From Canceling Variables to Building Polynomial Models

Talk I: How to Cancel Variables Using the Macaulay Matrix

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Wintercircus, Ghent

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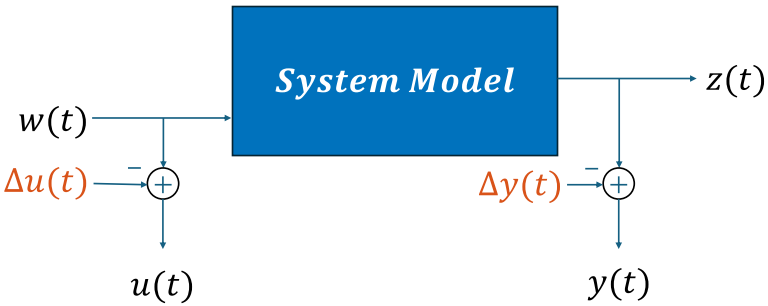
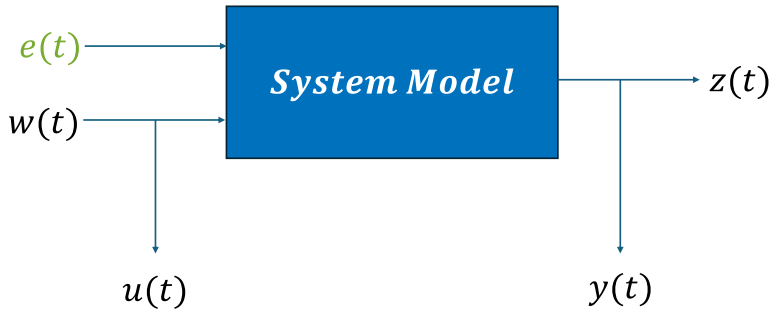
Why Eliminate in Modeling?

Misfit



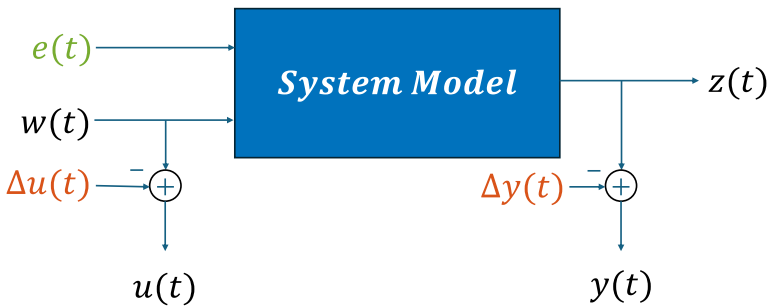
$$\min J_{misfit}$$

Latency



$$\min J_{latency}$$

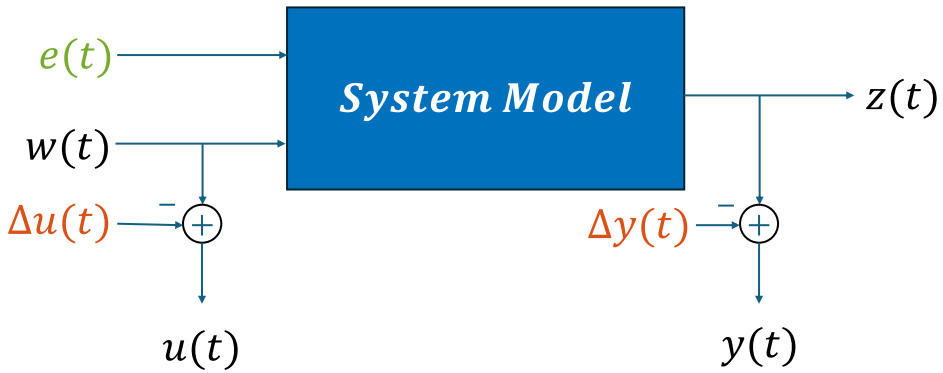
Misfit versus Latency



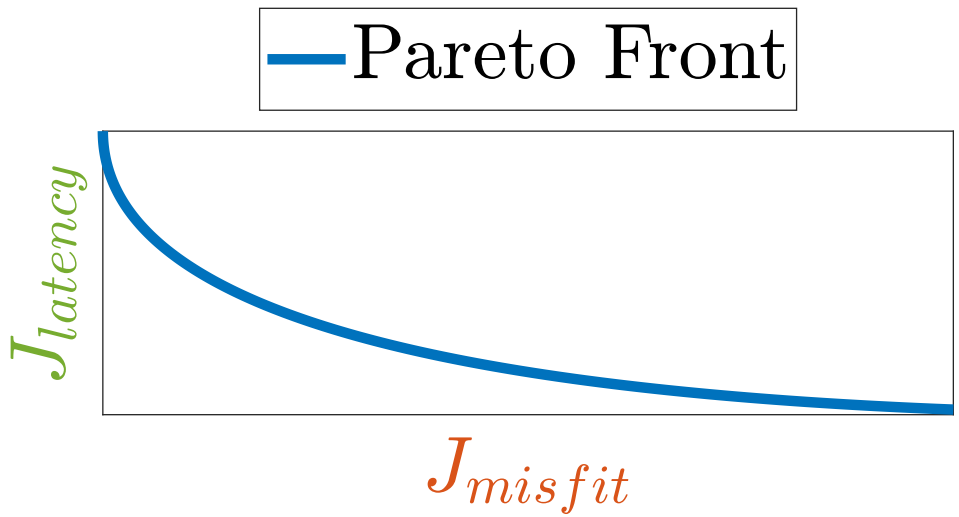
$$\min \alpha J_{misfit} + \beta J_{latency}$$

Why Eliminate in Modeling?

Misfit versus Latency

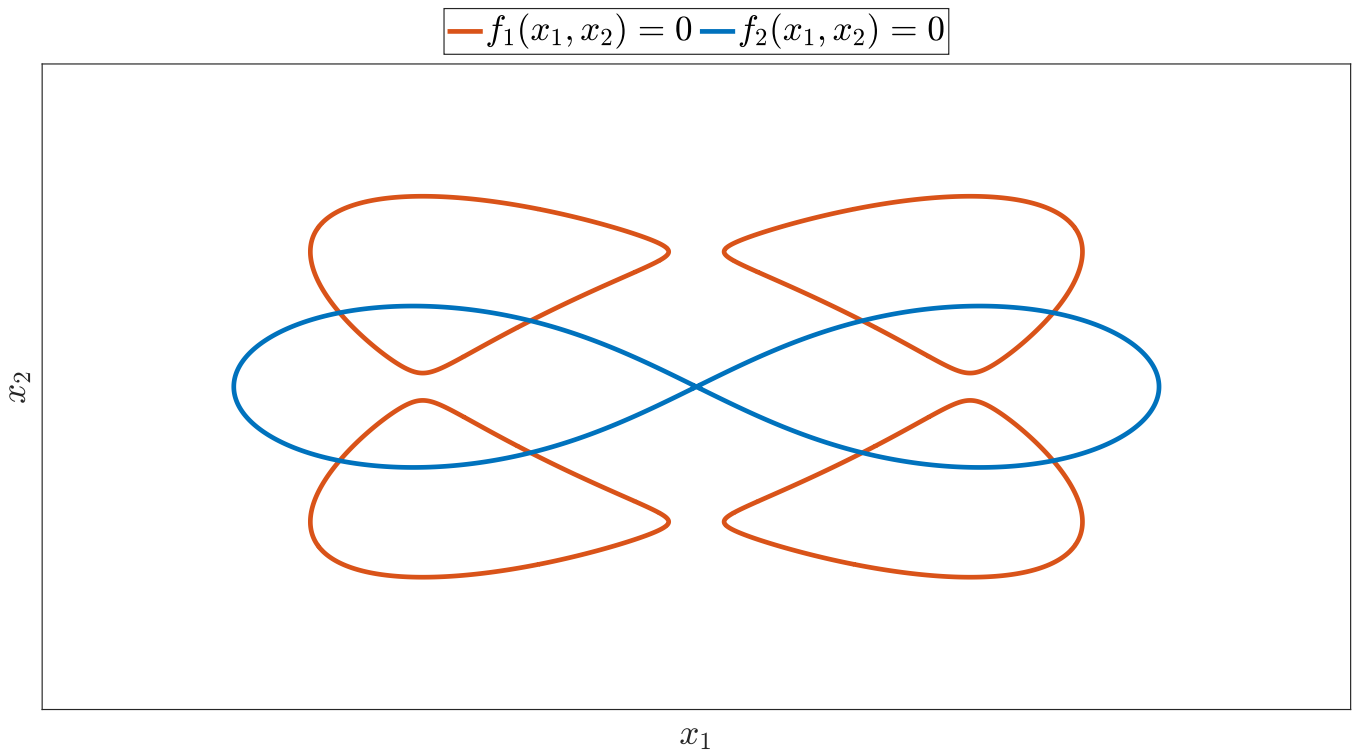


$$\min \alpha J_{misfit} + \beta J_{latency}$$



Bivariate Polynomial System

$$\begin{cases} f_1(x_1, x_2) = x_1^4 + x_2^4 - 1.4\,(x_1^2 + x_2^2) + 0.5 = 0 \\ f_2(x_1, x_2) = (x_1^2 + x_2^2)^2 - 2\,(x_1^2 - x_2^2) = 0 \end{cases}$$



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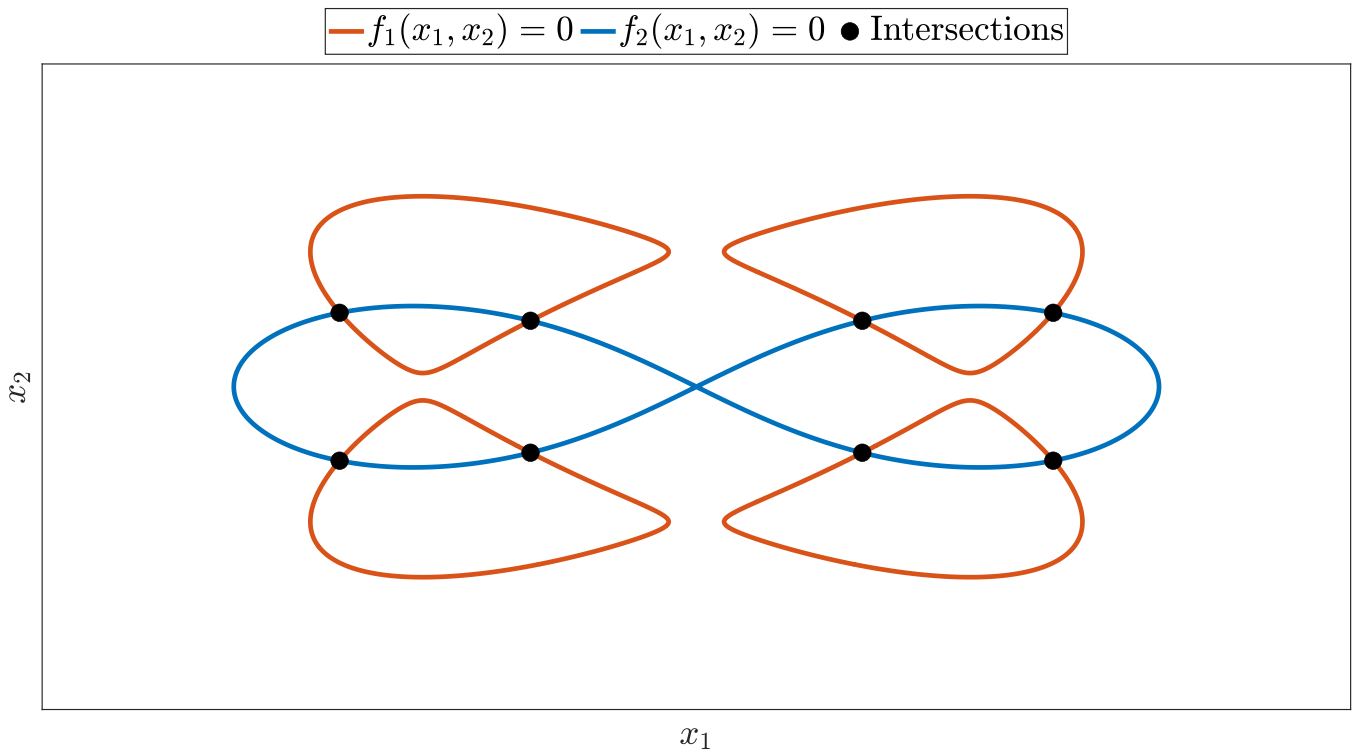

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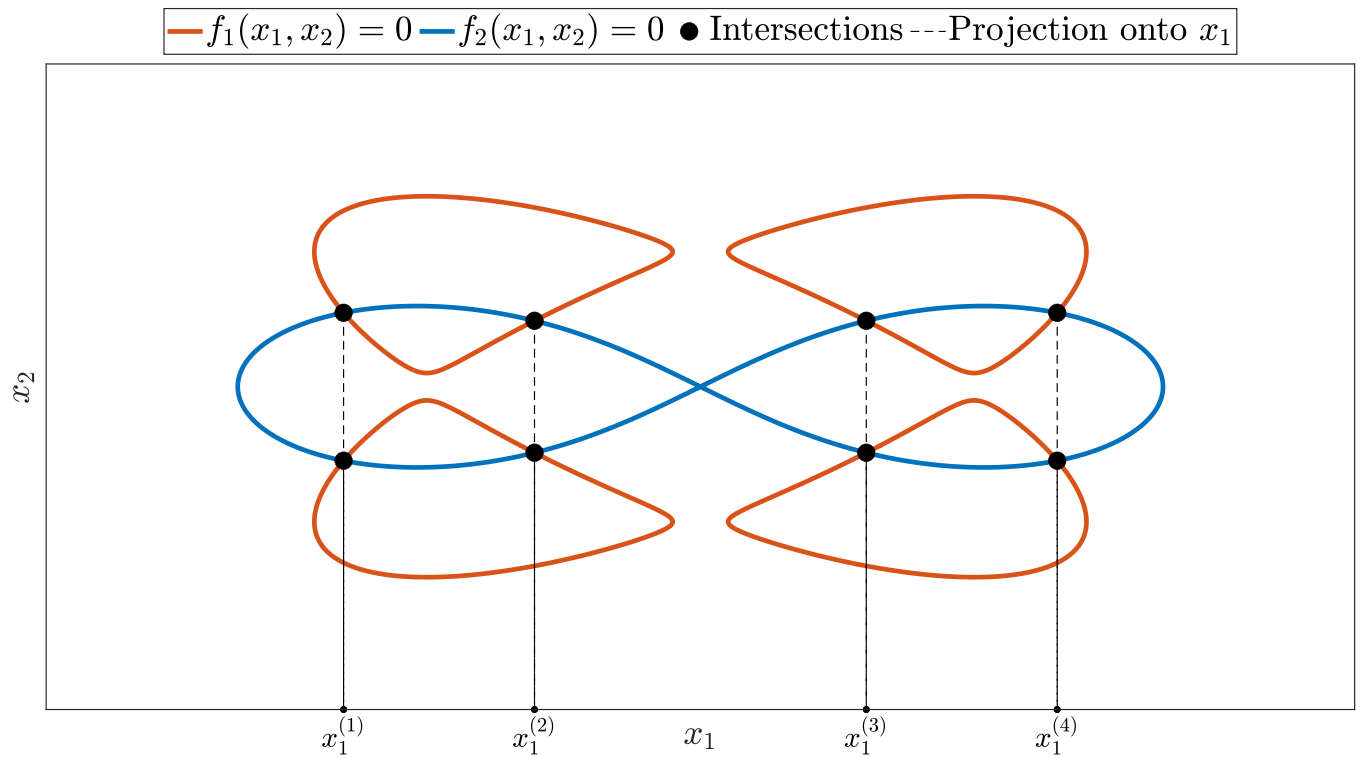


Bivariate Polynomial System

$$\begin{cases} f_1(x_1, x_2) = x_1^4 + x_2^4 - 1.4\,(x_1^2 + x_2^2) + 0.5 = 0 \\ f_2(x_1, x_2) = (x_1^2 + x_2^2)^2 - 2\,(x_1^2 - x_2^2) = 0 \end{cases}$$

$$\begin{aligned} f_{\text{elim}}(x_1) = {} & \left(0.415 + 0.762\,x_1^2 + 0.080\,x_2^2 + 0.170\,x_1^4 - 0.057\,x_2^4\right) f_1(x_1, x_2) + \\ & \left(0.270 + 0.046\,x_1^2 - 0.273\,x_2^2 + 0.057\,x_1^4 - 0.114\,x_1^2 x_2^2 + 0.057\,x_2^4\right) f_2(x_1, x_2) \\ = {} & 0.227\,x_1^8 + 0.455\,x_1^6 - 0.386\,x_1^4 - 0.741\,x_1^2 + 0.207 \;=\; 0 \end{aligned}$$

Key note

Elimination in polynomial systems combines equations using polynomial coefficients to project the problem onto fewer variables.



Bivariate Polynomial System

Key note

Elimination in polynomial systems combines equations using polynomial coefficients to project the problem onto fewer variables.

$$\begin{bmatrix} 0.415 & 0.762 & \cdots & -0.114 & 0.057 \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2) \\ x_1^2 f_1(x_1, x_2) \\ x_2^2 f_1(x_1, x_2) \\ x_1^4 f_1(x_1, x_2) \\ x_2^4 f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ x_1^2 f_2(x_1, x_2) \\ x_2^2 f_2(x_1, x_2) \\ x_1^4 f_2(x_1, x_2) \\ x_1^2 x_2^2 f_2(x_1, x_2) \\ x_2^4 f_2(x_1, x_2) \end{bmatrix} = 0$$



Bivariate Polynomial System

Key note

Elimination in polynomial systems combines equations using polynomial coefficients to project the problem onto fewer variables.

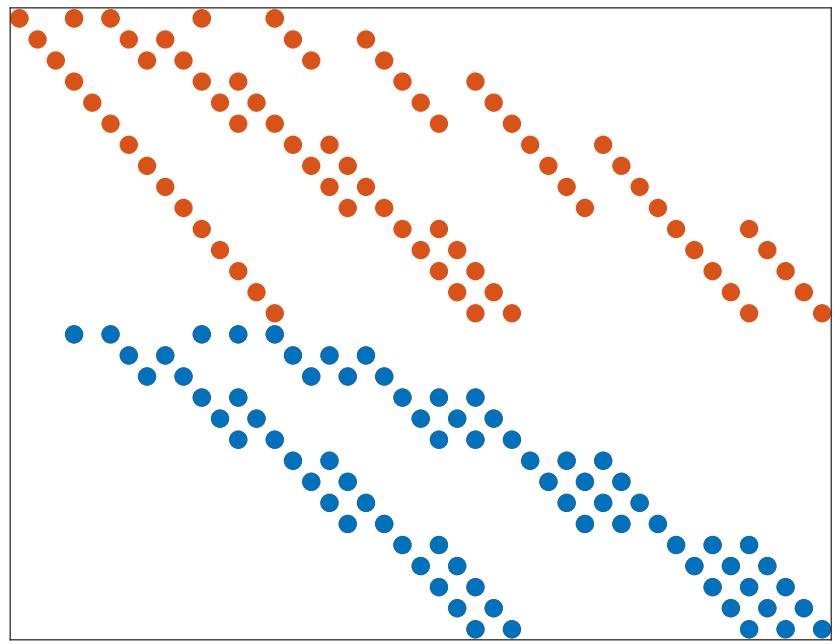
$$\underbrace{\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & \cdots & c_i & c_{i+1} & c_{i+2} & c_{i+3} & \cdots \end{bmatrix}}_{c^T} \begin{bmatrix} f_1(x_1, x_2) \\ x_1 f_1(x_1, x_2) \\ x_2 f_1(x_1, x_2) \\ x_1^2 f_1(x_1, x_2) \\ \vdots \\ f_2(x_1, x_2) \\ x_1 f_2(x_1, x_2) \\ x_2 f_2(x_1, x_2) \\ x_1^2 f_2(x_1, x_2) \\ \vdots \end{bmatrix} = 0$$



Bivariate Polynomial System

$$\begin{cases} f_1(x_1, x_2) = x_1^4 + x_2^4 - 1.4\,(x_1^2 + x_2^2) + 0.5 = 0 \\ f_2(x_1, x_2) = (x_1^2 + x_2^2)^2 - 2\,(x_1^2 - x_2^2) = 0 \end{cases}$$

$$\begin{bmatrix} f_1(x_1,x_2) \\ x_1\, f_1(x_1,x_2) \\ x_2\, f_1(x_1,x_2) \\ x_1^2\, f_1(x_1,x_2) \\ \vdots \\ f_2(x_1,x_2) \\ x_1\, f_2(x_1,x_2) \\ x_2\, f_2(x_1,x_2) \\ x_1^2\, f_2(x_1,x_2) \\ \vdots \end{bmatrix} = \underbrace{\text{Macaulay Matrix}}_{M} \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \\ \vdots \end{bmatrix}}_{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$



Row in Macaulay Matrix with Specified Zero Pattern

Goal

Identify a vector z within the row space of M that possesses a predetermined zero pattern.

Predetermined zero pattern for $n = 2$

In case $x = (x_1, x_2)^T \in \mathbb{R}^2$ and we want to eliminate x_2, the predetermined zero pattern to be found in the row space of M is

$$z^T = \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & \cdots \\ [\bullet & \bullet & 0 & \bullet & 0 & 0 & \bullet & 0 & 0 & 0 & \cdots] \end{matrix},$$

where $\bullet$ stands for any arbitrary real number.



Row in Macaulay Matrix with Specified Zero Pattern

Goal

Identify a vector z within the row space of M that possesses a predetermined zero pattern.

Predetermined zero pattern for $n = 3$

In case $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ and we want to eliminate x_3, the predetermined zero pattern to be found in the row space of M is

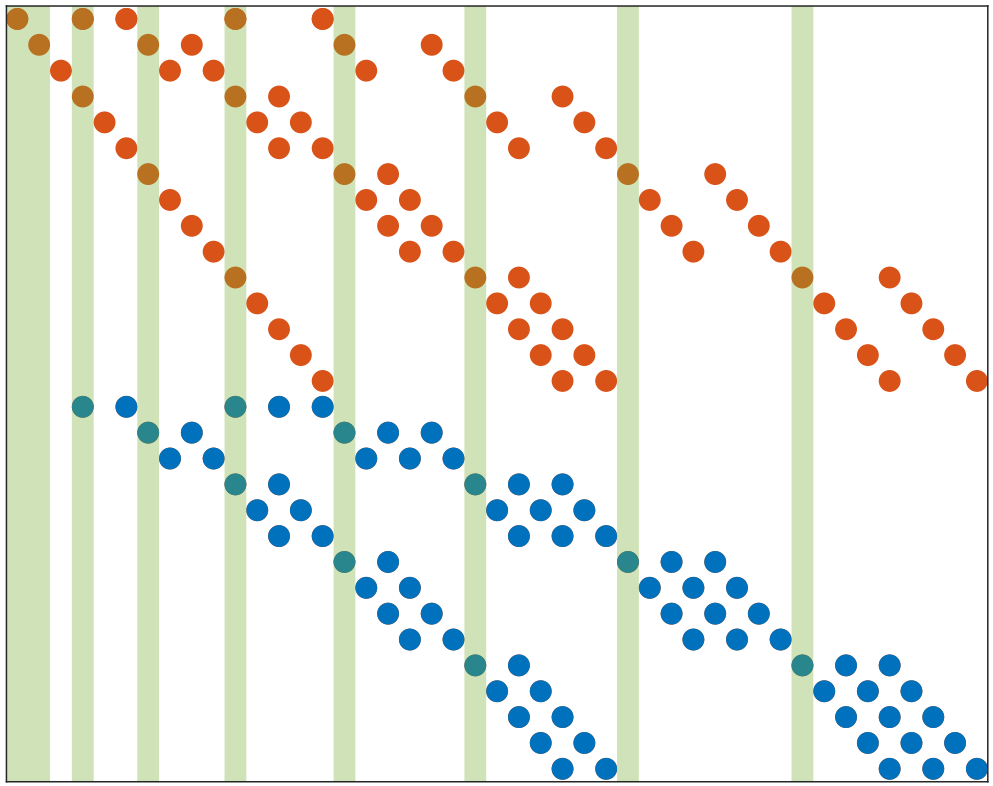
$$z^T = \begin{matrix} 1 & x_1 & x_2 & x_3 & x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 & \cdots \\ [\bullet & \bullet & \bullet & 0 & \bullet & \bullet & 0 & \bullet & 0 & 0 & \cdots] \end{matrix},$$

where $\bullet$ stands for any arbitrary real number.

Zero Pattern

Find a vector with the following zero pattern:

$$z^T = \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & \cdots \\ [\bullet & \bullet & 0 & \bullet & 0 & 0 & \bullet & 0 & 0 & 0 & \cdots] \end{matrix}$$



Macaulay Matrix

Zero Pattern

Find a vector with the following zero pattern:

$$z^T = \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 & \cdots \\ [\bullet & \bullet & 0 & \bullet & 0 & 0 & \bullet & 0 & 0 & 0 & \cdots] \end{array}$$

This vector can be parametrized as

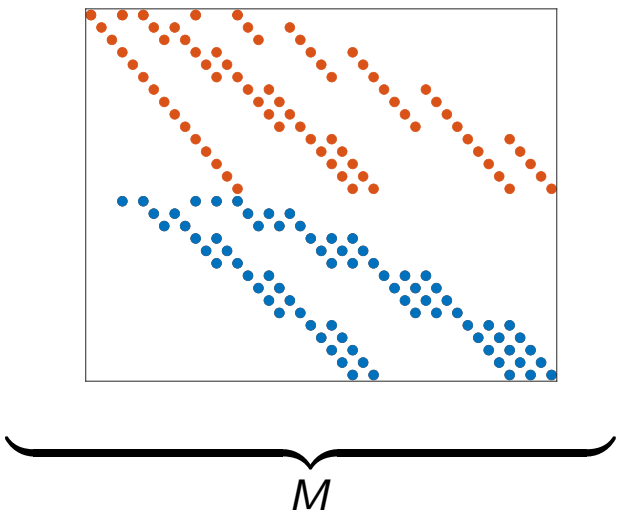
$$z^T = \begin{bmatrix} \bullet & \bullet & \bullet & \cdots \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{E}$$



Elimination as Intersection of Two Row Spaces

Find an intersection between the row space of M and E.

$$t = \dim(\mathrm{row}(M) \cap \mathrm{row}(E)) > 0$$



M

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{E}$$

Eliminant Existence Condition

Find an intersection between the row space of M and E.

$$t = \dim(\mathrm{row}(M) \cap \mathrm{row}(E)) > 0$$

By Graßmann's dimension theorem, the dimension is computed as

$$\begin{aligned}\dim(\mathrm{row}(M) \cap \mathrm{row}(E)) &= \mathrm{rank}(M) + \mathrm{rank}(E) - \mathrm{rank}\begin{bmatrix} E \\ M \end{bmatrix} \\ &= \mathrm{rank}(M) - \mathrm{rank}(\bar{M}).\end{aligned}$$

Eliminant existence condition: $\mathrm{rank}(M) > \mathrm{rank}(\bar{M})$



Eliminant Existence Condition

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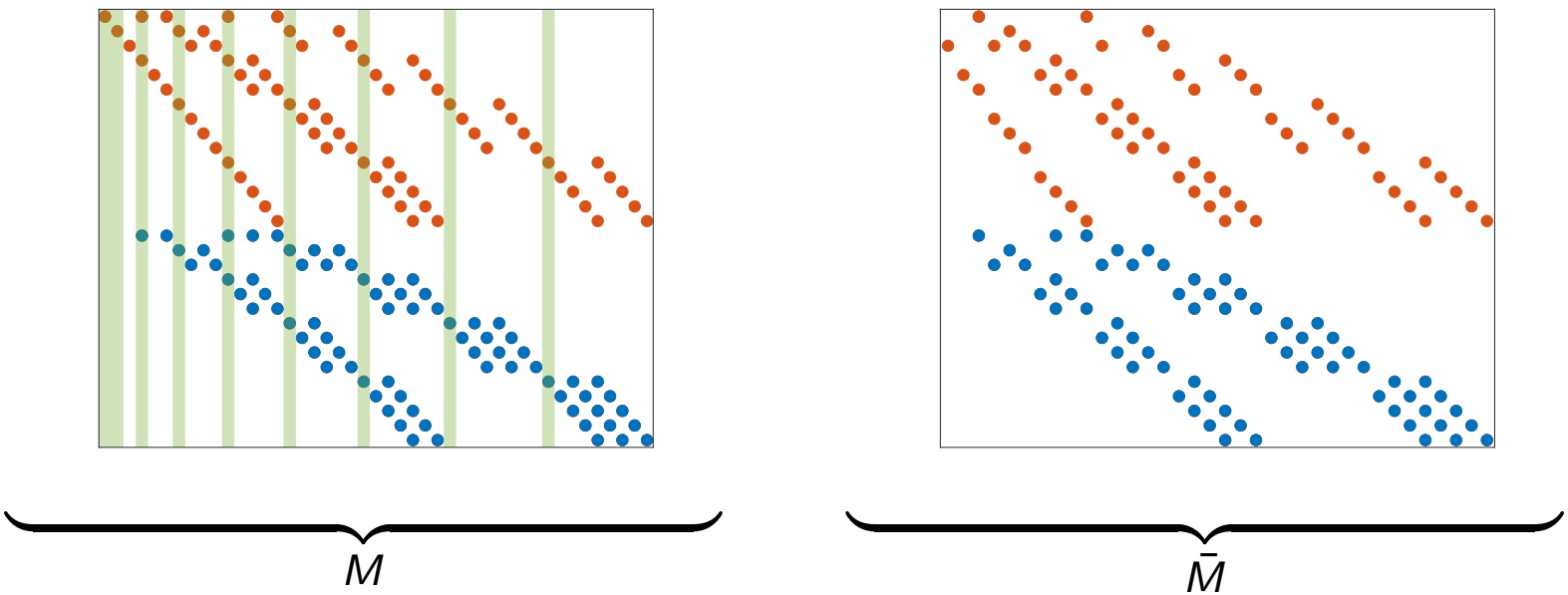
$$\operatorname{rank}(M) > \operatorname{rank}(\bar{M})$$

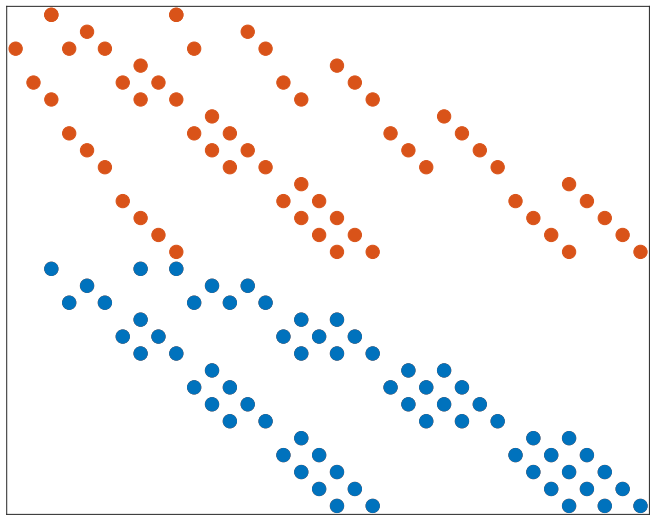


Computing the Eliminant Polynomial

Compute a vector c in the left null space of $\bar{M}$:

$$c^T \bar{M} = 0$$

$$\underbrace{\begin{bmatrix} 0.415 & 0 & 0 & 0.762 & \cdots \end{bmatrix}}_{c^T} \underbrace{\quad}_{\bar{M}} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

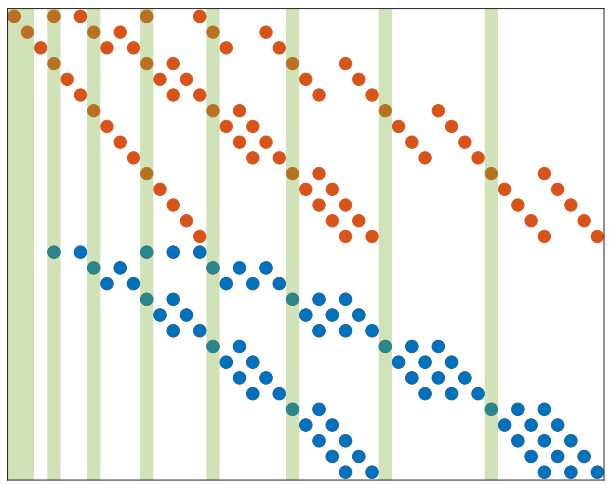


Computing the Eliminant Polynomial

Use c to obtain the vector z satisfying the predetermined zero-pattern:

$$z^T = c^T M$$

$$\underbrace{\begin{bmatrix} 0.415 & 0 & 0 & 0.762 & \cdots \end{bmatrix}}_{c^T} \underbrace{\quad}_{M} = \underbrace{\begin{bmatrix} 0.207 & 0 & 0 & -0.741 & \cdots \end{bmatrix}}_{z^T}$$



Computing the Eliminant Polynomial

Use c to obtain the vector z satisfying the predetermined zero-pattern:

$$z^T = c^T M$$

$$f_{\text{elim}}(x_1) = \underbrace{\begin{bmatrix} 0.207 & 0 & 0 & -0.741 & 0 & \cdots \end{bmatrix}}_{z^T} \underbrace{\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ \vdots \end{bmatrix}}_{v}$$

$$= 0.207 - 0.741\, x_1^2 - 0.386\, x_1^4 + 0.455\, x_1^6 + 0.227 x_1^8 \;=\; 0$$




QUESTIONS ?