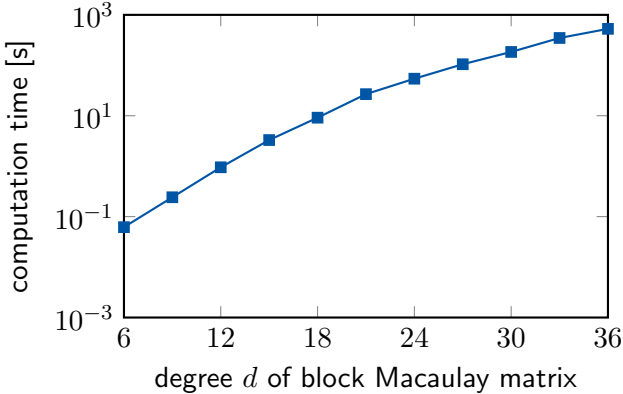
Three important issues!

- No rigourous theory
- Only one rectangular equation
- Computational complexity

Identifying some open problems

Three important issues!

- No rigourous theory
- Only one rectangular equation
- Computational complexity



Comutation time of the block Macaulay algorithm



Outline

1 | Multiparameter Eigenvalue Problems
2 | Current Block Macaulay Algorithms
  a | Shift-Invariance in the (Right) Null Space
  b | Complementary Approch in the Column Space
3 | Paths for Future Research
  a | Similarities with the Kronecker Algorithm
  b | Bihomogeneous Structure and Lifting Approach
  c | Other problem structures

Comparison with the Kronecker algorithm

After constructing Δ-matrices,

$$\boldsymbol{\Delta}_0 = \begin{vmatrix} \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_m \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_m \end{vmatrix}_{\otimes}, \boldsymbol{\Delta}_1 = \begin{vmatrix} \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{i-1} & \boldsymbol{A}_0 & \boldsymbol{A}_{i+1} & \cdots & \boldsymbol{A}_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{i-1} & \boldsymbol{A}_0 & \boldsymbol{A}_{i+1} & \cdots & \boldsymbol{A}_m \end{vmatrix}_{\otimes}$$

the MEP can also be solved via the so-called Kronecker algorithm:

$$\begin{cases} \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{w} = \boldsymbol{\Delta}_1 \boldsymbol{w}, \\ \vdots \\ \boldsymbol{\Delta}_0 \lambda_m \boldsymbol{w} = \boldsymbol{\Delta}_m \boldsymbol{w}, \end{cases}$$

which is a joint generalized eigenvalue problem (joint GEP) with eigenvectors $\boldsymbol{w} = \boldsymbol{z} \times \cdots \times \boldsymbol{z}$.

Comparison with the Kronecker algorithm

The joint GEP has l^m eigenvalues, while the MEP only has $\binom{l+m-1}{l}$ solutions.

A reduction/projection $\boldsymbol{\Delta}_i \to \widetilde{\boldsymbol{\Delta}}_i = \boldsymbol{L}\boldsymbol{\Delta}_i\boldsymbol{T}$ resolves this issue!

$$\begin{cases} \widetilde{\boldsymbol{\Delta}}_0\lambda_1\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{\Delta}}_1\widetilde{\boldsymbol{w}} \\ \vdots \\ \widetilde{\boldsymbol{\Delta}}_0\lambda_m\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{\Delta}}_m\widetilde{\boldsymbol{w}} \end{cases}$$

Hochstenbach et al. (2024); Alsubaie (2019)

Exploiting similarities with the Kronecker algorithm

Kronecker algorithm
(implemented in MULTIPAREIG)

$$\Downarrow$$

$$\begin{cases} \boldsymbol{T}\boldsymbol{D}_{\lambda_1}\boldsymbol{T}^{-1} = \widetilde{\boldsymbol{\Delta}}_0^{-1}\widetilde{\boldsymbol{\Delta}}_1 \\ \vdots \\ \boldsymbol{T}\boldsymbol{D}_{\lambda_m}\boldsymbol{T}^{-1} = \widetilde{\boldsymbol{\Delta}}_0^{-1}\widetilde{\boldsymbol{\Delta}}_m \end{cases}$$

block Macaulay algorithm
(implemented in MACAULAYLAB)

$$\Downarrow$$

$$\begin{cases} \boldsymbol{T}\boldsymbol{D}_{\lambda_1}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_1}\boldsymbol{Z}_d) \\ \vdots \\ \boldsymbol{T}\boldsymbol{D}_{\lambda_m}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_m}\boldsymbol{Z}_d) \end{cases}$$

Both algorithms solve the (linear) MEP via a joint GEP

- Link between both joint GEPs is still unknown
- Different ways of dealing with solutions at infinity



Outline

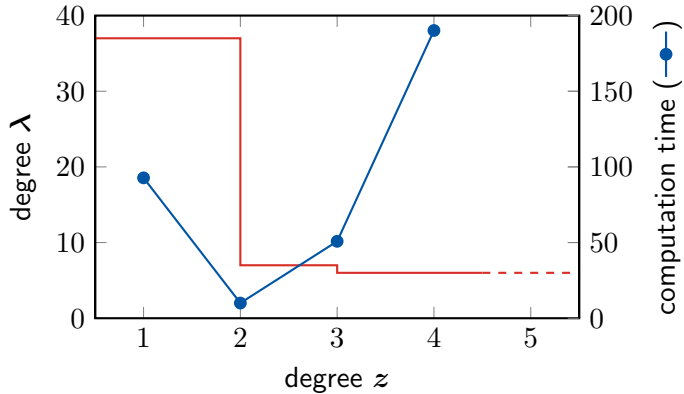
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Considering the bihomogeneous structure

From an algebraic geometry viewpoint, an MEP can be **modeled as a bihomogeneous system** of polynomial equations in $\mathbb{C}[\boldsymbol{\lambda}][\boldsymbol{z}]$.

- Solutions define a biprojective variety in $\mathbb{P}^{m-1} \times \mathbb{P}^{l-1}$
- Could be studied in the setting of bi/multihomogeneous polynomial systems
- What about special sparse Macaulay matrices?

Lifting the multiparameter eigenvalue problem



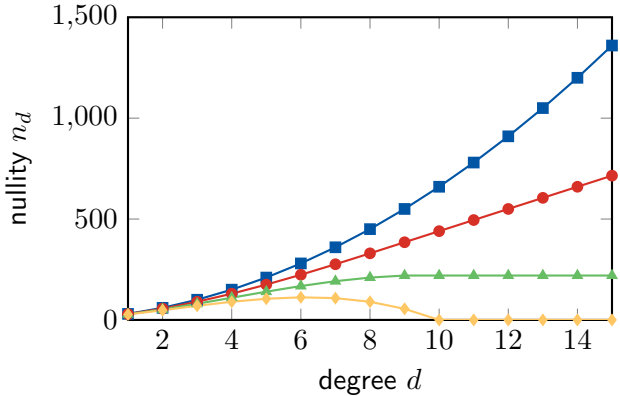
Degree of regularity, d^*, w.r.t. the eigenvalues (——) when varying the degree of eigenvector elements for the same two-parameter eigenvalue problem



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Can we use the block Macaulay algorithm for square problems?



Change of nullity for a three-parameter eigenvalue problem with coefficient matrices of size 10×10 (—■—), 11×10 (—●—), 12×10 (—▲—), 13×10 (—◆—)



Any questions?

This work was supported in part by the KU Leuven Research Fund (grants IBOF/23/064, C3/20/117, and C3I/21/00316); in part by the FWO (grants S005319 and T001919N); in part by the Flemish government via the Flanders AI Research Program; in part by the VLAIO (grant HBC/2021/0076); and in part by the European Research Council (grant 885682). Research stays during this work were supported by a Short Stay grant of the FWO (grant K250225N) and by a Tournesol mobility project (grant VS00326N).

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