From Canceling Variables to Building Polynomial Models

Talk II: Realizing First Order Polynomial State-Space Model

Flanders AI Research Day 2025

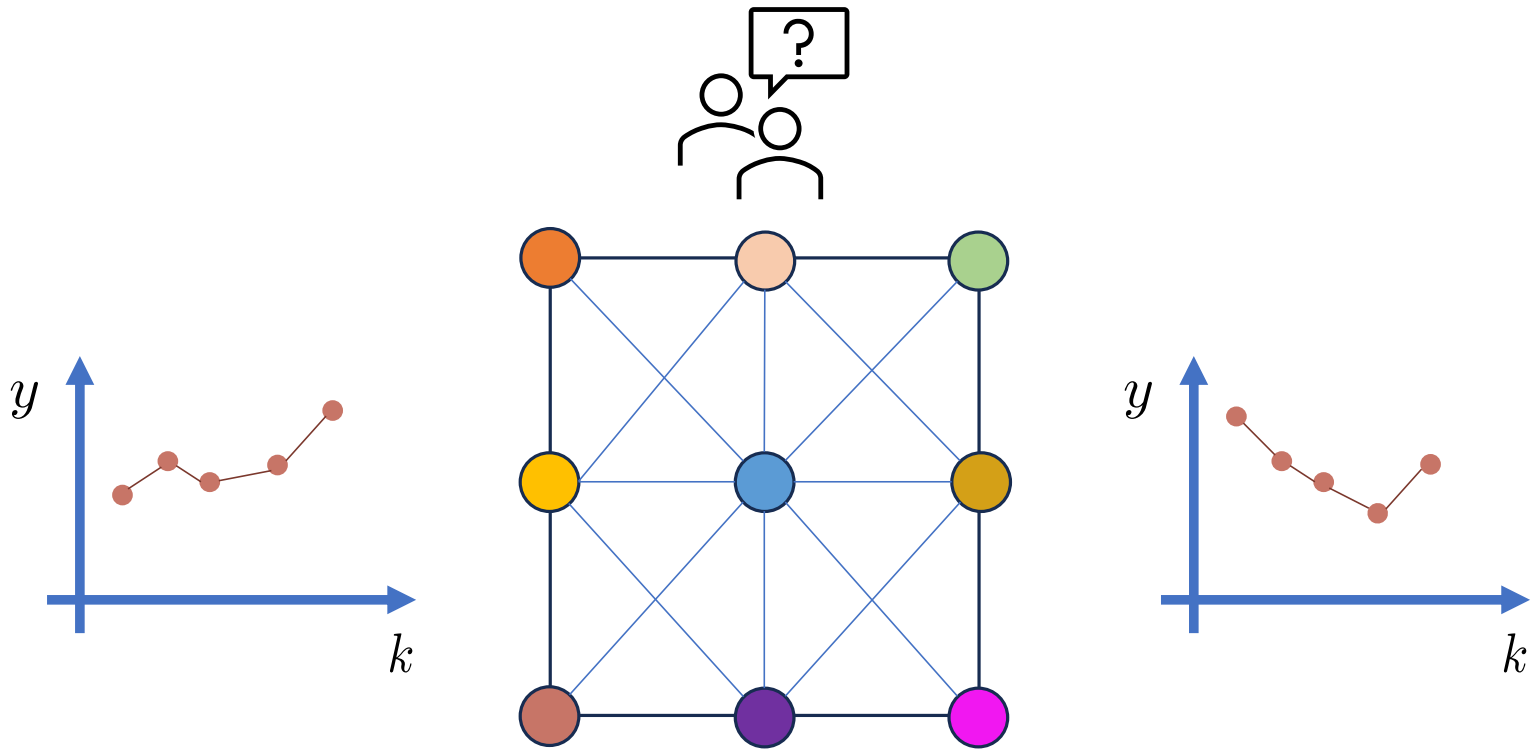
Wintercircus, Ghent

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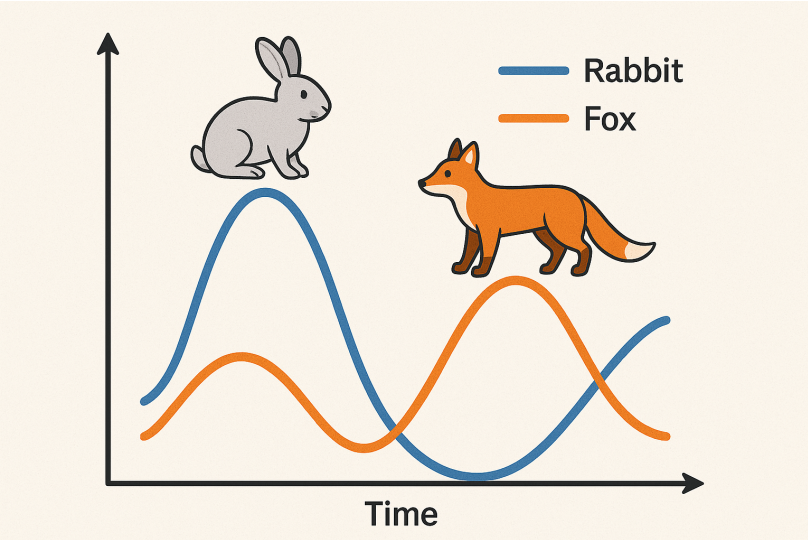
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State of the art



Tackling nonlinearity at the cost of explainability ?

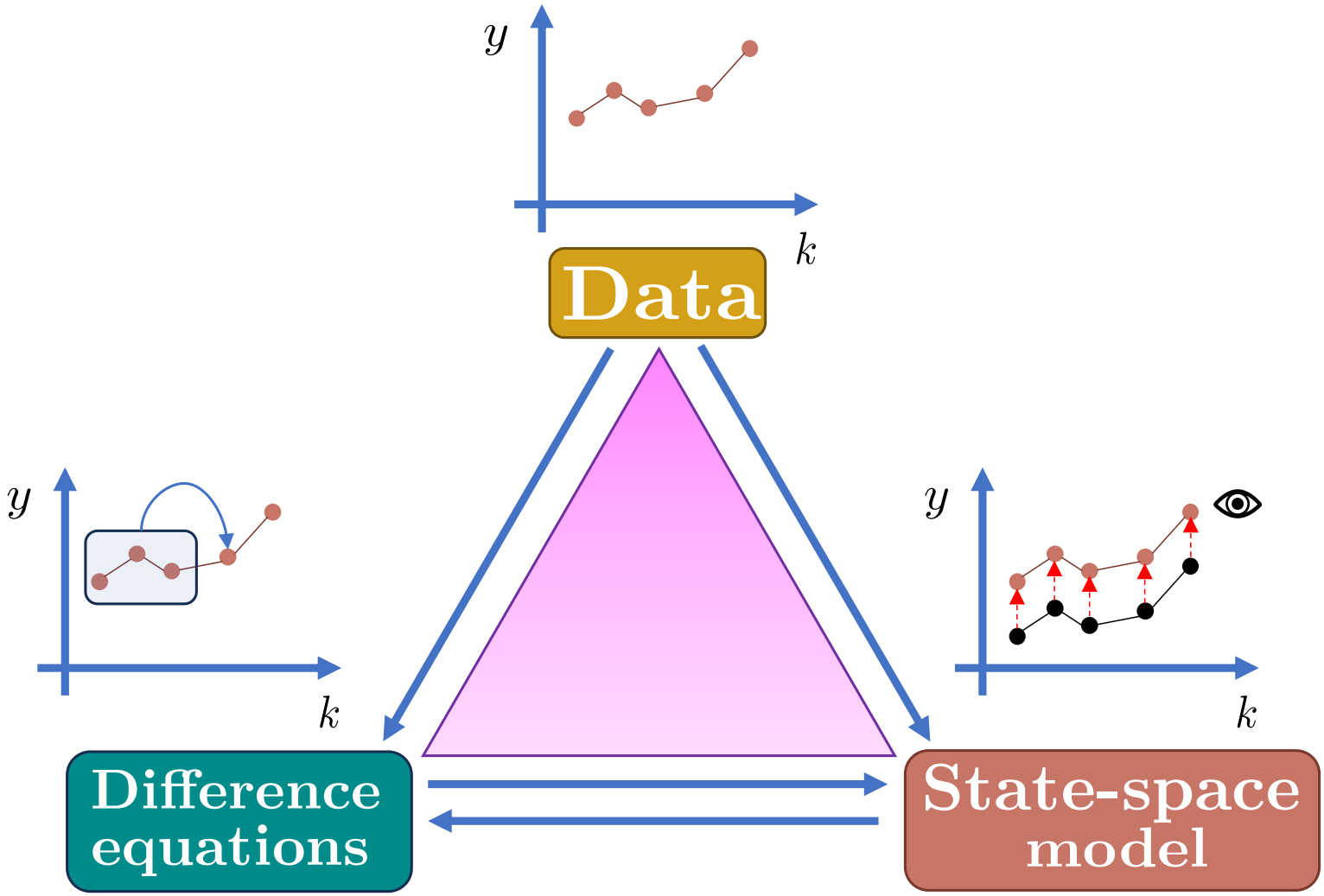
Polynomial Dynamical Systems





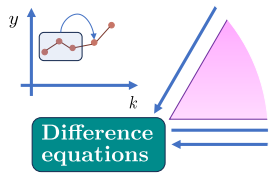
(Left) Lotka–Volterra predator–prey cycles (Right) Lorenz attractor, each arises from a polynomial state-space model.

Model Representations



The three representations of a model top: model trajectories, left: difference equations and right: state-space model

Difference equations



Nonlinear autoregressive models

$y[k+n] = \phi(y[k+n-1], \ldots, y[k])$ where ϕ is a nonlinear function.

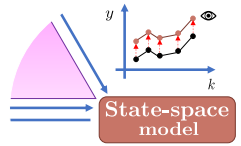
Polynomial difference equations

$\phi(y[k], \ldots, y[k+n]) = 0$ where ϕ is a polynomial in $n+1$ consecutive outputs from k.

Predator population according to the Lotka-Volterra model

$$-7y[k]y[k+1] + 100y[k+2]y[k] - 110y[k+1]^2 + 0.70y[k]^2y[k+1] + y[k]y[k+1]^2 = 0$$

State-space model



Polynomial state-space models

$$\Sigma : \begin{cases} \boldsymbol{x}[k+1] &= \boldsymbol{f}(\boldsymbol{x}[k]) \\ y[k] &= g(\boldsymbol{x}[k]) \end{cases},$$

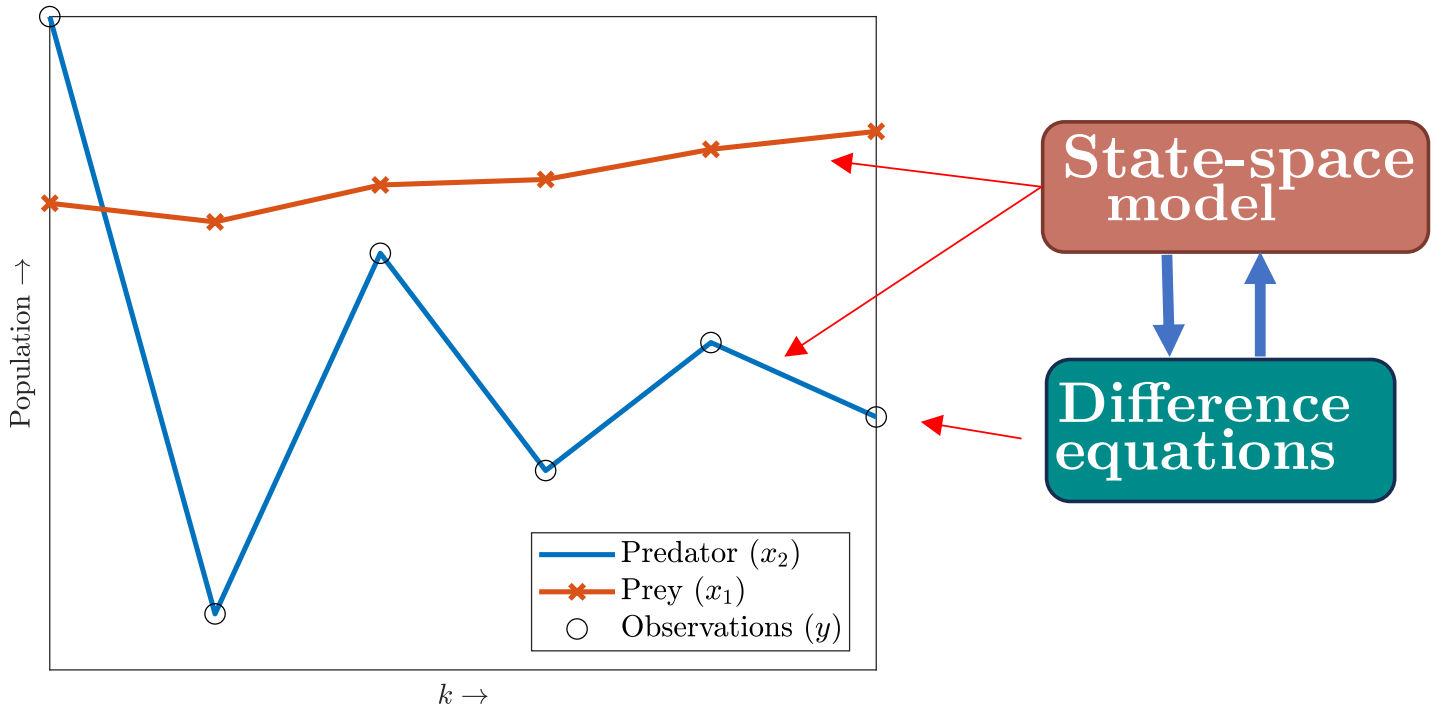
where $\boldsymbol{x}[k] \in \mathbb{R}^{n_f}$, $\boldsymbol{f} = \begin{bmatrix} f_1 & \dots & f_{n_f} \end{bmatrix}$ and f_i, g are polynomials.

Lotka-Volterra predator-prey state-space model

$$\Sigma_{\text{PP}} : \begin{cases} x_1[k+1] = 1.1x_1[k] - 0.01x_1[k]x_2[k] \\ x_2[k+1] = -0.7\,x_2[k] + 0.01x_1[k]x_2[k] \\ y[k] = x_2[k] \end{cases}$$

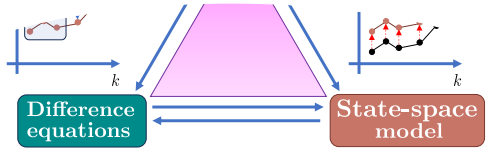


Traversing the edges



Simulation of the Lotka-Volterra model

Traversing the edges



Difference equation

$$
\begin{aligned}
\phi_{\mathrm{W}} = \; & 512\,y[k]^6 + 704\,y[k]^5 \\
& - 192\,y[k]^4 y[k{+}1] + 600\,y[k]^4 \\
& - 280\,y[k]^3 y[k{+}1] + 24\,y[k]^2 y[k{+}1]^2 \\
& + 281\,y[k]^3 - 191\,y[k]^2 y[k{+}1] \\
& + 23\,y[k] y[k{+}1]^2 - y[k{+}1]^3 \\
& + 80\,y[k]^2 - 56\,y[k] y[k{+}1] \\
& + 12\,y[k] - 12\,y[k{+}1] = 0
\end{aligned}
$$

Realization

Elimination

State-space model

$$
\Sigma_{\mathrm{W}} \begin{cases} x[k+1] = x[k] + 2x[k]^2 \\ \quad y[k] = x[k] + x[k]^2 + x[k]^3 \end{cases}
$$

We will recruit linear algebraic tools to move between the different model representations.



Model matrix - I

- Given Σ_W, the model outputs in terms of $x[k]$,

$$\begin{bmatrix} y[k] \\ y[k+1] \end{bmatrix} = \begin{array}{c} \begin{matrix} 1 & x[k] & x[k]^2 & x[k]^3 & x[k]^4 & x[k]^5 & x[k]^6 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 5 & 10 & 12 & 8 \end{bmatrix} \end{array} \boldsymbol{x}[k]^6$$

  where $\boldsymbol{x}[k]^6 = \begin{bmatrix} 1 & x[k] & \cdots x[k]^6 \end{bmatrix}^\mathsf{T}$.

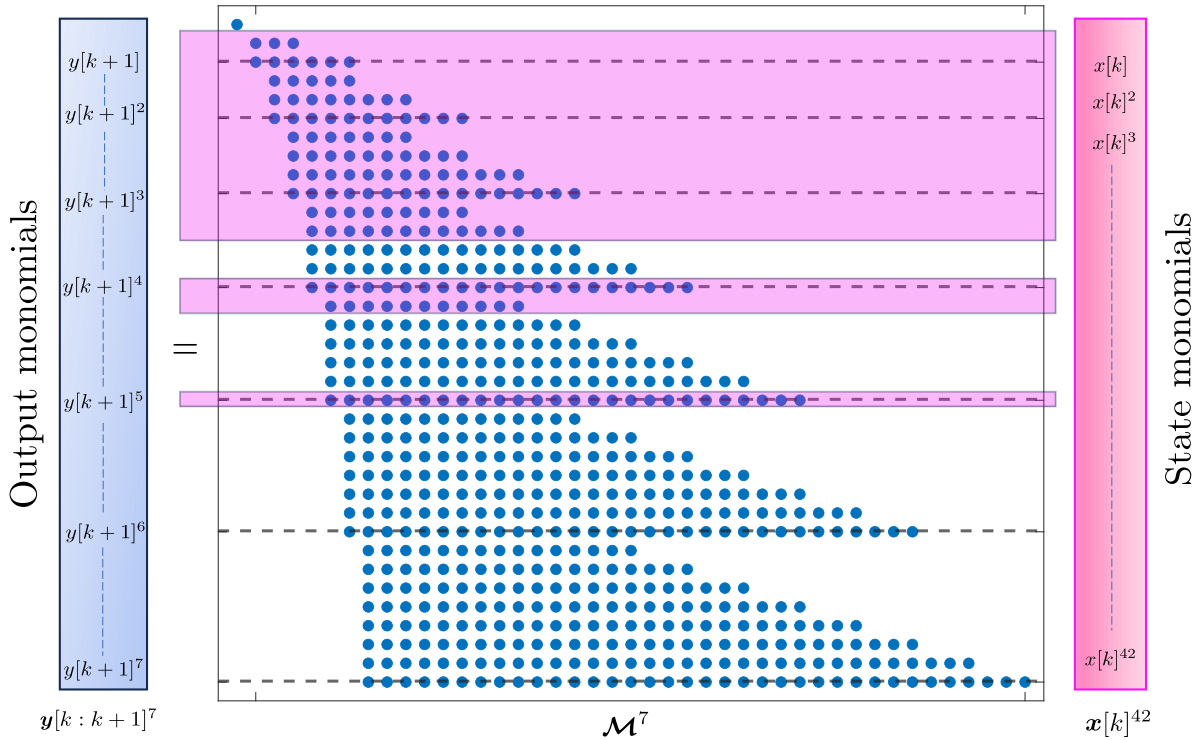
- Output monomials $y[k]^{\alpha_0} y[k+1]^{\alpha_1}$ are also polynomials in $x[k]$,

$$y[k]^{\alpha_0} y[k+1]^{\alpha_1} = (x[k] + x[k]^2 + x[k]^3)^{\alpha_0} (x[k] + 3x[k]^2 + 5x[k]^3 + 10x[k]^4 + 12x[k]^5 + 8x[k]^6)^{\alpha_1}$$



Model matrix - II

Model matrix maps the state monomials to the output monomials, here $\boldsymbol{y}[k:k+1]^7 = \boldsymbol{\mathcal{M}}^7 \boldsymbol{x}[k]^{42}$

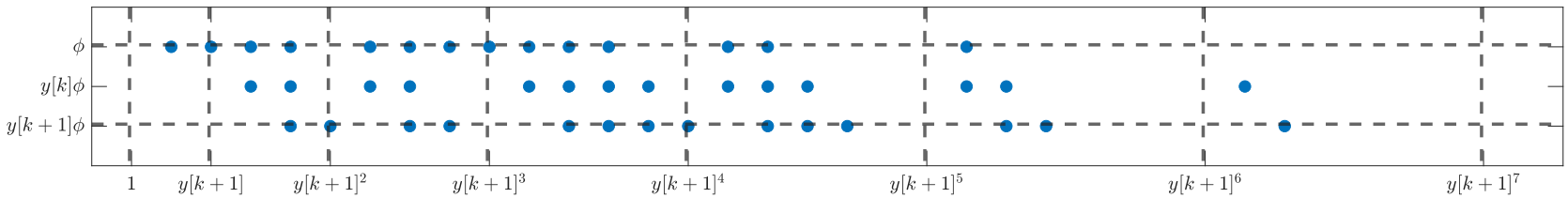


- There exists ϕ_{W}, s.t., $\phi_{\mathsf{W}}^{\mathsf{T}} \boldsymbol{y}[k:k+1]^7 = \left(\phi_{\mathsf{W}}^{\mathsf{T}} \boldsymbol{\mathcal{M}}^7\right) \boldsymbol{x}[k]^{42} = 0.$

difference equation(s) from the left null space of the model matrix!

Nullspace of the Macaulay matrix

- Other linear relations are captured by the Macaulay matrix ,

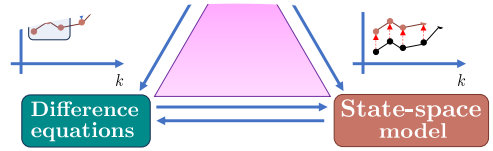


Macaulay matrix $\boldsymbol{M}^7_{\phi_W}$ (degree $= 7$), generated by ϕ_W (degree=6).

- Here $\boldsymbol{M}^7_{\phi_W}\boldsymbol{\mathcal{M}}^7 = \mathbf{0}$.

The right null space of the Macaulay matrix spans the column space of the model matrix !

Traversing the edges



Difference equation

$$
\begin{aligned}
\phi_{\mathrm{W}} = \; & 512\,y[k]^6 + 704\,y[k]^5 \\
& - 192\,y[k]^4 y[k{+}1] + 600\,y[k]^4 \\
& - 280\,y[k]^3 y[k{+}1] + 24\,y[k]^2 y[k{+}1]^2 \\
& + 281\,y[k]^3 - 191\,y[k]^2 y[k{+}1] \\
& + 23\,y[k] y[k{+}1]^2 - y[k{+}1]^3 \\
& + 80\,y[k]^2 - 56\,y[k] y[k{+}1] \\
& + 12\,y[k] - 12\,y[k{+}1] = 0
\end{aligned}
$$

Realization

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State-space model

$$
\Sigma_{\mathrm{W}} \begin{cases} x[k+1] = x[k] + 2x[k]^2 \\ \qquad y[k] = x[k] + x[k]^2 + x[k]^3 \end{cases}
$$

Realization - I



- Given $\phi_W(y[k], y[k+1]) = 0$
- Determine coefficients of the state-space model[1] Σ_W with one state $x[k]$, where

$$\Sigma_W : \begin{cases} x[k+1] &= f(x[k]) = f_1\, x[k] + f_2\, x[k]^2 \\ y[k] &= g(x[k]) = g_1\, x[k] + g_2\, x[k]^2 + g_3\, x[k]^3 \end{cases}$$

[1] The number of state variables and the degree of polynomials in f, g are determined directly from ϕ_W [1].



Realization - II

Consider the ansatz

$$\Sigma_W : \begin{cases} x[k+1] &= f(x[k]) = f_1\, x[k] + f_2\, x[k]^2 \\ y[k] &= g(x[k]) = g_1\, x[k] + g_2\, x[k]^2 + g_3\, x[k]^3 \end{cases}$$

which in turn yields an ansatz for the model matrix,

	1	$x[k]$	$x[k]^2$	$x[k]^3$	$\cdots$
1	1	0	0	0	$\cdots$
$y[k]$	0	g_1	g_2	g_3	$\cdots$
$y[k+1]$	0	$f_1 g_1$	$g_2 f_1^2 + f_2 g_1$	$g_3 f_1^3 + 2 f_2 g_2 f_1$	$\cdots$
$y[k]^2$	0	0	g_1^2	$2 g_1 g_2$	$\cdots$
$y[k]y[k+1]$	0	0	$f_1 g_1^2$	$g_2 f_1^2 g_1 + g_2 f_1 g_1 + f_2 g_1^2$	$\cdots$
$y[k+1]^2$	0	0	$f_1^2 g_1^2$	$2 g_2 f_1^3 g_1 + 2 f_2 f_1 g_1^2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	0	$\ddots$	

Observe the zero patterns in the columns !



Realization - III

- Given ϕ_W
- Construct $\boldsymbol{M}_{\phi}^{7}$.
- Compute the numerical basis matrix $\boldsymbol{Z}$ of the right null space of the Macaulay matrix via the SVD.

Realization by reconstruction

Find vectors in the col($\boldsymbol{Z}$) having the same zero pattern as the columns of the ansatz of the model matrix.



Reconstruction - $x[k]$

Determine the coefficients (f_1, g_1) at the intersection,

$$\text{col}(\boldsymbol{Z}) \cap \text{col}\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.7071 \\ 0.7071 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \overset{x[k]}{\begin{bmatrix} 0 \\ g_1 \\ g_1 f_1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}$$

- Any scalar multiple of a vector in the intersection remains in the intersection.
- Set $g_1 = 1$, then the ansatz for the column of $x[k]$ becomes $\begin{bmatrix} 0 & 1 & f_1 & 0 & \cdots \end{bmatrix}^\mathsf{T}$ then we have, $(g_1, f_1) = (1, 1)$.



Reconstruction - $x[k]^2$

Determine the coefficients (g_2, f_2) at the intersection,

$$\overset{x[k]^2}{\begin{bmatrix} 0 \\ g_2 \\ g_2 f_1^2 + f_2 g_1 \\ g_1^2 \\ f_1 g_1^2 \\ f_1^2 g_1^2 \\ 0 \\ \vdots \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \in \text{col}(\boldsymbol{Z}) \cap \text{col} \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \right)$$

- Set $g_2 = 1$ and we know that $(g_1, f_1) = (1, 1)$
- The ansatz of $x[k]^2$ is,

$$\begin{bmatrix} 0 & 1 & 1 + f_2 & 1 & 1 & 1 & 0 & \cdots \end{bmatrix}^\mathsf{T}$$

- Here, $1 + f_2 = 3$, thus, $f_2 = 2$



Reconstruction - $x[k]^3$



Repeat for $x[k]^3$,

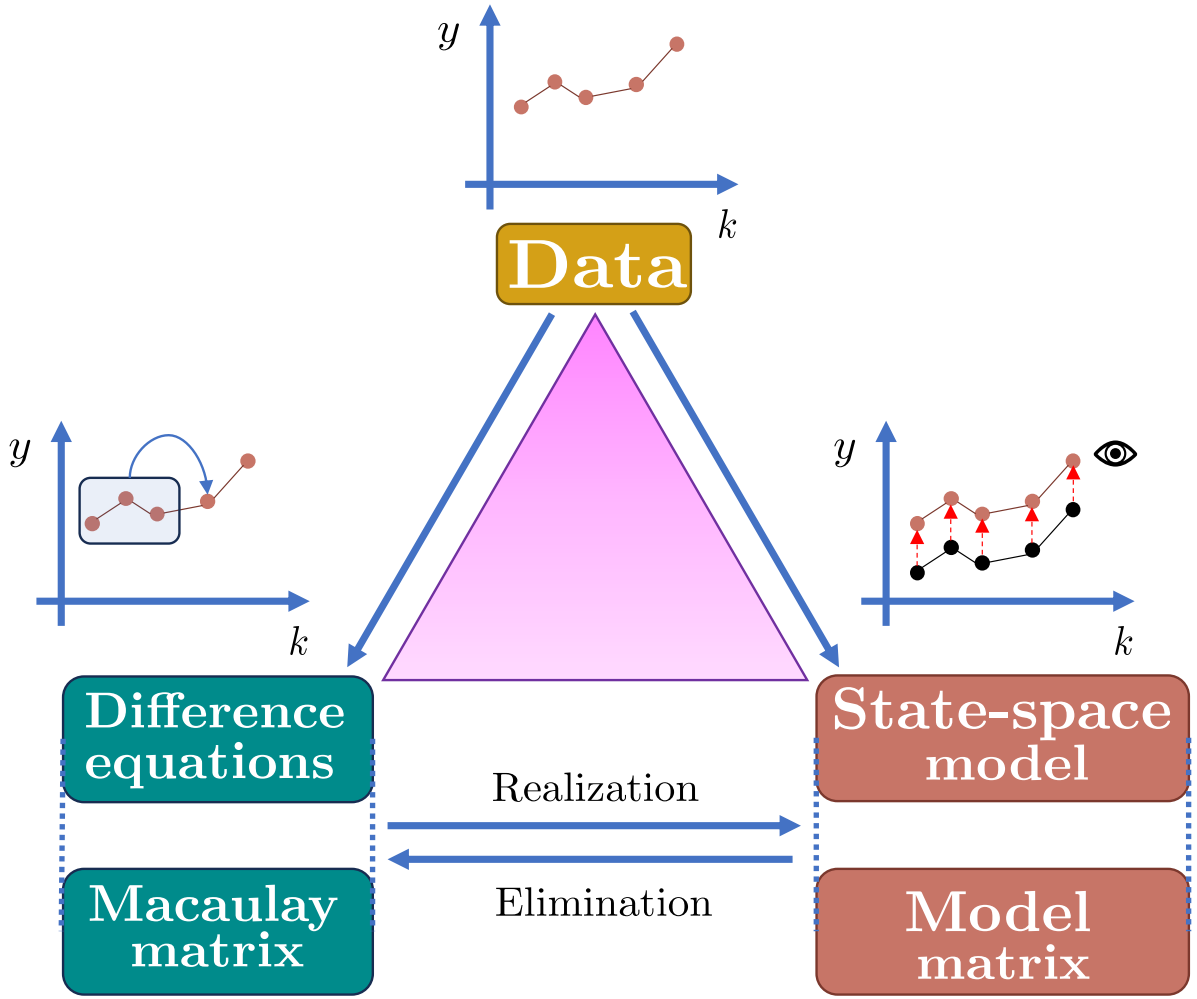
- Determine the intersection of col($\boldsymbol{Z}$) and col($\boldsymbol{E}$).
- From the ansatz for $x[k]^3$, we can determine $g_3 = 1$

We recover the state-space model from the difference equation!

$$\Sigma_W \begin{cases} x[k+1] = x[k] + 2x[k]^2 \\ \quad y[k] = x[k] + x[k]^2 + x[k]^3 \end{cases}$$



Summary



Building polynomial models with linear algebra!

Questions



References I


J Rafael Sendra, Franz Winkler, and Sonia Pérez-Díaz. *Rational algebraic curves: a computer algebra approach*. Springer, 2008.



Appendix: Reconstruction - $x[k]^2$

Determine the coefficients (g_2, f_2) at the intersection,

$$\operatorname{col}(\boldsymbol{Z}) \cap \operatorname{col}\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\right) = \operatorname{col}\left(\underbrace{\begin{bmatrix} 0 & 0 \\ 0.0818 & -0.8326 \\ -0.7172 & -0.4308 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}}_{\xi}\right)$$

- The dimension of intersection is two.
- The coefficients are *not yet* uniquely determined.



Appendix: Reconstruction - $x[k]^2$

Find $\boldsymbol{c} \in \mathbb{R}^2$, such that (g_2, f_2) are uniquely recovered from $\xi \boldsymbol{c}$

$$\underbrace{\begin{bmatrix} 0 & 0 \\ 0.0818 & -0.8326 \\ -0.7172 & -0.4308 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}}_{\xi} \boldsymbol{c} = \begin{matrix} x[k]^2 \\ \begin{bmatrix} 0 \\ g_2 \\ g_2 f_1^2 + f_2 g_1 \\ g_1^2 \\ f_1 g_1^2 \\ f_1^2 g_1^2 \\ \vdots \end{bmatrix} \end{matrix}$$

Appendix: Reconstruction - $x[k]^2$

- Set $g_2 = 1$, and substitute $(g_1, f_1) = (1, 1)$, in the ansatz of $x[k]^2$, we have $\begin{bmatrix} 0 & 1 & 1+f_2 & 1 & 1 & 1 & 0 & \cdots \end{bmatrix}^\mathsf{T}$.
- Find $\boldsymbol{c}$, such that

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \end{bmatrix}}_{\boldsymbol{S}} \begin{bmatrix} 0 & 0 \\ 0.0818 & -0.8326 \\ -0.7172 & -0.4308 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 0.0818 & -0.8326 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\boldsymbol{S}$ selects the rows corresponding to the known components of the ansatz.



Appendix: Reconstruction - $x[k]^2$

- We find that $\boldsymbol{c} = \begin{bmatrix} -3.2685 & -1.5222 \end{bmatrix}^{\mathsf{T}}$

$$\begin{bmatrix} 0 & 0 \\ 0.0818 & -0.8326 \\ -0.7172 & -0.4308 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ -0.3995 & 0.2009 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} -3.2685 \\ -1.5222 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

- Since $1 + f_2 = 3$, we have $f_2 = 2$.

