Realizing first-order polynomial state-space models

Using a linear algebraic framework

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Problem statement

Given a difference equation describing the algebraic relation between two consecutive model outputs, $k \in \mathbb{N}$

$$\phi = \sum_{\boldsymbol{\omega} \in \Omega} c_{\boldsymbol{\omega}} \boldsymbol{y}_1[k]^{\boldsymbol{\omega}} = 0$$

Here,

- The irreducible difference polynomial $\phi \in \mathbb{R}[\boldsymbol{y}_1[k]]$ over $\mathbb{R}$ in the components of $\boldsymbol{y}_1[k] = \begin{bmatrix} y[k] & y[k+1] \end{bmatrix}^{\mathrm{T}}$
- The output monomials are given as,

$$\boldsymbol{y}_1[k]^{\boldsymbol{\omega}} = \prod_{i=0}^{1} y[k]^{\omega_i}$$

where $\boldsymbol{\omega} = (\omega_0, \omega_1) \in \Omega$ are the multi indices such that $\Omega = \{\boldsymbol{\omega} \in \mathbb{N}^2 : c_{\boldsymbol{\omega}} \in \mathbb{R} \setminus \{0\}\}$

Find the corresponding discrete-time, autonomous, time-invariant single output polynomial state-space model

$$\Sigma : \begin{cases} x[k+1] = f(x[k]) \\ y[k] = g(x[k]) \end{cases}$$

Where $f, g \in \mathbb{R}[x[k]]$ are univariate polynomials over $\mathbb{R}$ in the state variable $x[k] \in \mathbb{R}$

Example

Given the difference equation $\phi_{\mathrm{W}} = 0$ defined by an irreducible difference polynomial of degree 6

$$\begin{aligned} \phi_{\mathrm{W}} = {} & 512y[k]^6 + 704y[k]^5 - 192y[k]^4y[k+1] + 600y[k]^4 \\ & - 280y[k]^3y[k+1] + 24y[k]^2y[k+1]^2 + 281y[k]^3 - 191y[k]^2y[k+1] \\ & + 23y[k]y[k+1]^2 - y[k+1]^3 + 80y[k]^2 - 56y[k]y[k+1] \\ & + 12y[k] - 12y[k+1] \end{aligned}$$

Find the corresponding discrete-time, autonomous, time-invariant single output polynomial state-space model,

$$\Sigma_{\mathrm{W}} : \begin{cases} x[k+1] = x[k] + 2x[k]^2 \\ y[k] = x[k] + x[k]^2 + x[k]^3 \end{cases}$$

example environment

Elimination using the model matrix

Consider the reverse problem of state-elimination

$$\Sigma \to \phi$$

The model outputs $\boldsymbol{y}_1[k] = \begin{bmatrix} y[k] & y[k+1] \end{bmatrix}^{\mathrm{T}}$ of Σ can be evaluated at $x[k]$ using $\boldsymbol{y}_1[k] = \begin{bmatrix} g(x[k]) & g(f(x[k])) \end{bmatrix}^{\mathrm{T}}$. The vector representation of $g(x[k]), g(f(x[k]))$ corresponding to Σ_{W} is,

$$\begin{bmatrix} y[k] \\ y[k+1] \end{bmatrix} = \begin{matrix} 1 & x[k] & x[k]^2 & x[k]^3 & x[k]^4 & x[k]^5 & x[k]^6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 14 & 64 & 120 & 80 \end{bmatrix} \boldsymbol{x}[k]^6$$

We denote $\boldsymbol{x}[k]^d = \begin{bmatrix} 1 & x[k] & x[k]^2 & \cdots & x[k]^d \end{bmatrix}^{\mathrm{T}}$ for $d \in \mathbb{N}$

Now consider output monomials for some $\boldsymbol{\alpha} \in \mathbb{N}^2$, that can be evaluated at $x[k]$ as,

$$\boldsymbol{y}_1[k]^{\boldsymbol{\alpha}} = y[k]^{\alpha_0} y[k+1]^{\alpha_1} = g(x[k])^{\alpha_0} g(f(x[k]))^{\alpha_1}.$$

For all output monomials $\boldsymbol{\alpha} \in \mathcal{A}$ where $\mathcal{A} = \{\boldsymbol{\alpha} \in \mathbb{N}^2 : \boldsymbol{\alpha} \leq d\}$, we can write the matrix-vector product,

$$\boldsymbol{y}_1[k]^d = \boldsymbol{\mathcal{M}}_1^d \boldsymbol{x}^{md}$$

Here, $\boldsymbol{y}_1[k]^d = \begin{bmatrix} 1 & y[k] & y[k+1] & y[k]^2 & y[k]y[k+1] & \cdots & y[k+1]^d \end{bmatrix}$, $\boldsymbol{\mathcal{M}}_1^d \in \mathbb{R}^{p \times q}$ is the **model matrix**, whose rows contain the coefficients of the polynomial functions of the output monomials and m is the maximum degree Between $g(x[k])$ and $g(f(x[k]))$

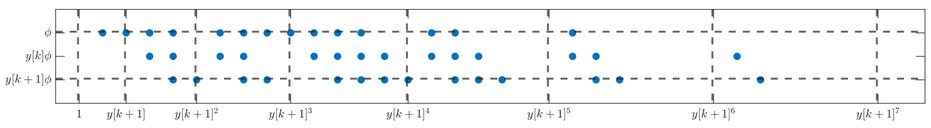
Orthogonal complementarity

- Consider the basis matrix $\boldsymbol{W}_1^d \in \mathbb{R}^{p \times \ell}$, the columns of which span the left null space of the model matrix $\mathcal{L}_{\mathrm{null}}(\boldsymbol{\mathcal{M}}_1^d)$, then we have,

$$\boldsymbol{\phi} = (\boldsymbol{W}_1^d)^{\mathrm{T}} \boldsymbol{y}_1[k]^d = \left((\boldsymbol{W}_1^d)^{\mathrm{T}} \boldsymbol{\mathcal{M}}_1^d \right) \boldsymbol{x}^{md} \\ = \boldsymbol{0}$$

- The columns of $\boldsymbol{W}_1^d = \begin{bmatrix} \phi_1^{\mathrm{T}} & \cdots & \phi_\ell^{\mathrm{T}} \end{bmatrix}$ for all $d \geq d^\circ$ with $\dim\left(\mathcal{L}_{\mathrm{null}}(\boldsymbol{\mathcal{M}}_1^d)\right) = \ell \geq 1$ are the vector representations of $\phi_i \in \mathbb{R}[\boldsymbol{y}_1[k]]$
- For strongly locally observable models [1], there exists a difference equation described by a unique, irreducible first order difference polynomial [2].
- The row space $\mathrm{row}(\boldsymbol{W}_1^d)$ is spanned by the vector representation of ϕ and its monomial multiples $\boldsymbol{y}^{\boldsymbol{\alpha}}\phi$, i.e., the $\mathrm{row}(\boldsymbol{W}_1^d) = \mathrm{row}(\boldsymbol{M}_\phi^d)$
- $\boldsymbol{M}_\phi^d$ is the Macaulay matrix generated using ϕ. We have, $\mathrm{row}\left(\boldsymbol{M}_\phi^d\right) \perp \mathrm{col}\left(\boldsymbol{\mathcal{M}}_1^d\right)$ and $\dim(\mathrm{row}(\boldsymbol{M}_\phi^d)) + \dim(\mathrm{col}(\boldsymbol{\mathcal{M}}_1^d)) = p$

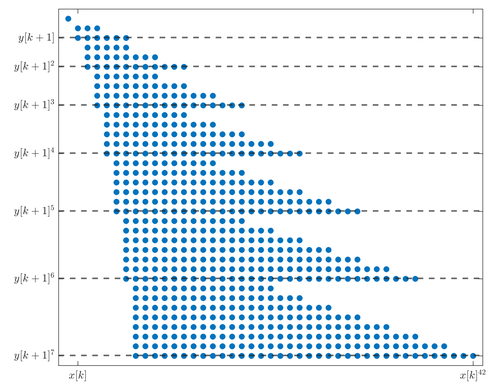
For Σ_{W}, the model matrix $\boldsymbol{\mathcal{M}}_1^7$ of degree 7, and the corresponding Macaulay matrix of degree 7 $\boldsymbol{M}_{\phi_{\mathrm{W}}}^7$ are as follows



$\boldsymbol{M}_{\phi_{\mathrm{W}}}^7$

- Here, $||\boldsymbol{M}_{\phi_{\mathrm{W}}}^7 \boldsymbol{\mathcal{M}}_1^7||_2 \approx 0$
- The size of $\boldsymbol{\mathcal{M}}_1^7$ is 36 x 43, and $\dim(\mathrm{col}(\boldsymbol{\mathcal{M}}_1^7)) = 33$

$\boldsymbol{\mathcal{M}}_1^7$



Realization

Given, $\phi = 0$, where $\phi \in \mathbb{R}[\boldsymbol{y}_1[k]]$, an ansatz for the state-space model is,

$$\Sigma : \begin{cases} x[k+1] = f(x[k]) & = f_1 x[k] + \ldots + f_r x[k]^r \\ y[k] = g(x[k]) & = g_1 x[k] + \ldots + g_s x[k]^s \end{cases}$$

where, $s = \deg_{y[k+1]}(\phi)$ and $r = \deg_{y[k]}(\phi)/s$ (*cf.* [3]).

An ansatz for $y[k+1] = g_1 f_1 x[k] + (g_1 f_2 + g_2 f_1^2) x[k]^2 + \ldots$ and the model matrix follow

For, $\phi_{\mathrm{W}} = 0$ we have $(r, s) = (2, 3)$ and a subsection of the model matrix is,

$$\begin{matrix} & 1 & x[k] & x[k]^2 & x[k]^3 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ y[k] & 0 & g_1 & g_2 & g_3 & \cdots \\ y[k+1] & 0 & f_1 g_1 & g_2 f_1^2 + f_2 g_1 & g_3 f_1^3 + 2 f_2 g_2 f_1 & \cdots \\ y[k]^2 & 0 & 0 & g_1^2 & 2 g_1 g_2 & \cdots \\ y[k]y[k+1] & 0 & 0 & f_1 g_1^2 & g_2 f_1^2 g_1 + g_2 f_1 g_1 + f_2 g_1^2 & \cdots \\ y[k+1]^2 & 0 & 0 & f_1^2 g_1^2 & 2 g_2 f_1^3 g_1 + 2 f_2 f_1 g_1^2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Observe that the columns adhere to a pattern in which the non-zero elements appear (non-zero pattern).

By determining the coefficients $f_i \in \mathbb{R} \, \forall \, i \in \{1, \ldots, r\}$ and $g_i \in \mathbb{R} \, \forall \, i \in \{1, \ldots, s\}$ the state-space model can be found

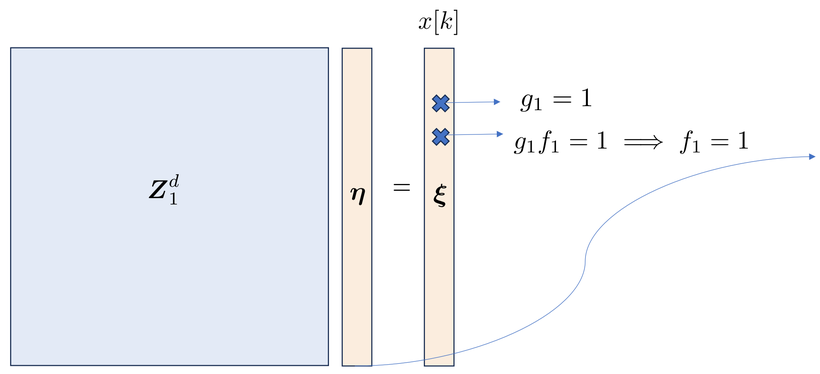
Reconstruction of the model matrix – Part 1

- Given the difference equation $\phi = 0$, construct the Macaulay matrix $\boldsymbol{M}_\phi^d$ of a sufficiently large degree.
- From the orthogonal complementarity property, we know $\mathrm{col}(\boldsymbol{\mathcal{M}}_1^d) = \mathcal{R}_{\mathrm{null}}(\boldsymbol{M}_\phi^d)$ for which the numerical basis matrix is $\boldsymbol{Z}_1^d \in \mathbb{R}^{p \times \ell}$
- Identify vectors in $\mathrm{col}(\boldsymbol{\mathcal{M}}_1^d)$ which adhere to the non-zero pattern of the ansatz for the model matrix.
- Consider the index set $\mathcal{I}(q)$, with the indices of the rows, with non-zero entries corresponding to the column $x[k]^q$ in the ansatz with $q \in \{1, \ldots, md\}$
- Define $\boldsymbol{E} \in \mathbb{R}^{p \times \mathrm{size}(\mathcal{I}(q))}$ as the identity matrix only containing the columns in $\mathcal{I}(q)$
- Find the coefficients (f_i, g_j) in the intersection $\mathrm{col}(\boldsymbol{\mathcal{M}}_1^d) \cap \mathrm{col}(\boldsymbol{E})$. The intersection is found when,

$$\dim\left(\mathrm{col}(\boldsymbol{\mathcal{M}}_1^d) \cap \mathrm{col}(\boldsymbol{E})\right) = \mathrm{rank}(\boldsymbol{Z}_1^d) - \mathrm{rank}(\boldsymbol{Z}_1^d(\mathcal{I}(q)', :)) > 0$$

- Find $\boldsymbol{\eta}$ in $\mathcal{R}_{\mathrm{null}}\left(\boldsymbol{Z}_1^d(\mathcal{I}(q)', :)\right)$, such that $\boldsymbol{Z}_1^d(\mathcal{I}(q)', :)\boldsymbol{\eta} = \boldsymbol{0}$. The columns of the model matrix corresponding to $x[k]^q$ are recovered as $\boldsymbol{\xi} = \boldsymbol{Z}_1^d \boldsymbol{\eta}$.

Construct $\boldsymbol{M}_{\phi_{\mathrm{W}}}^7 \in \mathbb{R}^{3 \times 36}$, and compute $\boldsymbol{Z}_1^d \in \mathbb{R}^{36 \times 33}$. There exists a unique $\boldsymbol{\eta} \in \mathbb{R}^{33}$ such that,



The dimension of intersection can be greater than one, then: $\boldsymbol{\xi}$ is the basis matrix which generates coefficients in the $x[k]^q$ column of the model matrix

For example, the dimension of intersection for the column of $x[k]^2$ is two.

Reconstruction of the model matrix – Part 2

$\boldsymbol{\xi}_2 \in \mathbb{R}^{36 \times 2}$ is the basis matrix of the coefficients in the column $x[k]^2$. Substituting the ansatz column with $(g_1, f_1) = (1, 1)$ the column of $x[k]^2$ is $\begin{bmatrix} 0 & 1 & 1 + f_2/g_2 & 1/g_2 & 1/g_2 & 1/g_2 & \cdots \end{bmatrix}$. Assuming, $g_2 = 1$ we can solve for f_2

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}} = \boldsymbol{\eta}_2 (\begin{bmatrix} 2 & 4 & 5 & 6 \end{bmatrix}, :) \boldsymbol{v}$$

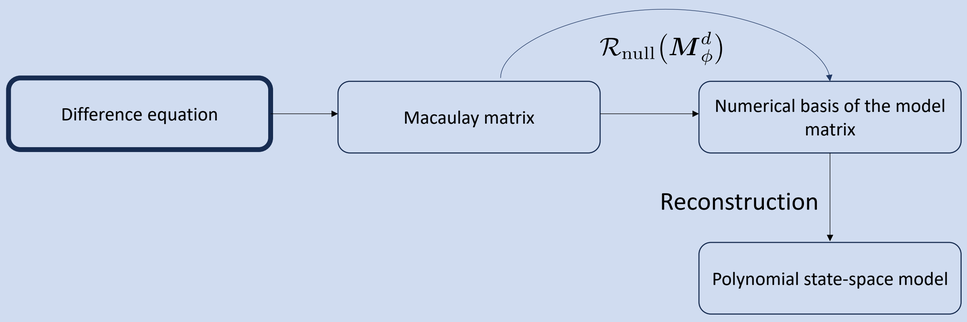
From $\boldsymbol{\eta}_2 \boldsymbol{v}$, we have $f_2 = 2$

Repeating this, with the basis $\boldsymbol{\xi}_3 \in \mathbb{R}^{36 \times 3}$ of the coefficients in the column $x[k]^3$, we recover the polynomial state-space model,

$$\Sigma_{\mathrm{W}} : \begin{cases} x[k+1] = x[k] + 2x[k]^2 \\ y[k] = x[k] + x[k]^2 + x[k]^3 \end{cases}.$$

Realization: summary

Given a difference equation , ***find*** the polynomial state-space model.



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[1] Nijmeijer, H., 1982. Observability of autonomous discrete time non-linear systems: a geometric approach. *International journal of control*, *36*(5), pp.867-874.
[2] De S, De Moor B. 2025. State elimination in polynomial models: A linear algebraic approach. (*In Preparation)*
[3] Sendra, J.R., Winkler, F. and Pérez-Díaz, S., 2008. *Rational algebraic curves: a computer algebra approach*. Springer Berlin Heidelberg.