Using Numerical Elimination Theory for Polynomial Optimization with Positive-Dimensional Solution Sets

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Abstract

Optimization problems in engineering, such as system identification and control, often lead to systems of multivariate polynomial equations. In many cases, these equations yield not only isolated solutions but also positive-dimensional solution sets, where infinitely many solutions lie on curves or surfaces. Classical tools from algebraic geometry, such as resultants and Gröbner bases, can be used to study such sets. However, in this poster, we present a numerical elimination approach based on the Macaulay matrix and linear algebra techniques, which provides an alternative way to eliminate variables and extract implicit relations between them. This perspective offers new insights into the structure of positive-dimensional solution sets and their role in polynomial optimization problems. As an illustration, we highlight how this approach can be applied to parametric optimization, a setting relevant for applications such as model predictive control.

Parametric Polynomial Optimization

Many optimization problems involve parameters and can be posed as parametric polynomial optimization problems:

$$\underset{\boldsymbol{x}}{\text{minimize}}\ J(\boldsymbol{x},\boldsymbol{y}),\ \ \boldsymbol{x}\in\mathbb{R}^n,\ \boldsymbol{y}\in\mathcal{Y}\subset\mathbb{R}^p,$$
$$\text{subject to } g_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\mu})=0,\quad i=1,\dots,n_g,\quad \boldsymbol{\mu}\in\mathbb{R}^{n_h}.$$

Here $J(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}[\boldsymbol{x},\boldsymbol{y}]$ is the polynomial cost function, the equality constraints are collected in $\mathbf{g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\mu})=[g_1,\dots,g_{n_g}]^{\mathrm{T}}$, $\boldsymbol{x}$ are the decision variables, $\boldsymbol{\mu}$ are the slack variables, $\boldsymbol{y}$ are the parameters, and $\mathcal{Y}$ is the parameter set.

By introducing Lagrange multipliers $\boldsymbol{\lambda}=[\lambda_1,\dots,\lambda_{n_g}]^{\mathrm{T}}\in\mathbb{R}^{n_g}$, the Langrangian $\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{y}):\mathbb{R}^{n+p+n_g+n_h}\to\mathbb{R}$ can be formulated as:

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{y})=J(\boldsymbol{x},\boldsymbol{y})+\boldsymbol{\lambda}^T\mathbf{g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\mu}),$$

The first-order conditions form a structured system of polynomial equations in $(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu})$ that depend parametrically on $\boldsymbol{y}$:

$$\begin{cases}\nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{y})=0,\\ \nabla_{\boldsymbol{\lambda}}\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{y})=0,\\ \nabla_{\boldsymbol{\mu}}\mathcal{L}(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{y})=0.\end{cases}\tag{1}$$

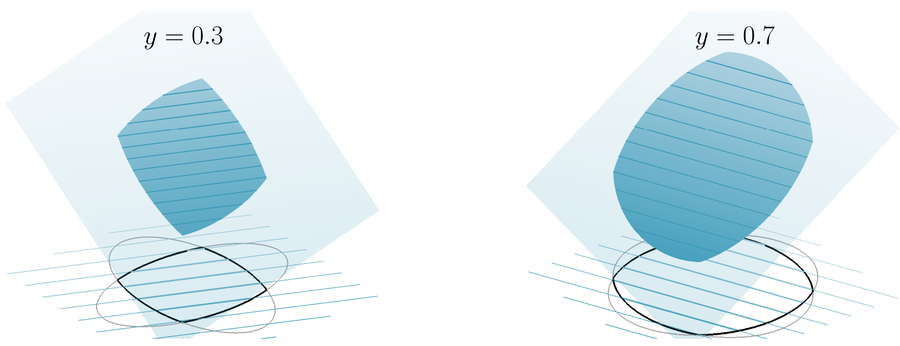
- Find the optimal critical points $\boldsymbol{x}$ as a function of $\boldsymbol{y}$.
- Find the optimal cost as a function of $\boldsymbol{y}$.
- Find the optimal multipliers $\boldsymbol{\lambda}$ as a function of $\boldsymbol{y}$.

Didactic Example

Consider the parametric optimization problem of a bilinear cost function constrained by two cylinders ([1], Example 14.10):

$$\underset{x_1,\,x_2}{\text{minimize}}\ J(x_1,x_2,y)=yx_1+(1-y)x_2,$$
$$\text{subject to}\quad yx_1^2+x_2^2-y\le 0\ \rightarrow\ yx_1^2+x_2^2-y+\mu_1^2=0,$$
$$x_1^2+yx_2^2-y\le 0\ \rightarrow\ x_1^2+yx_2^2-y+\mu_2^2=0.$$

The optimization landscape depends on the parameter y:

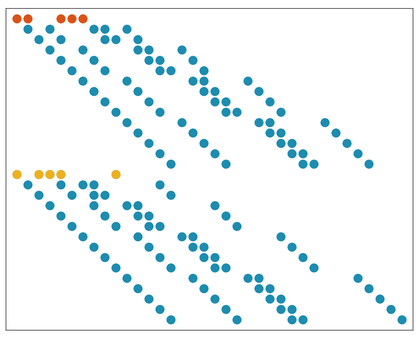


The Macaulay matrix $\boldsymbol{M}$ is built recursively by performing forward multi-shift recursions. Thus, the system of multivariate polynomial equations (1), together with their respective monomial shifts, can be represented as follows [2]:

$$\boldsymbol{M}\boldsymbol{\phi}=0,$$

where $\boldsymbol{\phi}$ represents a multivariate Vandermonde vector:

$$\boldsymbol{\phi}=\begin{bmatrix}1 & x_1 & x_2 & y & \lambda_1 & \lambda_2 & \mu_1 & \mu_2 & x_1^2 & x_1x_2 & x_1y & \cdots\end{bmatrix}^T.$$



Macaulay Matrix

Numerical Elimination Theory

Use numerical elimination [3, 4] to solve parametric polynomial optimization problems with a positive-dimensional solution set.

Identify a vector $\boldsymbol{\xi}$ within the row space of $\boldsymbol{M}$ that possesses a predetermined zero pattern.

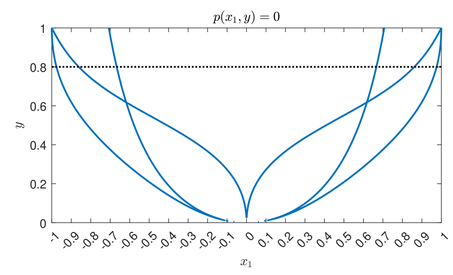
Suppose we want to find the parametric solution of x_1 as a function of the parameter y, then the predetermined zero pattern to be found in the row space of $\boldsymbol{M}$ is

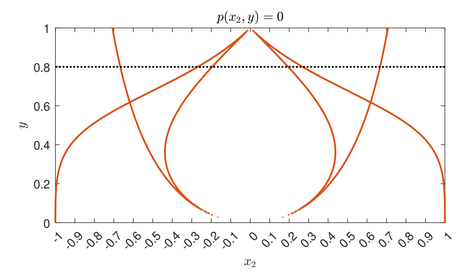
$$\begin{array}{c}\boldsymbol{\xi}=\big[\end{array}\begin{array}{cccccccccccc}1 & x_1 & x_2 & y & \lambda_1 & \lambda_2 & \mu_1 & \mu_2 & x_1^2 & x_1x_2 & x_1y & \cdots\\ \bullet & \bullet & 0 & \bullet & 0 & 0 & 0 & 0 & \bullet & 0 & \bullet & \cdots\end{array}\begin{array}{c}\big]^{\mathrm{T}}.\end{array}$$

where $\bullet$ stands for any arbitrary real number.

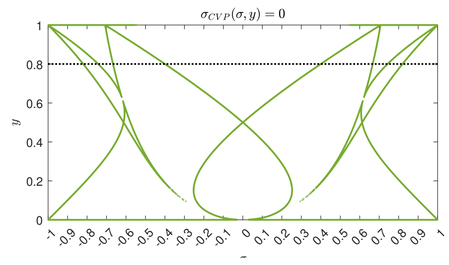
Numerical Results Didactic Example ($y=0.8$)

- Parametric critical points x_1 and x_2:

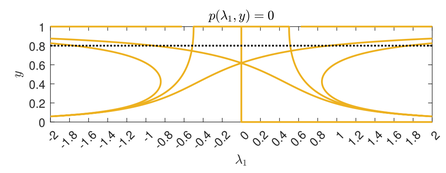


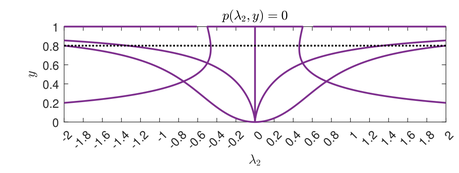


- Parametric critical value (cost):



- Parametric Lagrange multipliers λ_1 and λ_2:





λ_1	λ_2	x_1	x_2	σ
∓1.7500	±2.0000	∓0.6667	±0.6667	∓0.4000
±0.5123	±0.0000	∓0.9759	∓0.1952	∓0.8198
±0.5123	±0.0000	∓0.9759	∓0.1952	∓0.8198
±0.0000	±0.4644	∓0.8614	∓0.2692	∓0.7430
±0.0000	±0.4644	∓0.8614	∓0.2692	∓0.7430
∓0.9167	±1.3333	∓0.6667	∓0.6667	∓0.6667

References

[1] J. B. Lasserre, *An introduction to polynomial and semi-algebraic optimization*. Cambridge University Press, 2015, vol. 52.

[2] P. Dreesen, "Back to the roots - polynomial system solving using linear algebra," Ph.D. dissertation, KU Leuven (Leuven, Belgium), 2013.

[3] Z. Zeng, "A numerical elimination method for polynomial computations," *Theoretical Computer Science*, vol. 409, no. 2, pp. 318–331, 2008.

[4] K. Batselier, P. Dreesen, and B. De Moor, "The geometry of multivariate polynomial division and elimination," *SIAM Journal on Matrix Analysis and Applications*, vol. 34, no. 1, pp. 102–125, 2013.

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