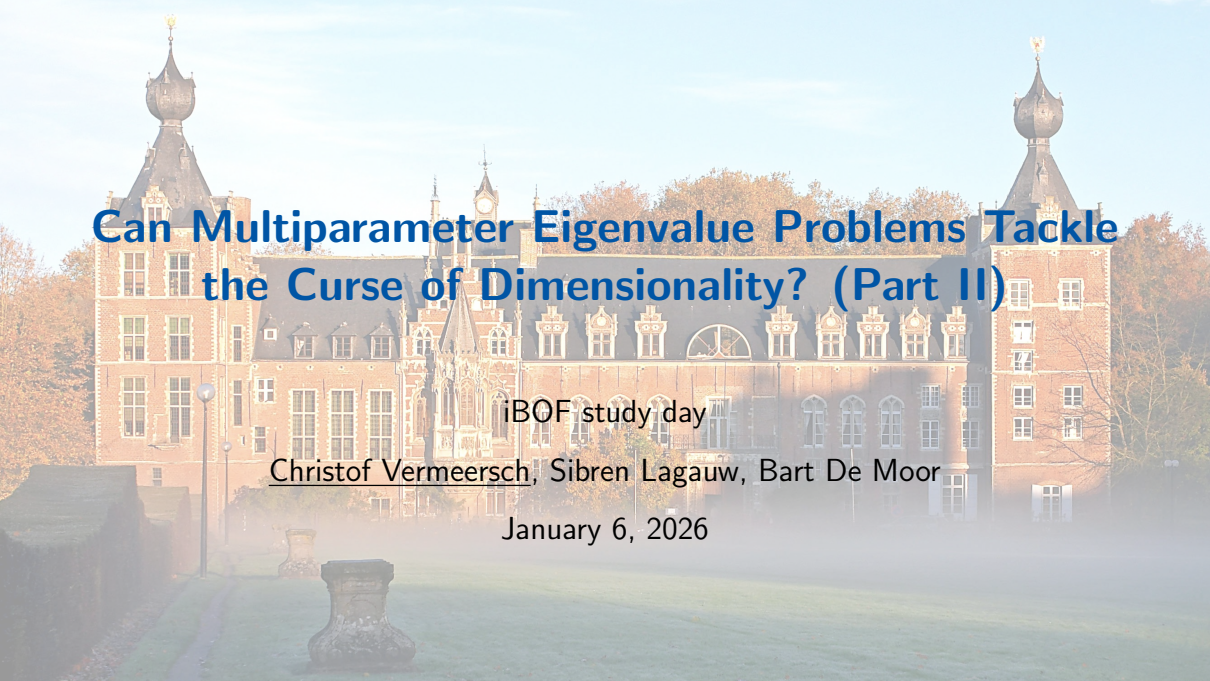


Can Multiparameter Eigenvalue Problems Tackle the Curse of Dimensionality? (Part II)

iBOF study day

Christof Vermeersch, Sibren Lagauw, Bart De Moor

January 6, 2026



Can Multiparameter Eigenvalue Problems Tackle the Curse of Dimensionality? (Part II)

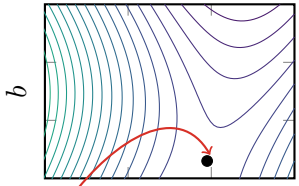
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STADIUS
Center for Dynamical Systems, Signal Processing and Data Analytics

The talk is the second part of a two-part presentation.

Introducing multiparameter eigenvalue reformulations

multivariate polynomial application



$$\min f(\boldsymbol{a}, \boldsymbol{b})$$
$$\text{s. t. } g(\boldsymbol{a}, \boldsymbol{b})$$
$$h(\boldsymbol{a}, \boldsymbol{b})$$

tackle optimization problem

traditional approach

characterization of the first-order necessary conditions

$$\begin{cases} p_1(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = 0 \\ \vdots \\ p_s(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = 0 \end{cases}$$

exploit structure

project's novel approach

compact multiparameter eigenvalue reformulation

$$\boldsymbol{\mathcal{A}}(\boldsymbol{a}, \boldsymbol{b}) \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \boldsymbol{0}$$

solve numerically

obtain insights

find solutions

achieve optimality

critical points

cost	$\boldsymbol{a}$	$\boldsymbol{b}$
0.8681	0.4751	0.4598
2.9340	−0.3940	−0.5342

The visualization is adapted from a figure in Lagauw (2026).



Outline

1 | Multiparameter Eigenvalue Problems
2 | Numerical Solution Algorithms
  a | Block Macaulay Algorithms
  b | Kronecker Algorithms
  c | Nonlinear Iterative Algorithms
3 | Practical Example



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What is a multiparameter eigenvalue problem?

A **two-parameter eigenvalue problem** (of degree d) is given by

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \left(\boldsymbol{A}_{00} + \boldsymbol{A}_{10}\lambda_1 + \boldsymbol{A}_{01}\lambda_2 + \cdots + \boldsymbol{A}_{0d}\lambda_2^d\right)\boldsymbol{z} = \boldsymbol{0}.$$

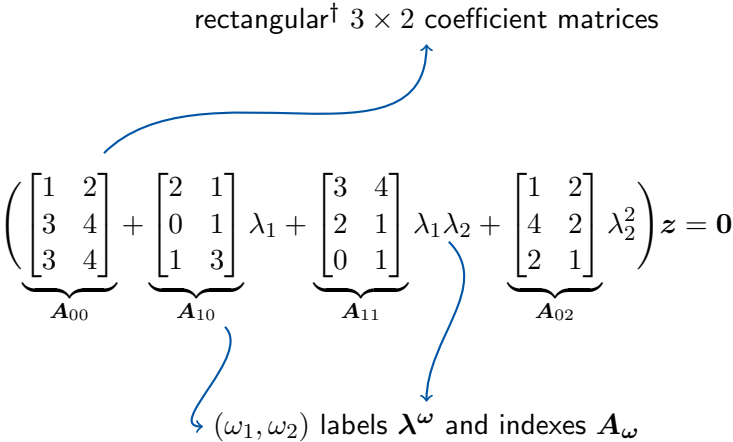
It has rectangular coefficient matrices $\boldsymbol{A}_{\boldsymbol{\omega}} = \boldsymbol{A}_{(\omega_1 \cdots \omega_m)} \in \mathbb{C}^{k \times l}$ with $k \geq l + m - 1$.

The **multiparameter eigenvalue problem (MEP)** $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \boldsymbol{0}$ consists in finding all m-tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$ and corresponding vectors $\boldsymbol{z} \in \mathbb{C}_0^l$, so that (with $\|\boldsymbol{z}\|_2 = 1$)

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} = \sum_{\{\boldsymbol{\omega}\}} \boldsymbol{A}_{\boldsymbol{\omega}} \boldsymbol{\lambda}^{\boldsymbol{\omega}} \boldsymbol{z} = \boldsymbol{0}.$$

Vermeersch and De Moor (2022); Vermeersch (2023); Vermeersch and De Moor (2025)

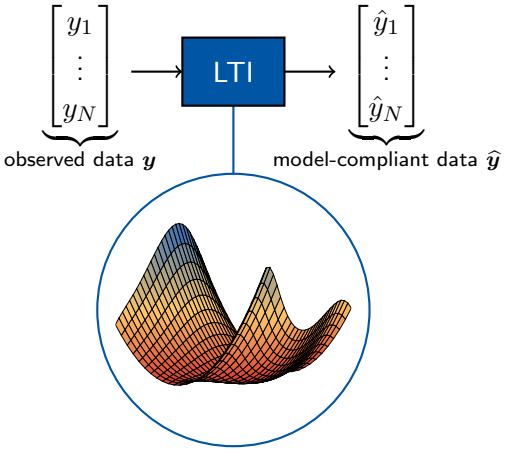
Example of a two-parameter eigenvalue problem



†We consider only rectangular problems in this presentation.

System identification as a potential application

mth-order realization model with N data points ($\boldsymbol{y}$) and m model parameters (α_i)



$$\min_{\boldsymbol{\alpha},\widehat{\boldsymbol{y}}} \|\boldsymbol{y}-\widehat{\boldsymbol{y}}\|_2^2$$

$$\text{subject to } \sum_{i=0}^{m} \alpha_i y_{k+i} = 0$$

$$\downarrow$$

$$p_1(\boldsymbol{\alpha},\widehat{\boldsymbol{y}}) = \cdots = p_s(\boldsymbol{\alpha},\widehat{\boldsymbol{y}}) = 0$$

or

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\alpha})\boldsymbol{z} = \boldsymbol{0}$$

$$\downarrow$$

parameters $\boldsymbol{\alpha}$ and model-compliant data $\widehat{\boldsymbol{y}}$

Lagauw et al. (2026)

Multiparameter eigentuple-finding algorithms

Two types of numerical solution algorithms:

optimization-based algorithms

- nonlinear optimization algorithms
- quickly track one solution path
- potential convergence issues and depend on initial guess
- gradient descent, Gauss–Newton, Newton–Raphson, **homotopy continuation**

matrix-based algorithms

- structured matrices
- good approximations of all solutions
- slower than optimization-based algorithms
- randomized sketching, **Kronecker matrices**, **block Macaulay matrices**

Vermeersch (2023); Vermeersch and De Moor (2025); Hochstenbach et al. (2024) and references therein



Outline

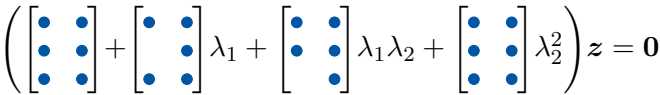
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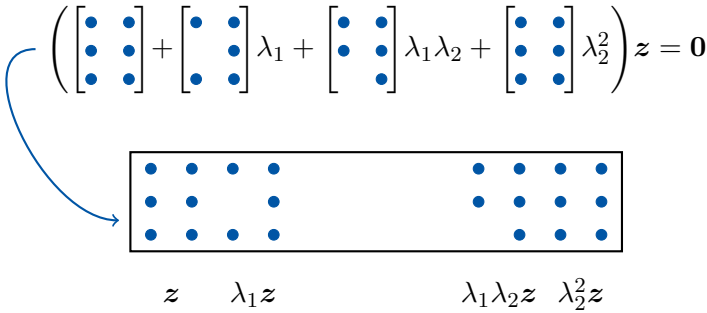
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Constructing the block Macaulay matrix

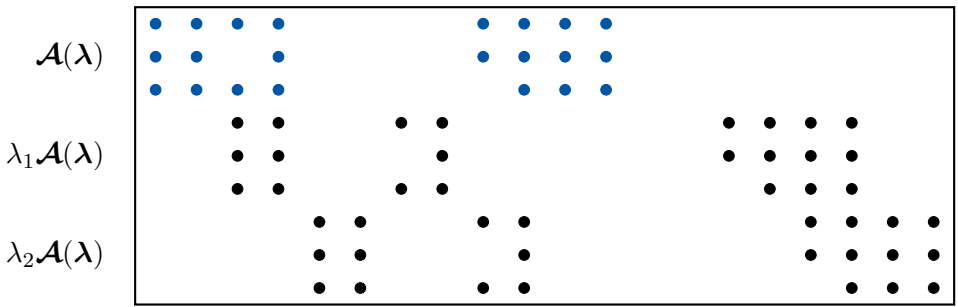


Constructing the block Macaulay matrix



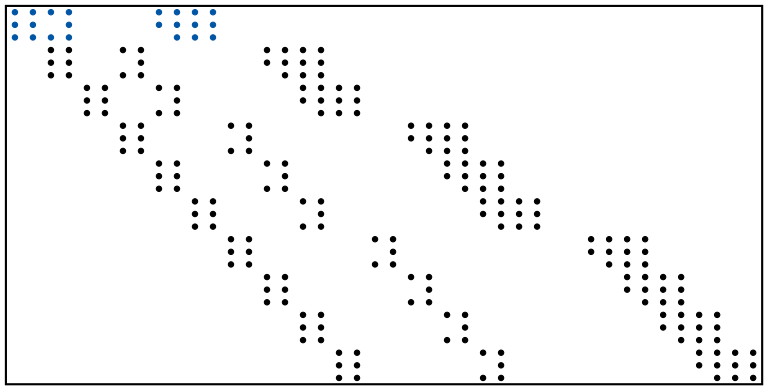
Constructing the block Macaulay matrix

$$\left(\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \lambda_1 + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \lambda_1\lambda_2 + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \lambda_2^2 \right) \boldsymbol{z} = \mathbf{0}$$



Constructing the block Macaulay matrix

$M_d =$



Exploiting the structure of the (right) null space

Every solution yields exactly one vector in the multivariate block Vandermonde basis matrix ($d \geq d^*$):

$$\boldsymbol{V}_d = \left[\begin{array}{cc} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \\ \\ \downarrow \\ \boldsymbol{v}_d|_{(1)} \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \\ \\ \downarrow \\ \boldsymbol{v}_d|_{(2)} \end{array}$$

Monomial multiplicative property

one affine solution

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}}_{\boldsymbol{S}_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1 \boldsymbol{z}|_{(1)} \\ \lambda_2 \boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2 \boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2 \boldsymbol{z}|_{(1)} \\ \lambda_2^2 \boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}} \lambda_1|_{(1)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots \end{bmatrix}}_{\boldsymbol{S}_{\lambda_1}} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1 \boldsymbol{z}|_{(1)} \\ \lambda_2 \boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2 \boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2 \boldsymbol{z}|_{(1)} \\ \lambda_2^2 \boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}}$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

one affine solution

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} \\ \lambda_2\boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}} \lambda_1|_{(1)} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} \\ \lambda_2\boldsymbol{z}|_{(1)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} \end{bmatrix}}_{\boldsymbol{v}_d|_{(1)}}$$

select the linearly independent rows of $\boldsymbol{V}_d$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

multiple affine solutions

$$\underbrace{\boldsymbol{S}_1 \begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d}$$

$$\begin{bmatrix} \lambda_1|_{(1)} & 0 \\ 0 & \lambda_1|_{(2)} \end{bmatrix}$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Monomial multiplicative property

multiple affine solutions

$$\boldsymbol{S}_1 \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d} \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \underbrace{\begin{bmatrix} \boldsymbol{z}|_{(1)} & \boldsymbol{z}|_{(2)} \\ \hline \lambda_1\boldsymbol{z}|_{(1)} & \lambda_1\boldsymbol{z}|_{(2)} \\ \lambda_2\boldsymbol{z}|_{(1)} & \lambda_2\boldsymbol{z}|_{(2)} \\ \hline \lambda_1^2\boldsymbol{z}|_{(1)} & \lambda_1^2\boldsymbol{z}|_{(2)} \\ \lambda_1\lambda_2\boldsymbol{z}|_{(1)} & \lambda_1\lambda_2\boldsymbol{z}|_{(2)} \\ \lambda_2^2\boldsymbol{z}|_{(1)} & \lambda_2^2\boldsymbol{z}|_{(2)} \end{bmatrix}}_{\boldsymbol{V}_d}$$

standard eigenvalue problem!

$$(\boldsymbol{S}_1\boldsymbol{V}_d)\,\boldsymbol{D}_{\lambda_1} = (\boldsymbol{S}_{\lambda_1}\boldsymbol{V}_d)$$

Note that we consider a two-parameter eigenvalue problem in this exposition.

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

- Solution vectors $\boldsymbol{v}_d$ in $\boldsymbol{V}_d$ not known in advanced:

numerical basis matrix $\boldsymbol{Z}_d$ of the right null space ($\boldsymbol{V}_d = \boldsymbol{Z}_d \boldsymbol{T}$)

$$\Downarrow$$

$$(\boldsymbol{S}_1 \boldsymbol{Z}_d)\, \boldsymbol{T} \boldsymbol{D}_{\lambda_1} = (\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d)\, \boldsymbol{T}$$

$$\boldsymbol{T} \boldsymbol{D}_{\lambda_1} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{\lambda_1} \boldsymbol{Z}_d)$$

Null space based solution approach

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{\lambda_1} = \boldsymbol{S}_{\lambda_1} \boldsymbol{V}_d$$

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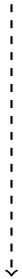
- Possible to **shift with any polynomial** in the variables:

$$\boldsymbol{T} \boldsymbol{D}_{\lambda_2} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{\lambda_2} \boldsymbol{Z}_d)\,,$$

which leads to the same $\boldsymbol{T}$

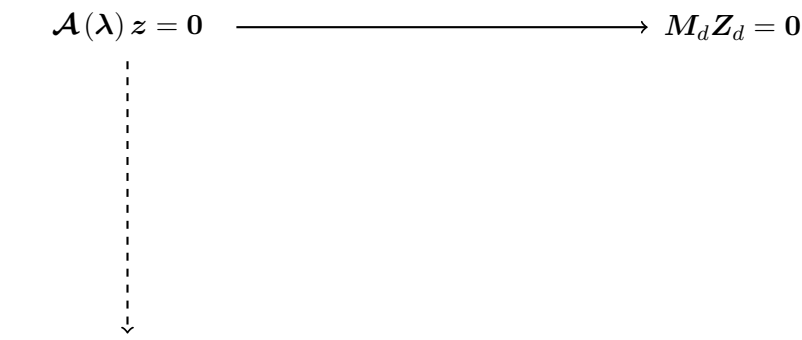
Null space based solution approach

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \mathbf{0}$$

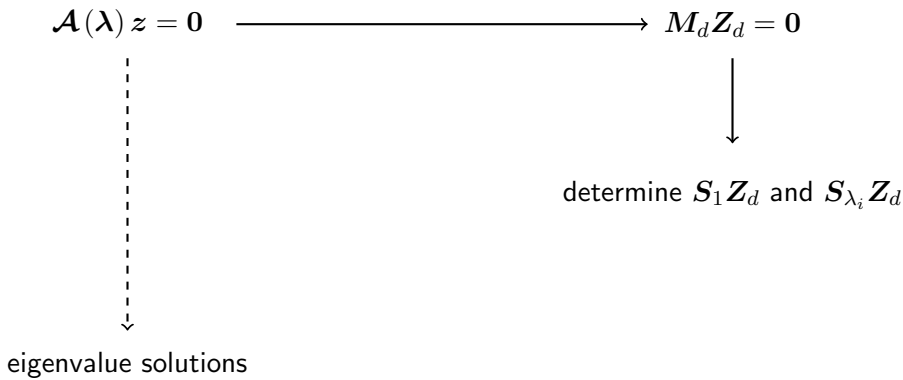


eigenvalue solutions

Null space based solution approach



Null space based solution approach



Vermeersch and De Moor (2022)

Null space based solution approach

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\,\boldsymbol{z} = \boldsymbol{0} \longrightarrow \boldsymbol{M_d Z_d} = \boldsymbol{0}$$

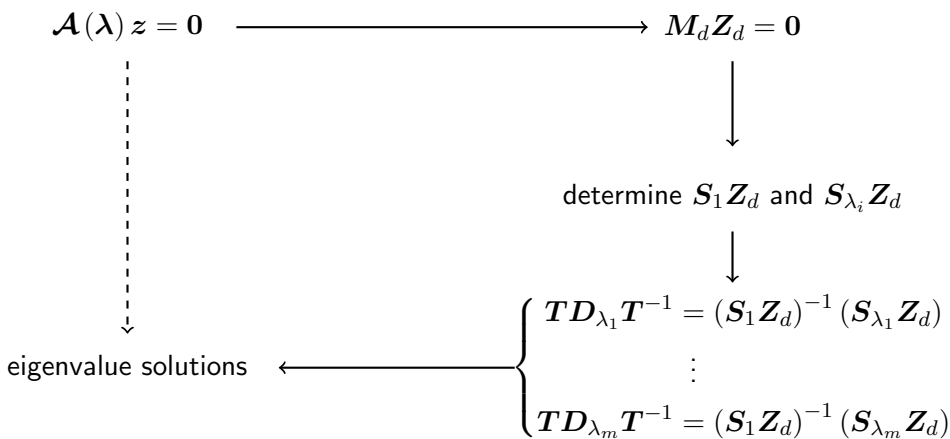
eigenvalue solutions

determine $\boldsymbol{S_1 Z_d}$ and $\boldsymbol{S_{\lambda_i} Z_d}$

$$\begin{cases} \boldsymbol{T D_{\lambda_1} T^{-1}} = \left(\boldsymbol{S_1 Z_d}\right)^{-1} \left(\boldsymbol{S_{\lambda_1} Z_d}\right) \\ \vdots \\ \boldsymbol{T D_{\lambda_m} T^{-1}} = \left(\boldsymbol{S_1 Z_d}\right)^{-1} \left(\boldsymbol{S_{\lambda_m} Z_d}\right) \end{cases}$$

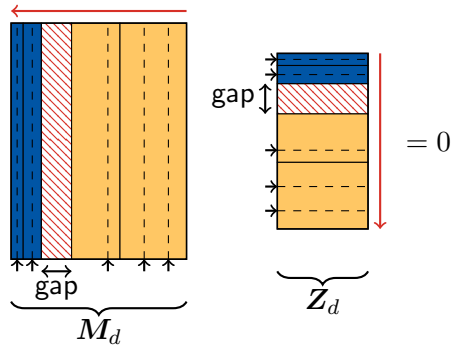
Vermeersch and De Moor (2022)

Null space based solution approach



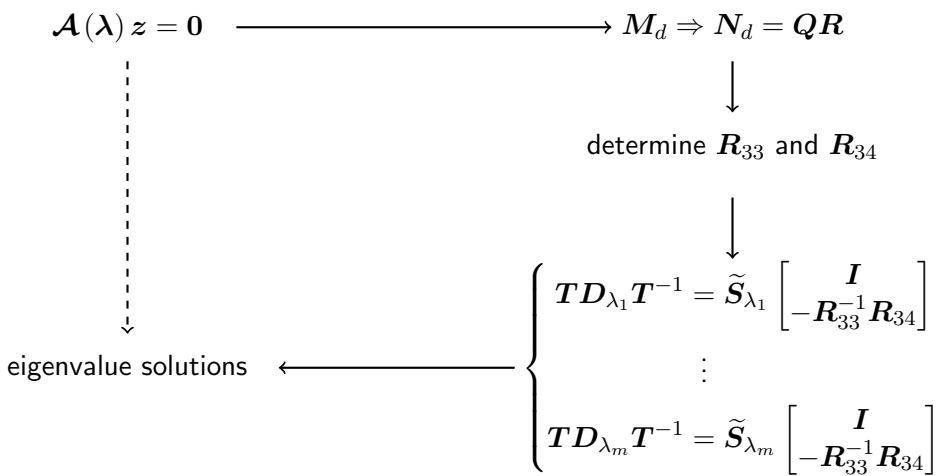
Vermeersch and De Moor (2022)

What about the column space?



The column space and (right) null space of the Macaulay matrix are complementary subspaces: linearly independent rows in the (right) null space are related to linearly dependent columns in the column space

Column space based solution approach



(Vermeersch and De Moor, 2022)



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Idea behind the Kronecker algorithm

After constructing **Kronecker Δ-matrices**,

$$\boldsymbol{\Delta}_0 = \begin{vmatrix} \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_m \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_m \end{vmatrix}_{\otimes}, \boldsymbol{\Delta}_i = \begin{vmatrix} \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{i-1} & \boldsymbol{A}_0 & \boldsymbol{A}_{i+1} & \cdots & \boldsymbol{A}_m \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{i-1} & \boldsymbol{A}_0 & \boldsymbol{A}_{i+1} & \cdots & \boldsymbol{A}_m \end{vmatrix}_{\otimes}$$

the MEP can also be solved via the so-called Kronecker algorithm:

$$\begin{cases} \boldsymbol{\Delta}_0 \lambda_1 \boldsymbol{w} = \boldsymbol{\Delta}_1 \boldsymbol{w}, \\ \vdots \\ \boldsymbol{\Delta}_0 \lambda_m \boldsymbol{w} = \boldsymbol{\Delta}_m \boldsymbol{w}, \end{cases}$$

which is a **joint generalized eigenvalue problem (joint GEP)** with eigenvectors $\boldsymbol{w} = \boldsymbol{z} \times \cdots \times \boldsymbol{z}$.

Idea behind the Kronecker algorithm

The joint GEP has l^m eigenvalues, while the MEP only has $\binom{l+m-1}{l}$ solutions.

A **reduction/projection** $\boldsymbol{\Delta}_i \to \widetilde{\boldsymbol{\Delta}}_i = \boldsymbol{L}\boldsymbol{\Delta}_i\boldsymbol{T}$ resolves this issue!

$$\begin{cases} \widetilde{\boldsymbol{\Delta}}_0\lambda_1\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{\Delta}}_1\widetilde{\boldsymbol{w}} \\ \vdots \\ \widetilde{\boldsymbol{\Delta}}_0\lambda_m\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{\Delta}}_m\widetilde{\boldsymbol{w}} \end{cases}$$

Hochstenbach et al. (2024); Alsubaie (2019)

Similarities with the block Macaulay algorithm

Kronecker algorithm
(implemented in MULTIPAREIG)

$$\Downarrow$$

$$\begin{cases} \boldsymbol{T}\boldsymbol{D}_{\lambda_1}\boldsymbol{T}^{-1} = \widetilde{\boldsymbol{\Delta}}_0^{-1}\widetilde{\boldsymbol{\Delta}}_1 \\ \vdots \\ \boldsymbol{T}\boldsymbol{D}_{\lambda_m}\boldsymbol{T}^{-1} = \widetilde{\boldsymbol{\Delta}}_0^{-1}\widetilde{\boldsymbol{\Delta}}_m \end{cases}$$

block Macaulay algorithm
(implemented in MACAULAYLAB)

$$\Downarrow$$

$$\begin{cases} \boldsymbol{T}\boldsymbol{D}_{\lambda_1}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_1}\boldsymbol{Z}_d) \\ \vdots \\ \boldsymbol{T}\boldsymbol{D}_{\lambda_m}\boldsymbol{T}^{-1} = (\boldsymbol{S}_1\boldsymbol{Z}_d)^{-1}(\boldsymbol{S}_{\lambda_m}\boldsymbol{Z}_d) \end{cases}$$

Both algorithms solve the (linear) MEP via a joint GEP

- Link between both joint GEPs is still unknown
- Different ways of dealing with solutions at infinity



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Issues with standard iterative techniques

Idea is to solve the nonlinear system

$$\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z}) = \begin{bmatrix} \boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z} \\ \boldsymbol{z}^{\mathrm{H}}\boldsymbol{z} - 1 \end{bmatrix} = \mathbf{0}$$

or minimize the residual vector

$$f(\boldsymbol{\lambda}, \boldsymbol{z}) = \|\boldsymbol{\mathcal{F}}(\boldsymbol{\lambda}, \boldsymbol{z})\| = \|\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z}\| + \left|\boldsymbol{z}^{\mathrm{H}}\boldsymbol{z} - 1\right|.$$

- Efficient iterative techniques (Gauss–Newton, quasi-Newton, etc.)
- Finds only one solution per run
- Depends on initial point and can get stuck in local optima

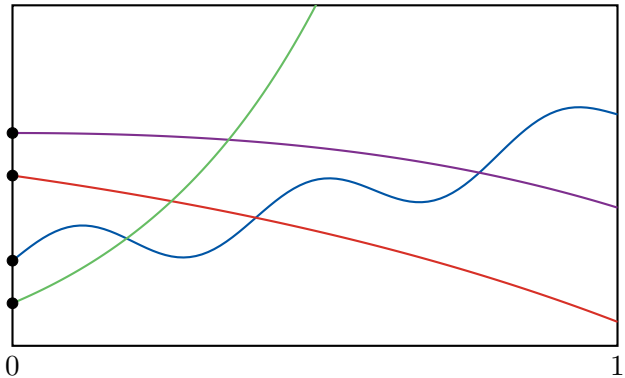
Homotopy continuation methods

- Homotopy continuation methods **start from a "solved" problem** $\boldsymbol{\mathcal{B}}(\boldsymbol{\lambda})\boldsymbol{z} = \mathbf{0}$
  - For (rectangular) MEPs, we **use "banded" start problems**, e.g.,

$$\left(\begin{bmatrix} \bullet & \\ \bullet & \bullet \\ & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \\ & \bullet \end{bmatrix} \lambda_1 + \begin{bmatrix} \bullet & \\ & \bullet \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \mathbf{0}$$

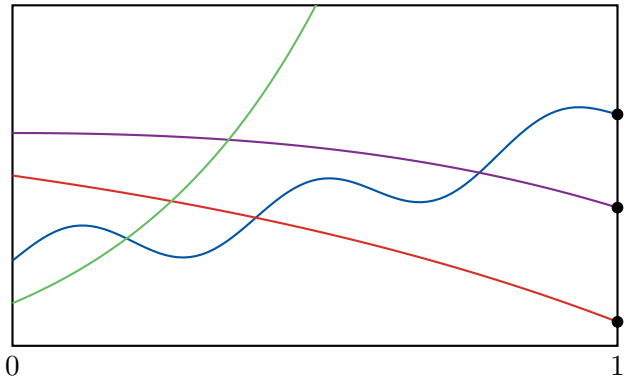
  - Reduced $\widetilde{\boldsymbol{\Delta}}$-matrices can be constructed directly
- It uses the "homotopy" $H(\boldsymbol{\lambda}, \boldsymbol{z}, t) = (1-t)\boldsymbol{\mathcal{B}}(\boldsymbol{\lambda})\boldsymbol{z} + t\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z}$ to compute the unkown solutions of $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})\boldsymbol{z}$
- The "homotopy" hopefully defines **nice paths** $\boldsymbol{\lambda}(t)$ as $t = 0 \to 1$

Homotopy continuation methods



$$\left(\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ & \bullet \end{bmatrix} + \begin{bmatrix} \bullet & \bullet \\ & \bullet \end{bmatrix} \lambda_1 + \begin{bmatrix} \bullet & \bullet \\ & \bullet \end{bmatrix} \lambda_2 \right) \boldsymbol{z} = \mathbf{0}$$

Homotopy continuation methods



$$\left(\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} & \bullet \\ & \bullet \\ \bullet & \bullet \end{bmatrix} \lambda_1 + \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ & \bullet \end{bmatrix} \lambda_1 \lambda_2 \begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix} \lambda_2^2 \right) \boldsymbol{z} = \mathbf{0}$$



Outline

1 | Multiparameter Eigenvalue Problems
2 | Numerical Solution Algorithms
  a | Block Macaulay Algorithms
  b | Kronecker Algorithms
  c | Nonlinear Iterative Algorithms
3 | Practical Example

Tackling the LTI problem from Part I

$$\alpha_0 \hat{y}_k + \alpha_1 \hat{y}_{k+1} + \cdots + \alpha_m \hat{y}_{k+m} = 0$$

- Discrete-time system generates signals $\boldsymbol{y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{N-1} \end{bmatrix} \in \mathbb{R}^N$
- Explain the observed data with a **mathematical model (a) on model-compliant data ($\widehat{y}$)**
- First-order necessary conditions lead to several critical points, one of which corresponds to the global optimal model

$N \setminus m$	**1**	**2**	**3**	**4**
2	1			
3	4			
4	7	1		
5	10	13		
6	13	34	1	
7	16	64	40	
8	19	103	142	1
9	22	151	334	121
10	25	208	643	547

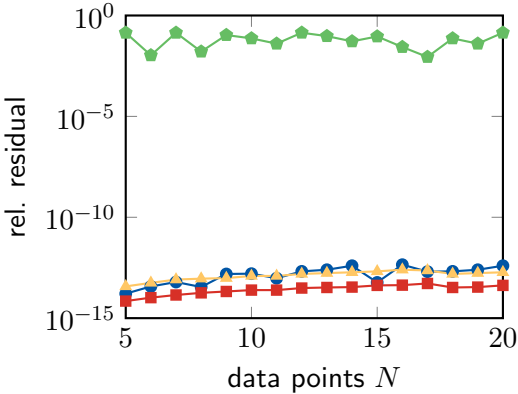
Tackling the LTI problem from Part I

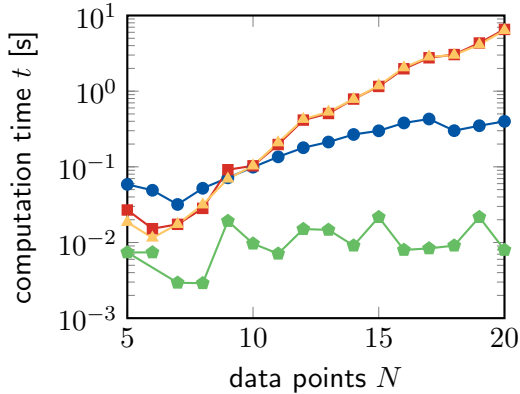
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Tackling the LTI problem from Part I





Legend: (—●—) block Macaulay algorithm, (—■—) Kronecker algorithm, (—▲—) randomized sketching algorithm, (—⬟—) Gauss–Newton algorithm

Can multiparameter eigenvalue problems tackle the curse of dimensionality?

multivariate polynomial system

- $N - r$ polynomials
- $N - r$ variables
- quartic polynomials

multiparameter eigenvalue problem

- $(N - r) \times (N - r - 1)$ matrices
- r (eigenvalue) variables
- cubic problem

for $N = 7$ and $r = 2$, it takes

- 35.7878 seconds
- $352\,716 \times 245\,157$ Macaulay matrix

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- 455×544 block Macaulay matrix

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Any questions?

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