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The Colored Branch Theorem and Its Applications in Circuit Theory

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Abstract—**The colored branch theorem (Minty 1960 [1]) is a result in graph theory, which essentially says that the existence (resp., nonexistence) of a certain loop immediately implies the nonexistence (resp., existence) of a certain cutset.**

Its relevance and use in circuit theory, however, has only recently been recognized. Since it is expected that many more applications in circuit theory will follow, the theorem is interpreted and proved in a network setting. Many graph-theoretic corollaries are derived, which may facilitate later use. It is illustrated that many results in circuit theory can be simplified or given a simpler proof using this theorem and its corollaries.

I. Introduction

In 1960 Minty obtained a powerful result [1] in graph theory, which is often called "the colored branch theorem." It should not be confused with the problem of coloring the branches such that no two branches incident at the same node have the same color (coloring problem). The colored branch theorem essentially says that for any graph the existence (resp., the nonexistence) of a certain loop immediately implies the nonexistence (resp., existence) of a certain cutset, and vice versa. Like Tellegen's theorem this does not depend on the type of elements or the coupling of elements in a network $\mathcal{N}$; it only depends on the graph $\mathcal{G}$ associated with $\mathcal{N}$.

Although Minty used his theorem first in his algorithm for solving monotone networks [1], [2] its main application was originally found in transportation and network flow problems [3], as well as in matroid theory [4], [5]. In recent years, however, it has been used more and more in circuit theory [6]–[12] and it seems likely that many more applications will follow.

Some of the reasons why this theorem is not so well known among electrical engineers are: (1) The theorem is usually formulated only in a graph-theoretic framework and therefore it is difficult to adapt to circuit problems; (2) The use of this theorem in several research papers is in many cases obscured by the mainstream of ideas in the paper.

In this paper we wish to convince the reader that the colored branch theorem is also a simple and powerful theorem in circuit theory. In particular, the following is presented: (1) The theorem is made plausible to the electrical engineer by substituting the branches by simple elements, such as open and short circuits, ideal diodes, or a resistor battery series combination. This interpretation produces a simple mnemonic tool for remembering the theorem; (2) A rigorous proof is given directly; (3) It is illustrated that this theorem is very general: It is valid for both linear and nonlinear networks as well as for resistive and dynamic networks. Even coupling among elements is allowed. It shares with KCL, KVL, and Tellegen's theorem the distinction of being among the most general theorems in circuit theory; (4) The great generality of the theorem still leaves much freedom to the user. We will show that some particular choices lead to many nice and surprising properties. In many cases this theorem can be used to give a painless proof of a known result, or to generate equivalent conditions, which are easier to check.

We only assume basic knowledge of the theory of directed graphs [13]. We also assume that the number of branches and the number of nodes are finite. Given a graph $\mathcal{G}$, we say that a branch b is *removed* if it is erased from the graph $\mathcal{G}$. We say that a branch b is *coalesced* or *shrunk* if the two nodes to which b is connected are made coincident and b is then removed. Given a finite set A, we call any set of subsets of A, (i.e., $\{A_1, A_2, \cdots, A_n\}$ with $A_i \subset A$ for $1 \leqslant i \leqslant n$), such that $A = A_1 \cup A_2 \cdots \cup A_n$, and $A_i \cap A_j = \varnothing$ for $i \neq j$ a *partition* of A, where $\cup$ denotes the union, $\cap$ denotes the intersection and $\varnothing$ denotes the empty set. In other words $\{A_1, A_2, \cdots, A_n\}$ is a partition of A if each element of A belongs to just one A_i.

II. The Colored Branch Theorem and Its Proof

Given a directed graph $\mathcal{G}$ let it be colored as follows: every branch of the graph is painted in one of three colors: *red*, *blue*, or *green*. Moreover, one green branch is singled out by coloring it dark green.[1] It is clear that any directed graph $\mathcal{G}$ can be colored in many different ways. Any such graph will henceforth be called a *directed colored graph* $\hat{\mathcal{G}}$. Given a directed colored graph and a loop L, any subset of branches in L is said to be "similarly directed" if they are all oriented in the same direction along the loop. Likewise, given a cutset C, any subset of branches in C are said to be "similarly directed" if they

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[1] The sets of red, blue, and green branches form a partition of the set of all branches of the graph. Any set may be empty except the set of green branches, which must contain at least the dark green branch.

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are all oriented in the same direction from one connected component created by the cutset to the other.

The general colored branch theorem is then stated as follows[2] [1]:

Theorem 1: Let $\hat{\mathcal{G}}$ be a (not necessarily connected) directed colored graph, then exactly one, but not both, of the following properties must hold:

(1) The dark green branch forms a *loop* exclusively with the *green* and/or *red* branches such that the green branches in the loop are similarly directed.

(2) The dark green branch forms a *cutset* exclusively with the green and/or blue branches such that the green branches in the cutset are similarly directed.

It is worthwhile to pause here for a while to assess the degree of generality of this theorem. The branches of the graph can be arbitrarily colored as long as (1) each branch is colored red, blue, or green and (2) exactly one branch is colored dark green. Under these two conditions the existence (resp., nonexistence) of a loop satisfying condition (1) immediately guarantees the nonexistence (resp., existence) of a cutset satisfying condition (2). Observe that the theorem does not say that such a loop (resp., cutset) is unique; in fact, there may exist many such loops (resp., cutsets).

Set-Theoretic Proof [1]: Construct recursively a sequence of sets of branches $V_1, V_2, V_3 \cdots$ and a sequence of sets of nodes $W_1, W_2, W_3, \cdots$ using the following *algorithm*.

Step 1: Set $V_1 = \varnothing$ and let W_1 only contain the node at the head (negative terminal) of branches g^*. Set $i = 1$.

Step 2: Given V_i and W_i obtain V_{i+1} and W_{i+1} by adding, respectively, one branch and one node as follows. The added branch must be a red branch *connecting* a node in W_i to a node not in W_i, or a green branch *oriented* from a node in W_i to a node not in W_i, in either case add the node not in W_i to W_i to obtain W_{i+1}.

Step 3: Stop if i) the node at the tail of g^* has been included in W_{i+1} or ii) no new branch could be included in V_{i+1}. Else increment i and go to Step 2.

It is easy to see that the two situations where this algorithm terminates are mutually exclusive and that one of the two has to occur since the algorithm terminates in a finite number of steps for a graph with a finite number of branches and nodes. It remains to be proven that (1) (resp., (2)) of the theorem implies termination of the algorithm at i) (resp., at (ii)) and vice versa.

If the graph $\mathcal{G}$ has a loop satisfying (1) then the algorithm has to terminate at (i). Conversely, if the algorithm terminates at (i) a loop $g^*, b_1, \cdots, b_l$ satisfying (1) can be constructed by tracing back from the positive node of branch g^* (called n_l) as follows. Form a sequence of nodes $n_0, n_1, \cdots, n_l$ and a sequence of branches

[2]For simplicity, the theorem is stated for a *directed* colored graph. It will be clear in the context that only the green branches need to be oriented.

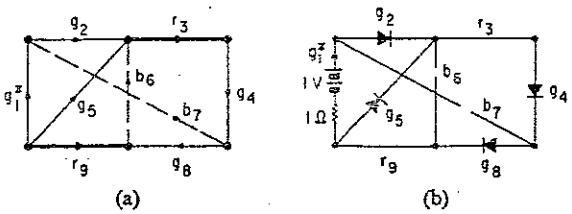


Fig. 1. (a) A directed colored graph: branch g_1^* is colored dark green, branches g_2, g_4, g_5, and g_8 are colored green branches r_3 and r_9 are colored red, and branches b_6 and b_7 are colored blue. (b) Network interpretation of the directed colored graph in (a).

$b_1, b_2, \cdots, b_l$ such that branch b_j is the branch which in the algorithm adds node n_j to W_k for some k, and such that n_{j-1} is the other node of b_j (i.e., the node in W_{k-1}). This procedure can only end when n_0 is the negative node of branch g^*.

If the graph has a cutset satisfying (2) then the algorithm has to terminate at (ii) since any W_i can only contain nodes belonging to that connected component created by the cutset which contains the negative node of g^*. Conversely, if the algorithm terminates at (ii) the following set S of branches contains a cutset satisfying (2). The set S consists of 1) g^*, 2) the blue branches connected with one terminal to a node of the final W_{i+1} and with the other to a node not in W_{i+1}, and 3) the green branches oriented from a node not in W_{i+1} to a node of W_{i+1}. □

This theorem, however, is quite intuitive if we look at the following *network interpretation*. Given a directed colored graph $\hat{\mathcal{G}}$, we replace the *red* branches by *short circuits*, the *blue* branches by *open circuits*, and the *dark green* branch by a series connection of a 1-Ω resistor and a 1-V battery with the positive terminal of the battery located at the arrowhead of the dark green branch. Let the remaining *green* branches be replaced by *ideal diodes* such that positive current flows in the same direction as that defined by the green branch. As an illustration we derive in Fig. 1(b) a network interpretation for the graph of Fig. 1(a), where branches r_3 and r_9 are colored red, branches b_6 and b_7 are colored blue and branches g_2, g_4, g_5, and g_8 are colored green with g_1^* chosen as the dark green branch. It is clear in this example (see Fig. 1(a)) that the branches $\{g_1^*, g_2, g_4, g_8\}$ in the loop formed by $\{g_1^*, g_2, r_3, g_4, g_8, r_9\}$ are similarly directed. This implies in Fig. 1(b) that the branches g_2, r_3, g_4, g_8, and r_9 offer no resistance to the battery in g_1^*, and the voltage across g_1^* is zero so that a current of 1 A must flow through the 1-Ω resistor. This implies immediately that there is no cutset of g_1^*, diodes and open circuits such that all diodes and g_1^* are similarly directed. For, suppose there exists such a cutset then every diode of the cutset must convey a nonnegative current, while g_1^* carried a current of 1 A, thereby contradicting KCL. Other applications of this theorem to hydraulic networks and street traffic networks have been mentioned by Minty [2]. The applications to street network routings is particularly enlightening. The branches here are two-way streets, one-way streets, and blocked streets. The conclusion of the theorem is quite intuitive from the trivial

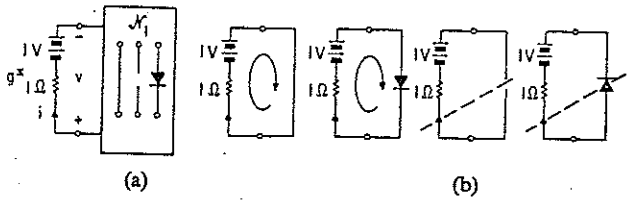


Fig. 2. (a) The network $\mathcal{N}$ corresponding to a general directed colored graph. (b) The four possible equivalents of $\mathcal{N}$.

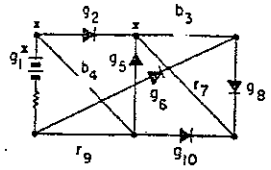


Fig. 3. Example illustrating the algorithm for detecting the existence (resp., nonexistence) of a loop satisfying condition (1), or a cutset satisfying condition (2).

statement: a point B can either be *reached* from a point A or *not at all.*

We will state shortly a theorem on ideal diode networks and then show that it is equivalent with the colored branch theorem. If the branch g^* forms a loop with ideal diodes and short circuits such that the diodes in this loop allow a positive current to flow out of the positive terminal of g^*, then we say that the diodes of the loop are *similarly directed.* Analogously a cutset containing g^*, ideal diodes and open circuits is said to have *similarly directed* ideal diodes if the diodes of this cutset allow a positive voltage to exist at all nodes $n_1^+ \cdots n_m^+$ with respect to the nodes $n_1^- \cdots n_m^-$, where the removal of the branches of the cutset creates two components: one component containing the node at the positive terminal of branch g^* and all the nodes $n_1^+ \cdots n_m^+$, and the second component containing the node at the negative side of branch g^* and all the nodes $n_1^- \cdots n_m^-$.

Theorem 2: Given a network $\mathcal{N}$ (Fig. 2(a)) consisting of a branch g^* made up of a 1-Ω resistor 1-V battery series connection and other branches made up of ideal diodes, open circuits and short circuits, then exactly one but not both of the following properties hold:

(1) There exists a loop containing only g^*, diodes and short circuits, where g^* and all diodes are similarly directed. In this case $i=1$ A and $v=0$ V.

(2) There exists a cutset containing only g^*, diodes and open circuits, where g^* and all diodes are similarly directed. In this case $i=0$ A and $v=-1$ V.

The basic strategy of the proof (see Appendix) is to reduce the number of internal nodes and branches of the one-port $\mathcal{N}_1$ (Fig. 2(a)) repeatedly while preserving the current and voltage in g^*. In the end we arrive at one of the four cases listed in Fig. 2(b) where the theorem is satisfied. The two circuits on the left contain a loop made up of the branch g^* and a short circuit, and a similarly directed diode, respectively. Observe that there is no cutset formed by g^* with an open circuit and/or a similarly directed diode. On the other hand, the two circuits on the right of Fig. 2(b) exhibit the opposite property. Observe that Theorems 1 and 2 imply each other for the following two reasons. First, each diode network $\mathcal{N}$ of Theorem 2 is mapped uniquely into a directed colored graph $\hat{\mathcal{G}}$ and vice versa by map

short circuit of $\mathcal{N}$ $\leftrightarrow$ red branch of $\hat{\mathcal{G}}$

open circuit of $\mathcal{N}$ $\leftrightarrow$ blue branch of $\hat{\mathcal{G}}$

diode of $\mathcal{N}$ $\leftrightarrow$ green branch of $\hat{\mathcal{G}}$.

Second, the alternatives (1) and (2) of Theorem 1 correspond one-to-one to the alternatives (1) and (2) of Theorem 2 and vice versa.

To conclude this section recall that the graph in Theorems 1 and 2 were not required to be connected. It follows, however, from the formulation of these theorems, and from the definitions of "loop" and "cutset," that only the connected component containing branch g^* can contain a loop satisfying condition (1), or a cutset satisfying condition (2). Note also that the algorithm described in the proof of Theorem 1 is an efficient algorithm to check whether case (1) or (2) applies. We have applied it to the graph of Fig. 3 by setting an asterisk "*" at all nodes which are included in some W_i. Since the "+" node of g^* is not included in a W_i we have case (2) and indeed $\{g_1^*, b_4, g_5, g_6, g_{10}\}$ is such a cutset.

III. Graph-Theoretic Corollaries of the Colored Branch Theorem

The colored branch theorem is extremely general. By a clever coloring (partition) of the branches and eventually a repeated use of the theorem, one can derive many surprising properties often painlessly.

Corollary 1: Colored Branch Corollary 1. Let g^* be any branch of a graph $\mathcal{G}$[3] and label it the dark green branch. Color each of the remaining branches arbitrarily either in red or in blue. Let B denote the set of the blue branches, and let R denote the set of the red branches. Then branch g^* either forms a loop exclusively with branches of R, or a *cutset* exclusively with branches of B, *but not both.*

Proof: This is a simple special case of Theorem 1. A simple independent proof is given in [8]. □

Corollary 2: Colored Branch Corollary 2. Let $\mathcal{G}$ be a directed graph and let g^* be any branch of $\mathcal{G}$. Then g^* *either* forms a *loop* with branches of $\mathcal{G}$ such that all branches of the loop are similarly directed, *or* a *cutset* with branches of $\mathcal{G}$ such that all branches of the cutset are similarly directed, *but not both.*

Proof: Obvious from Theorem 1. □

It is important to make the difference between the conclusions of the Colored branch corollaries 1 and 2: The branches of the loop and cutset in Corollary 1 need not be similarly directed. Observe that except for g^*, the branches in the loop and those in the cutset must come from different sets. In Corollary 2, on the other hand, all branches come from the same set. In this case, however,

[3]The graph $\mathcal{G}$ can be either directed or undirected.

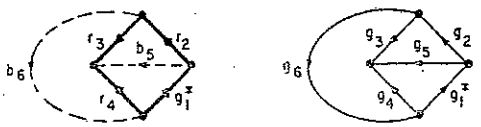


Fig. 4. An example illustrating the differences between Colored branch Corollary 1 for the graph in (a) and Colored branch Corollary 2 for the graph in (b).

the branches in the loop, and those in the cutset, must be similarly directed. For example, consider the graph shown in Fig. 4(a). If we choose $R=\{r_2, r_3, r_4\}$ and $B=\{b_5, b_6\}$ as the sets of blue and red branches, respectively, and identify the dark green branch as g_1^*, then the Colored branch corollary 1 asserts only that g_1^* must either form a loop with branches in R, or a cutset with branches in B. In this case the former holds, namely $\{g_1^*, r_2, r_3, r_4\}$ form a loop. Notice that this loop is not similarly directed. On the other hand, all branches are colored green in order to apply the Colored branch corollary 2. Identify g_1^* again as the dark green branch in Fig. 4(b). Observe that the branches of the loop $\{g_1^*, g_2, g_6\}$ are similarly directed. It can be checked exhaustively that there is no cutset containing g_1^* such that all branches are similarly directed, as predicted by the Colored branch corollary 2.

A repeated application of the colored branch theorem allows us to derive the following result:

Corollary 3: Colored Branch Corollary 3. Partition the branches of a directed graph $\mathcal{G}$ *arbitrarily* into three sets: A, B, and C. We define four subsets of A as follows: (1) Let D_1 be the maximal subset of A such that each branch of D_1 forms a *loop* (resp., cutset) exclusively with branches of A and/or C, and such that all branches of A in this loop are similarly directed. (2) Let D_2 be the maximal subset of A, such that each branch of D_2 *does not* form a *cutset* (resp., loop) exclusively with branches of A and/or B, and such that all branches of A in this cutset are similarly directed. (3) Let E_1 be the maximal subset of A such that each branch of E_1 forms a *cutset* (resp., loop) exclusively with branches of A and/or B, and such that all branches of A in this cutset are similarly directed. (4) Let E_2 be the maximal subset of A such that each branch of E_2 does not form a *loop* (resp., cutset) exclusively with branches of A and/or C, and such that all branches of A in this loop are similarly directed. *Then* we have

$$D_1=D_2\triangleq D \quad E_1=E_2\triangleq E \tag{1}$$

$$D\cup E=A \quad D\cap E=\emptyset. \tag{2}$$

In other words, A is partitioned into D and E.

Proof: First it has to be proved that the sets D_1, D_2, E_1, and E_2 *exist* and are *unique*. We prove it, for example, for D_1. Let D_1' and D_1'' be any two subsets of A such that each branch of D_1' (resp., D_1'') forms a loop exclusively with branches of A and/or C and such that all branches of A in the loop are similarly directed. Then, $D_1'\cup D_2''$ also satisfies this property. So the *maximal* subset D_1 is the union of all subsets of A satisfying the property, and is therefore unique. There exists at least one subset of A satisfying the property since the empty set $\emptyset$ satisfies the property by default. Hence D_1 exists. In order to prove (1), we show that $D_1\supset D_2$, $D_1\subset D_2$, $E_1\supset E_2$, and $E_1\subset E_2$. Let us prove, for example, $D_1\supset D_2$. Choose any branch g^* in D_2 and color it dark green. Color the remaining branches of A in green, the branches of B in blue and the branches of C in red. Since $g^*\in D_2$, by the Colored branch theorem it forms a loop with the green and/or red branches such that the green branches in the loop are similarly directed. Since D_1 is maximal this implies $g^*\in D_1$. Equation (2) then follows from the fact that D_1 and E_2 are by definition a partitioning of A. □

A similar corollary can be derived from the Colored branch corollary 1.

Corollary 4: Colored Branch Corollary 4. Partition the branches of a graph $\mathcal{G}$ arbitrarily into three sets: A, B, and C. We define four subsets of A as follows: (1) Let D_1 be the maximal subset of A such that each branch of D_1 forms a *loop* (resp., cutset) exclusively with branches of A and/or C. (2) Let D_2 be the maximal subset of A such that each branch of D_2 *does not* form a *cutset* (resp., loop) exclusively with branches of B. (3) Let E_1 be the maximal subset of A such that each branch of E_1 forms a *cutset* (resp., loop) exclusively with branches of B. (4) Let E_2 be the maximal subset of A, such that each branch E_2 *does not* form a *loop* (resp., cutset) exclusively with branches of A and/or C.

Then, we have

$$D=D_2\triangleq D \quad E_1=E_2\triangleq E \tag{3}$$

$$D\cup E=A \quad D\cap E=\emptyset. \tag{4}$$

In other words, each branch of A belongs either to $D_1=D_2\triangleq D$ or to $E_1=E_2\triangleq E$, but not to both.

Proof: The proof is analogous to that of the Colored branch corollary 3. □

Let us now compare Corollaries 3 and 4. Assume that a directed graph has been partitioned into three sets A, B, and C as specified in Corollaries 3 and 4. It is clear from the definitions that set D_1 in Corollary 3 must satisfy more conditions (all branches of A in the loop must be similarly directed) than set D_1 in Corollary 4. Thus set D in Corollary 3 is a subset of set D in Corollary 4. Analogously set E in Corollary 4 is a subset of set E in Corollary 3, as they should.

This corollary allows us to derive a nice property concerning the existence of certain loops and cutsets obtained by partitioning the twigs of a tree and the links of a cotree in a particular way. There are three alternative formulations.

Corollary 5a: Partition the twigs of a tree $\mathcal{T}$ *arbitrarily* into two sets $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\mathcal{L}_1$ be the maximal set of links of the corresponding cotree $\mathcal{L}$ such that each link of $\mathcal{L}_1$ forms a loop exclusively with twigs of $\mathcal{T}_1$. Let $\mathcal{L}_2$ be the set of the remaining links of $\mathcal{L}$. *Then* any twig of $\mathcal{T}_2$ forms a cutset exclusively with links of $\mathcal{L}_2$.

Proof: Set $A=\mathcal{L}$, $B=\mathcal{T}_1$, and $C=\mathcal{T}_2$. It follows from Corollary 4 (with $E_1=\mathcal{L}_1$) that $D_1=\mathcal{L}_2$ is the maximal subset of $\mathcal{L}$ such that each branch of $\mathcal{L}_2$ forms a cutset exclusively with branches of $\mathcal{L}$ and $\mathcal{T}_2$. However, each twig of $\mathcal{T}_2$ forms a fundamental cutset with $\mathcal{L}$. Since $\mathcal{L}_2$ is maximal, each twig of $\mathcal{T}_2$ forms a cutset exclusively with links of $\mathcal{L}_2$. □

An alternative statement of Corollary 5a can be formulated as follows:

Corollary 5b: Partition the links of a cotree $\mathcal{L}$ *arbitrarily* into two sets $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\mathcal{T}_2$ be the maximal set of twigs of the corresponding tree $\mathcal{T}$, such that each twig of $\mathcal{T}_2$ forms a cutset exclusively with links of $\mathcal{L}_2$. Let $\mathcal{T}_1$ be the set of the remaining twigs of $\mathcal{T}$. *Then* any link of $\mathcal{L}_1$ forms a loop exclusively with twigs of $\mathcal{T}_1$.

A third formulation of this property does not assume the tree or cotree to be given.

Corollary 5c: Partition the branches of a graph $\mathcal{G}$ arbitrarily into two sets S_1 and S_2. Let T_1 be the maximal subset of S_1 such that each branch of T_1 *does not* form a *loop* exclusively with branches of S_1. Let L_1 be the maximal subset of S_1 such that each branch of L_1 *does not* form a *cutset* exclusively with branches of S_2. Let T_2 be the maximal subset of S_2 such that each branch of T_2 *does not* form a loop exclusively with branches of S_1. Let L_2 be the maximal subset of S_2 such that each branch of L_2 *does not* form a *cutset* exclusively with branches of S_2. Then we have

(1)
$$S_1=T_1\cup L_1 \quad T_1\cap L_1=\emptyset \tag{5}$$

$$S_2=T_2\cup L_2 \quad T_2\cap L_2=\emptyset. \tag{6}$$

(2) Each branch of L_1 forms a loop exclusively with branches of L_1 and each branch of T_2 forms a cutset exclusively with branches of T_2.

(3) Let $\mathcal{T}_1\supset T_1$ be any maximal subset of S_1 such that each branch of $\mathcal{T}_1$ *does not* form a *loop* exclusively with branches of $\mathcal{T}_1$. Call $\mathcal{L}_1$ the remaining branches of S_1. Let $\mathcal{L}_2\supset L_2$ be any maximal subset of S_2 such that each branch of $\mathcal{L}_2$ does not form a cutset exclusively with branches of $\mathcal{L}_2$. Call $\mathcal{T}_2$ the remaining branches of S_2. Then $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2$ and $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2$ form a tree and cotree of the graph $\mathcal{G}$. Also each branch of $\mathcal{L}_1$ forms a loop exclusively with branches of $\mathcal{T}_1$ and each branch of $\mathcal{T}_2$ forms a cutset exclusively with branches of $\mathcal{L}_2$.

Proof: Apply the Colored branch corollary 4. Identify $A=S_1$, $B=S_2$, and $C=\emptyset$. Then (5) immediately follows. By identifying $A=S_2$, $B=S_1$, and $C=\emptyset$ and applying the Colored branch corollary 4 we obtain (6). This implies also part (2) of the corollary. Next we prove that $\mathcal{T}\triangleq\mathcal{T}_1\cup\mathcal{T}_2$ does not contain any loop and $\mathcal{L}\triangleq\mathcal{L}_1\cup\mathcal{L}_2$ does not contain any cutset. Suppose the contrary that there exists a cutset in $\mathcal{L}$. Then since $\mathcal{T}_1$ is a maximal subset of S_1 having no loops, any branch of $\mathcal{L}_1=S_1-\mathcal{T}_1$ must form a loop with $\mathcal{T}_1$. Hence, by the Colored branch corollary 1 any branch of $\mathcal{L}_1$ cannot form a cutset with branches of $\mathcal{T}_2\cup\mathcal{L}_1\cup\mathcal{L}_2$. Thus any cutset in $\mathcal{L}$ can only contain branches of $\mathcal{L}_2$. But this is impossible by the definition of $\mathcal{L}_2$. Analogously it can be proved that $\mathcal{T}$ contains no loops. The properties about the branches of $\mathcal{L}_1$ and $\mathcal{T}_2$ follow from the maximality of $\mathcal{L}_1$ and $\mathcal{T}_2$. □

Observe that by labelling the branches of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{T}_1$, and $\mathcal{T}_2$ consecutively, we obtain in each of the three cases a *fundamental loop matrix B* having the following structure [14]:

$$B=\begin{array}{c} \begin{matrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{T}_1 & \mathcal{T}_2 \end{matrix} \\ \begin{bmatrix} \mathbf{1}_{\mathcal{L}_1\mathcal{L}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{L}_2} & \boldsymbol{B}_{\mathcal{L}_1\mathcal{T}_1} & \mathbf{0}_{\mathcal{L}_1\mathcal{T}_2} \\ \mathbf{0}_{\mathcal{L}_2\mathcal{L}_1} & \mathbf{1}_{\mathcal{L}_2\mathcal{L}_2} & \boldsymbol{B}_{\mathcal{L}_2\mathcal{T}_1} & \boldsymbol{B}_{\mathcal{L}_2\mathcal{T}_2} \end{bmatrix} \end{array} \begin{matrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{matrix} \tag{7}$$

where the upper right submatrix is always a zero matrix.

Corollary 5c guarantees the existence of a tree and a cotree having certain properties in the case where the branches of a graph are partitioned into two sets. The next corollary deals with the case that the branches are partitioned into three sets.

Corollary 6: Let the branches of a graph $\mathcal{G}$ be partitioned *arbitrarily* into three sets X, Y, and Z. Let n_X be the number of branches of X minus the number of independent loops in X; and n_Z the number of branches of Z minus the number of independent cutsets in Z. Then there exists a tree $\mathcal{T}$ and a cotree $\mathcal{L}$ of $\mathcal{G}$ such that $\mathcal{T}$ contains n_X branches of X, and $\mathcal{L}$ contains n_Z branches of Z. Furthermore, let $X_2\mathcal{T}$ be the maximal subset of branches of X, such that each branch of $X_2\mathcal{T}$ *does not* form a *loop* exclusively with branches of X, and let $Z_1\mathcal{L}$ be the maximal subset of Z, such that each branch of $Z_1\mathcal{L}$ *does not* form a *cutset* exclusively with branches of Z. Then for any of the above tree cotree we have $X_2\mathcal{T}\subset\mathcal{T}$ and $Z_1\mathcal{L}\subset\mathcal{L}$. Partition the branches of $\mathcal{T}$ into the following four sets $X_1\mathcal{T}$, $X_2\mathcal{T}$, $Y\mathcal{T}$, and $Z\mathcal{T}$ such that:

$$X_1\mathcal{T}\cup X_2\mathcal{T}=X\cap\mathcal{T} \quad \text{and} \quad X_1\mathcal{T}\cap X_2\mathcal{T}=\emptyset \tag{8}$$

$$Y\mathcal{T}=Y\cap\mathcal{T} \tag{9}$$

$$Z\mathcal{T}=Z\cap\mathcal{T}. \tag{10}$$

Partition the branches of the corresponding cotree $\mathcal{L}$ into the following four sets $X\mathcal{L}$, $Y\mathcal{L}$, $Z_1\mathcal{L}$, and $Z_2\mathcal{L}$:

$$X\mathcal{L}=X\cap\mathcal{L} \tag{11}$$

$$Y\mathcal{L}=Y\cap\mathcal{L} \tag{12}$$

$$Z_1\mathcal{L}\cup Z_2\mathcal{L}=Z\cap\mathcal{L} \quad \text{and} \quad Z_1\mathcal{L}\cap Z_2\mathcal{L}=\emptyset. \tag{13}$$

Then each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$, and each branch of $Z\mathcal{T}$ forms a cutset exclusively with branches of $Z_2\mathcal{L}$.

Proof: Recall that a tree $\mathcal{T}$ and the corresponding cotree $\mathcal{L}$ form a partition of the branches of a graph such that $\mathcal{T}$ contains no loops and $\mathcal{L}$ contains no cutsets. Therefore the number n_X (resp., n_Z) of branches of X (resp., Z) in any tree (resp., cotree) as defined in the corollary is the maximum that can be attained. We construct an arbitrary tree satisfying the specifications of Corollary 7 by partitioning first the branches of X into three sets as follows: Remove any branch from each independent loop of X. Denote this set of removed branches by $X\mathcal{L}$. The remaining branches of X of course contain the set $X_2\mathcal{T}$ defined in

the corollary, and can therefore be considered as the union of two disjoint sets $X_1\mathcal{T}$ and $X_2\mathcal{T}$. Thus X is partitioned into $X_1\mathcal{T}$, $X_2\mathcal{T}$, and $X\mathcal{L}$. Next a dual partitioning of Z is made into $Z\mathcal{T}$, $Z_1\mathcal{L}$, and $Z_2\mathcal{L}$. It follows then that each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$ and that each branch of $Z\mathcal{T}$ forms a cutset exclusively with branches of $Z_2\mathcal{L}$. It remains for us to prove that the tree $\mathcal{T}$ contains all branches of $X_1\mathcal{T}\cup X_2\mathcal{T}\cup Z\mathcal{T}$, and the corresponding cotree $\mathcal{L}$ contains all branches of $X\mathcal{L}\cup Z_1\mathcal{L}\cup Z_2\mathcal{L}$. First we claim that $\mathcal{T}^1 \triangleq X_1\mathcal{T}\cup X_2\mathcal{T}\cup Z\mathcal{T}$ does not contain any loop and in a dual way that $\mathcal{L}^1 \triangleq X\mathcal{L}\cup Z_1\mathcal{L}\cup Z_2\mathcal{L}$ does not contain any cutset. Since each branch of $Z\mathcal{T}$ forms a cutset exclusively with branches of $Z_2\mathcal{L}$, by the Colored branch corollary 1, it cannot form a loop with branches of $X_1\mathcal{T}\cup X_2\mathcal{T}\cup Z\mathcal{T}$. Since $X_1\mathcal{T}\cup X_2\mathcal{T}$ does not contain any loop by construction, $\mathcal{T}^1$ does not contain any loop. We now reconstruct the tree and corresponding cotree inductively starting from $\mathcal{T}=\mathcal{T}^1$ and $\mathcal{L}=\mathcal{L}^1$ by adding branches of Y to $\mathcal{T}$ or $\mathcal{L}$ as follows. Consider one branch y of Y at a time. If y forms a loop with $\mathcal{T}$, then by the Colored branch corollary 1 it forms no cutset with $\mathcal{L}$ and hence y is added to $\mathcal{L}$. In a dual way if y forms a cutset with $\mathcal{L}$ then add it to $\mathcal{T}$. In all other cases y can be added to $\mathcal{L}$ or $\mathcal{T}$. □

Observe that by labelling the branches of $X\mathcal{L}$, $Y\mathcal{L}$, $Z_1\mathcal{L}$, $Z_2\mathcal{L}$, $X_1\mathcal{T}$, $X_2\mathcal{T}$, $Y\mathcal{T}$, and $Z\mathcal{T}$ consecutively the *fundamental loop matrix* B assumes the following form:

$$B = \begin{array}{c} \begin{matrix} X\mathcal{L} & Y\mathcal{L} & Z_1\mathcal{L} & Z_2\mathcal{L} & X_1\mathcal{T} & X_2\mathcal{T} & Y\mathcal{T} & Z\mathcal{T} \end{matrix} \\ \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & B_{15} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & B_{25} & B_{26} & B_{27} & 0 \\ 0 & 0 & 1 & 0 & B_{35} & B_{36} & B_{37} & 0 \\ 0 & 0 & 0 & 1 & B_{45} & B_{46} & B_{47} & B_{48} \end{array}\right] \end{array} \begin{matrix} X\mathcal{L} \\ Y\mathcal{L} \\ Z_1\mathcal{L} \\ Z_2\mathcal{L} \end{matrix} \tag{14}$$

where the zero submatrices in the upper right part of B are caused by the fact that each branch of $X\mathcal{L}$ forms a loop exclusively with branches of $X_1\mathcal{T}$ and each branch of $Z\mathcal{T}$ forms a cutset exclusively with branches of $Z_2\mathcal{L}$.

Before we close this section, a general comment about equivalent formulations of the statement of these graph-theoretic results is in order. In many cases it turns out that a "reduced" graph obtained by shrinking some branches and removing some others can just as well be used to express these results. This equivalent formulation may be conceptually simpler or may enhance the efficiency of the associated algorithms. For example, we could check the Colored branch theorem immediately on the reduced graph $\mathcal{G}_r$ obtained by shrinking the red branches and removing the blue branches. If we do the same in Colored branch corollary 1, then $\mathcal{G}_r$ only contains one branch and this branch forms either a self-loop or a self-cutset. For Colored branch corollaries 3 and 4, we would obtain the same sets D_1, D_2, E_1, and E_2 if the branches of B (resp., C) were removed and those of C (resp., B) were shrunk. In defining T_1, L_1, $\mathcal{T}_1$, and $\mathcal{L}_1$, in Corollary 5c, the branches of S_2 can be removed from the graph, such that $\mathcal{T}_1$ and $\mathcal{L}_1$ are merely a tree and a cotree in the resulting graph. Analogously the branches of S_1 can be shrunk from the graph in defining T_2, L_2, $\mathcal{T}_2$, and $\mathcal{L}_2$. This corresponds to the formulation in [21].

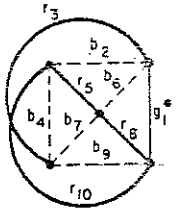


Fig. 5. By simply checking that g_1^* forms no loop with the set of red branches $\{r_3, r_5, r_8, r_{10}\}$, it follows that g_1^* forms a cutset with the set of blue branches $\{b_2, b_4, b_6, b_7, b_9\}$.

IV. Applications of the Colored Branch Theorem and Its Corollaries in Circuit Theory

In this section we compile a list of applications of the Colored branch theorem in circuit theory. Some applications are new while others allow simpler proofs of known properties. Whenever an application or a proof is new, a detailed treatment is given. Otherwise, we simply refer to the relevant literature.

1. The Colored branch theorem and its Corollaries 1 and 2, imply that the existence (resp., nonexistence) of certain loops or cutsets is equivalent to the nonexistence (resp., existence) of certain cutsets or loops. Now, in an *algorithm* or in the *formulation of a property*, one of the two equivalent properties may turn out to be simpler. This observation allows us to find equivalent conditions for the nonexistence of a loop of capacitors or inductors (or a cutset of capacitors or inductors) in an *RLC* circuit. It also allows us to check whether a digital filter has a delay free loop or not. One simple illustration of this idea is as follows. Check graphically whether a branch g^* forms a cutset with the blue branches of a nonplanar colored graph. A simple example if given in Fig. 5. It is difficult to see at first sight that g_1^* forms a cutset with the blue branches b_2, b_4, b_6, b_7 and b_9. However, it is trivial to see that g_1^* forms no loop with any subset of the red branches $\{r_3, r_5, r_8, r_{10}\}$.

2. The Colored branch theorem was discovered by Minty [1] as a lemma for the proof of the existence of a solution of a network with uncoupled strictly increasing[4] (monotone) resistors. He afterwards used it in a network programming technique to solve such networks [2]. More recently Desoer and Wu [12] extended these results to more general resistive and *RLC* networks. Again the Colored branch theorem was a main tool in the proof and in an algorithm for checking the topological conditions.

3. The Colored branch theorem is the key for obtaining a no-gain property [6], [7], [15], [16] used in finding fundamental limits of dc-to-dc converters and in deriving

[4] A two terminal resistor is said to be strictly increasing if for any two distinct admissible pairs (v', i') and (v'', i''), $(v'-v'')(i'-i'')>0$.

qualitative properties of resistive networks containing three-terminal [10] and multiterminal elements [7]. We describe here the generalized formulation found in [7] and present a unified proof.

Proposition 1: No-Gain Property for Networks Containing Two-Terminal Elements. Given a network containing independent voltage and current sources, strictly passive two-terminal resistors,[5] short-circuit elements which do not form loops among each other, and open-circuit elements which do not form cutsets among each other. Then for any solution of this network we have:

(1) The current magnitude through any element is not greater than the sum of the current magnitudes through all independent voltage and current sources.

(2) The voltage magnitude across any element is not greater than the sum of the voltage magnitudes across all independent voltage and current sources.

Proof: Consider one solution of the network. The proposition is trivially satisfied in the case where the branch considered (henceforth called branch l) is a voltage source or a current source, or where its branch voltage $v_l=0$ and branch current $i_l=0$. In order to prove part (1) of the proposition for the case where branch l is a resistor, an open circuit, or a short circuit, and has $v_l\neq 0$ or $i_l\neq 0$, we proceed as follows. All branches with $v=0$ and $i=0$ can be removed, since they contribute nothing to the sum in (1). Moreover, they do not alter the solution nor do they introduce any new loops. Let S be the set of the remaining sources and let X be the set of the remaining resistors, short-circuit and open-circuit elements. Since all resistors are strictly passive we can define associated reference directions for all branches in X such that the currents and the voltages in these branches are nonnegative. The set up for the Colored branch theorem is now as follows: The branches in X are colored green, those in S blue while there are no red branches. We now prove that branch l does not form a similarly directed loop exclusively with branches of X. If $v_l>0$ the voltages along such a loop would violate KVL. If $v_l=0$, then by KVL all branches of such a loop have zero voltage. Since none of them has zero current, all must be short-circuit elements. But this is impossible since there are no loops containing only short-circuit elements. Thus by the Colored branch theorem there exists a cutset C of branch l and some branches from X and S such that all branches of X are similarly directed. Applying KCL to this cutset, we obtain

$$\sum_{k\in X\cap C} i_k+\sum_{k\in S\cap C}\delta_k i_k=0 \tag{15}$$

for some $\delta_k=\pm 1$. Since all i_k with $k\in X$ are nonnegative, we have

$$|i_l|=i_l\leqslant \sum_{k\in X\cap C} i_k=-\sum_{k\in S\cap C}\delta_k i_k\leqslant \sum_{k\in S\cap C}|i_k|. \tag{16}$$

□

[5]A two terminal resistor is said to be strictly passive if for any nonzero admissible pair (v, i) we have $vi>0$.

4. Wolaver's three-basket theorem [15], [16] is a similar result concerning a partitioning of the branches of a network into 3 sets called baskets. Using the Colored branch theorem a considerably simpler proof than that of [15], [16] can be given.

Proposition 2: Wolaver's Three-Basket Theorem. Let the branches of a resistive network be partitioned into three sets (baskets) S_1, S_2, and S_3, such that each branch k of S_3 has $v_k i_k\geqslant 0$. Then for any solution of the network and any branch l of S_3, both of the following expressions cannot be simultaneously satisfied:

$$\text{(a)}\quad |i_l|>\sum_{k\in S_1}|i_k| \tag{17a}$$

$$\text{(b)}\quad |v_l|>\sum_{k\in S_2}|v_k|. \tag{17b}$$

Proof: Consider a solution of this network. Reverse the reference directions of any branch in S_3 if necessary until all voltages and currents of branches in S_3 are nonnegative. This entails no loss of generality since $v_k i_k\geqslant 0$ for $k\in S_3$. Let the branches of S_1 be colored blue, those of S_2 red, and those of S_3 green with branch l being the dark green branch. Then it follows from the Colored branch theorem that branch l forms either a similarly directed loop L exclusively with branches in S_2 and S_3, or a similarly directed cutset C exclusively with branches in S_1 and S_3, but not both. If the first property holds, then by applying KVL along the loop L we obtain

$$\sum_{k\in L\cap S_3} v_k+\sum_{k\in L\cap S_2}\delta_k v_k=0 \tag{18}$$

for some $\delta_k=\pm 1$. Using the fact that $v_k\geqslant 0$ for $k\in S_3$, we have

$$|v_l|\leqslant v_l+\sum_{\substack{k\in L\cap S_3\\ k\neq l}} v_k=-\sum_{k\in L\cap S_2}\delta_k v_k\leqslant\sum_{k\in S_2}|v_k| \tag{19}$$

which clearly contradicts (17b). Analogously it can be proved that the existence of a similarly directed cutset C contradicts (17a). Hence (17a) and (17b) cannot both be satisfied. □

5. In circuit theory it is important to know whether a certain property P is *closed under interconnection*, i.e., an arbitrary interconnection of multiports, each having property P results in a multiport which inherits this property. For some useful properties such as strict passivity, this is in general not true; but by imposing an additional *topological condition* on the ports, the closure property is valid [8], [11]. The Colored branch corollary 1 is essential for proving this conditional closure property.

6. In order to find properties of *RLC* networks it is important to be able to express the voltages and currents associated with the inductor and capacitor ports at any time and as a function of the *voltages and currents* at the remaining ports at time t (resp., as a function of the *voltages and currents* at the remaining ports and *their derivatives* at time t). It is shown in [8] that it is necessary and sufficient that the inductor and capacitor ports form

no loops or cutsets among themselves (resp., that it is sufficient that a topological condition called the *LC*-hypothesis is satisfied). The proof is once again based on the Colored branch corollary 1 (resp., Colored branch corollary 4). The reader will easily recognize the repeated implicit use that is made of the Colored branch corollary 4 in the proof of theorem 10 of [8]. Recently necessary and sufficient conditions have been found also for the second case which will be reported later.

7. In many cases state equations have to be derived for *RLC* networks from KVL, KCL, and the constitutive relations. Such a derivation is usually based on the existence of a tree containing the maximum number of capacitors (resp., inductors) and a corresponding cotree containing a maximum number of inductors (resp., capacitors). Such a tree is called a *C-normal tree* [17] (resp., *L-normal tree* [9], [18]). The existence of such a tree and its properties are a direct consequence of Corollary 6. As usual, we assume that the independent voltage sources do not form loops exclusively with each other or with capacitors (resp., inductors) and that the independent current sources do not form cutsets exclusively with each other or with inductors (resp., capacitors). Identify the sets X, Y, and Z of Corollary 6 as follows: Choose X to be the set consisting of all capacitors (resp., inductors) and all independent voltage sources, choose Y to be the set consisting of all resistors, and choose Z to be the set consisting of all inductors (resp., capacitors) and all independent current sources. It follows then from the preceding assumptions that all independent voltage sources (resp., current sources) are included in $X_2\mathcal{T}$ (resp., $Z_1\mathcal{L}$). Hence, it follows from Corollary 6 that there exists a tree which contains the maximum number of capacitors (resp., inductors), all independent voltage sources and a minimum number of inductors (resp., capacitors). Similarly, there exists a corresponding cotree which contains the maximum number of inductors (resp., capacitors), all independent current sources and the minimum number of capacitors (resp., inductors).

8. Corollaries 5a, 5b, and 5c provide 3 equivalent methods for generating a *complete set of variables* for a graph in the following sense. A set of branch voltages *and* currents is called a complete set of variables for a graph if they can be assigned arbitrary values without violating KVL and KCL and if they determine at least one of the two variables (the voltage or the current) in each branch by using KVL and KCL and without invoking the element constitutive equations [18]–[20]. Such a complete set of variables is essential in the hybrid analysis of a nonlinear network. By labelling the branch voltages and currents in $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{L}_1$, $\mathcal{L}_2$ as $v_{\mathcal{T}_1}$, $v_{\mathcal{T}_2}$, $v_{\mathcal{L}_1}$, $v_{\mathcal{L}_2}$ and $i_{\mathcal{T}_1}$, $i_{\mathcal{T}_2}$, $i_{\mathcal{L}_1}$, $i_{\mathcal{L}_2}$, respectively, and using the fundamental loop matrix B of (7), $v_{\mathcal{T}_1}$ and $i_{\mathcal{L}_2}$ emerge as a complete set of variables. Indeed KVL and KCL are not violated by assigning arbitrary values to $v_{\mathcal{T}_1}$ and $i_{\mathcal{L}_2}$. Moreover, $v_{\mathcal{L}_1}$ and $i_{\mathcal{T}_2}$ are determined uniquely via KVL and KCL as follows:

$$v_{\mathcal{L}_1} = -B_{\mathcal{L}_1\mathcal{T}_1} v_{\mathcal{T}_1} \qquad i_{\mathcal{T}_2} = B^T_{\mathcal{L}_2\mathcal{T}_2} i_{\mathcal{L}_2} \tag{20}$$

where the submatrices of B are defined in (7). The hybrid analysis proceeds then as follows: If the resistors in $\mathcal{T}_1$ and $\mathcal{L}_1$ are voltage controlled and those in $\mathcal{T}_2$ and $\mathcal{L}_2$ are current controlled, then the KVL and KCL equations, and the constitutive relations completely determine the solution of the resistive network via a set of $n_{\mathcal{T}_1}+n_{\mathcal{L}_2}$ equations with $n_{\mathcal{T}_1}+n_{\mathcal{L}_2}$ unknown variables $v_{\mathcal{T}_1}$ and $i_{\mathcal{L}_2}$, where $n_{\mathcal{T}_1}$ (resp., $n_{\mathcal{L}_2}$) is the dimension of $v_{\mathcal{T}_1}$ (resp., $i_{\mathcal{L}_2}$). The three different techniques for finding a complete set of variables, which follow from Corollary 5a, 5b, and 5c have been used extensively. The approach taken in Corollary 5a and 5b corresponds with that of [14], [19], [20], while the approach of Corollary 5c corresponds with that of [21]. In general there are many distinct ways for partitioning the branches as in Corollary 5. Any partition which minimizes the number of $n_{\mathcal{T}_1}+n_{\mathcal{L}_2}$ also minimizes the number of equations required in the hybrid analysis. This unique number is called the *topological degree* of freedom [21] and is completely determined by the graph of the network.

V. Concluding Remarks

This paper is entirely devoted to a simple, yet not so well-known result called the Colored branch theorem. Perhaps the easiest method for applying this theorem is via its network interpretation. In particular, consider a network $\mathcal{N}$ (Fig. 2a) consisting of branch g^* in parallel with a one-port $\mathcal{N}_1$ containing only diodes, open-circuit elements, and short-circuit elements. Then exactly one, but not both, of the following properties hold:

(1) $\mathcal{N}_1$ offers no resistance to the current i, in which case g^* forms a loop with short-circuit elements and forward-biased diodes.

(2) $\mathcal{N}_1$ offers an infinite resistance to the current i, in which case there exists a cutset of g^* with open-circuit elements and reverse-biased diodes. To obtain the theorem, one simply replaces the short-circuit elements of $\mathcal{N}$ by red branches, the open-circuit elements by blue branches, the diodes by "oriented" green branches, and g^* by an "oriented" dark green branch.

Due to its graph-theoretic nature this theorem is very general: It applies to both linear and nonlinear networks, as well as to resistive dynamic networks. Even coupling among elements is allowed. A clever use of the freedom in the choice of the sets of red, blue and green branches allowed us to derive many useful graph-theoretic corollaries. These corollaries greatly simplify the use of the Colored branch theorem in circuit theory. Some new applications are discussed and simpler proofs are given for existing properties. Many more applications are expected to follow in nonlinear circuit theory, analog and digital filter theory, and power networks. In other fields such as mechanical, thermodynamic, and hydraulic systems, the corollaries are equally applicable. In short, we conclude that the Colored branch theorem and its corollaries are among the most general and fundamental tools in circuit theory. We believe that the Colored branch theorem has not yet been fully exploited and that its potential for

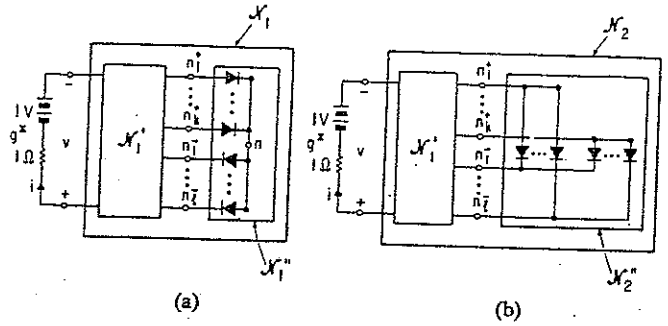


Fig. 6. By replacing $\mathcal{N}_1''$ of (a) by its equivalent $\mathcal{N}_2''$ of (b) the internal node n is eliminated.

future unconventional applications in circuit theory should be quite promising.

Appendix

Proof of Theorem 2: We perform an algorithm to reduce the number of internal nodes. Then we show that in this reduction process, the existence or absence of a loop satisfying condition (1), or a cutset satisfying condition (2) is preserved.

Step 1: Remove all open-circuit branches and shrink all short-circuit branches. Replace each parallel or series connection of two ideal diodes by its equivalent, which is again an ideal diode, an open circuit, or a short circuit. Remove branches which form hinged loops or self-cutsets. Repeat these reduction techniques until there are no more open- or short-circuit branches, and until there are no more parallel or series connections of ideal diodes.

Step 2: Reduce the number of internal nodes in $\mathcal{N}_1$ by one by substituting $\mathcal{N}_1$ with an equivalent one-port $\mathcal{N}_2$ as follows. Single out any internal node "n" to be removed (Fig. 6(a)). Node n is connected to nodes $n_1^+, n_2^+ \cdots n_k^+$ with the diodes pointing toward n, and to nodes n_1^-, $n_2^- \cdots n_l^-$ with the diodes pointing away from n. It is easy to check that $\mathcal{N}_1''$ of Fig. 6(d) is equivalent to $\mathcal{N}_2''$ of Fig. 6(b) (see for example [22], [23]). This equivalence transformation removes node n and may introduce some new parallel or series connections of diodes.

Step 3: Return to Step 1 as long as there remain any parallel or series connections of diodes. Return to Step 2 as long as there remain any internal nodes. If there are no more internal nodes, parallel and series connection of diodes, hinged loops, or self-cutsets, in $\mathcal{N}_1$, then the resulting network must necessarily assume one of the four cases shown in Fig. 2(b).

It remains for us to show that during the reduction steps the validity of the theorem is preserved. Only the equivalence transformation in Step 2 requires additional investigation. Of course, a loop satisfying condition (1) (resp., a cutset satisfying condition (2)) and containing no branches of $\mathcal{N}_1''$ is preserved in $\mathcal{N}_2$. It follows from the comparison of Fig. 6(a) and (b) that for a loop which contains the diode between nodes n_i^+ and n and the diode between nodes n and n_j^- in Fig. 6(a), there exists a loop containing the diode between n_i^+ and n_j^- in Fig. 6(b). Moreover, if the first loop satisfies condition (1), then likewise the second loop must satisfy the same condition.

A cutset satisfying condition (2) and containing diodes of $\mathcal{N}_1''$ can only contain some diodes connected to nodes $n_1^+ \cdots n_k^+$, or some diodes connected to nodes $n_1^- \cdots n_l^-$, but not both, since the diodes in each group are similarly directed. Assume that the diodes connecting $n_{i_1}^+ \cdots n_{i_s}^+$ to n belong to a cutset of Fig. 6(a) satisfying condition (2). Now, by definition, the removal of all branches of this cutset creates two connected components. One connected component containing, among other nodes, the "$-$" node of g^* and nodes $n_{i_1}^+ \cdots n_{i_s}^+$. The other component contains, among other nodes, node n and the "$+$" node of g^*. It is now easy to see that the network of Fig. 6(b) has a cutset satisfying condition (2) where all diodes of $\mathcal{N}_2''$ connecting nodes $n_{i_1}^+ \cdots n_{i_s}^+$ to nodes $n_1^- \cdots n_l^-$ are members of the cutset formed by g^* along with those branches of $\mathcal{N}_1'$ belonging to the original cutset in Fig. 6(a).

Conversely, the existence of a loop satisfying condition (1) in Fig. 6(b) implies the existence of a loop satisfying condition (1) in Fig. 6(a), and analogously for a cutset. □

Observe that the number of loops satisfying condition (1) (resp., cutsets satisfying condition (2)) in $\mathcal{N}_2$ can be smaller than in $\mathcal{N}_1$. Hence in general a network may have several loops satisfying condition (1) (resp., cutsets satisfying condition (2)).

The algorithm described in Theorem 2 can be rendered more efficient by the following two rules: (1) If $\mathcal{N}_1$ contains a loop of diodes all biased in the same direction, then all branches of this loop can be coalesced. (2) If $\mathcal{N}_1$ contains a cutset of diodes all biased from one component to the other then only that component which contains branch g^* needs to be considered in the algorithm.

References

[1] G. J. Minty, "Monotone networks," *Proc. Royal Society (London) A*, vol. 257, Sept. 1960, pp. 194–212.
[2] ——, "Solving steady-state nonlinear networks of monotone elements," *IRE Trans. Circuit Theory*, vol. CT-8, pp. 99–104, June 1961.
[3] C. Berge and A. Ghouila-Houri, *Programmes, Jeux et Reseaux de Transport*, Paris, France: Dunod, 1962; (In English) "*Programming, Games and Transportation Networks*," London, England: Methuen, 1965.
[4] G. J. Minty, "On the axiomatic foundations of the theories of directed linear graphs, electrical circuits and network-programming," *J. Mathematics and Mechanics*, vol. 15, no. 3, pp. 485–520, 1966.
[5] E. L. Lawler, *Combinatorial Optimization: Networks and Matroids*. New York: Holt, Rinehart and Winston, 1976.
[6] D. H. Wolaver, "Proof in graph theory of the no gain property of resistor networks," *IEEE Trans. Circuit Theory*, vol. CT-17, pp. 436–437, Aug. 1970.
[7] L. O. Chua, Y-F Lam, and L. A. Stromsmoe, "Qualitative properties of resistive networks containing multiterminal nonlinear elements," *IEEE Trans. Circuits Syst.*, vol. CAS-24, pp. 93–118, Mar. 1977.
[8] L. O. Chua and D. M. Green, "Graph-theoretic properties of dynamic nonlinear networks," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 292–312, May 1976.
[9] T. Matsumoto, L. O. Chua, and A. Makino, "On the implications of capacitor-only cut sets and inductor-only loops in nonlinear networks," *IEEE Trans. Circuits Syst.*, vol. CAS-26, pp. 828–845, Oct. 1979.
[10] A. N. Willson, Jr., "The no-gain property for networks containing three-terminal elements," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 678–687, Aug. 1975.
[11] L. O. Chua, "Nonlinear network theory," lecture notes for EECS 223, University of California, Berkeley, winter 1978.
[12] C. A. Desoer and F. F. Wu, "Nonlinear monotone networks," *SIAM J. Appl. Math.*, vol. 26, no. 2, pp. 315–333, Mar. 1974.
[13] C. A. Desoer and E. S. Kuh, *Basic Circuit Theory*. New York: McGraw-Hill, 1969.

[14] L. O. Chua, "Stationary principles and potential functions for nonlinear networks," *J. Franklin Inst.*, vol. 296, no. 2, pp. 91–114, Aug. 1973.
[15] D. M. Wolaver, "Some applications in electrical network theory of a linear graph theorem by Berge and Ghouila-Houri," in *Mathematical Aspects of Electrical Network Analysis*, Providence, Rhode Island, in *SIAM-AMS Proc.*, vol. III, 1971.
[16] P. Penfield, Jr., R. Spence, and S. Duinker, *Tellegen's Theorem and Electrical Networks.* Cambridge, Mass.: MIT Press, 1970.
[17] P. R. Bryant, "The order of complexity of electrical networks," *Proc. Inst. Elect. Eng., (London)* Monograph 335E, vol. 106C, pp. 174–188, June 1959.
[18] T. Ohtsuki and H. Watanabe, "State variable analysis of RLC networks containing nonlinear coupling elements," *IEEE. Trans. Circuit Theory*, vol. CT-16, pp. 26–39, Feb. 1969.
[19] R. K. Brayton and J. K. Moser, "A theory of nonlinear networks —I and II," *Quart. Appl. Math.*, vol. 22, pp. 1–33, Apr. 1964, and pp. 81–104, July 1964.
[20] L. O. Chua and N. N. Wang, "Complete stability of autonomous reciprocal nonlinear networks," *Circuit Theory and Appl.*, vol. 6, pp. 211–241, 1978.
[21] T. Ohtsuki, Y. Ishizaki, and H. Watanabe, "Topological degree of freedom and mixed analysis of electrical networks," *IEEE Trans. Circuit Theory*, vol. CT-17, pp. 491–499, Nov. 1970.
[22] M. Iri, *Network Flow, Transportation and Scheduling, Theory and Algorithms.* New York, Academic, 1969.
[23] J. B. Dennis, *Mathematical Programming and Electrical Networks.* Cambridge, MA.: M.I.T. Tech. Press, and New York: Wiley, 1959.



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