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Generalized invariant polynomials and the generalized companion form

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A precise algebraic basis is constructed for the so-called generalized rational canonical form that was recently introduced by Tan and Vandewalle (1986 a). The development is based on the concept of generalized invariant polynomials. The introduction of the generalized companion form is crucial for the whole theory.

1. Introduction

In a recent paper (Tan and Vandewalle 1986 a), we have presented a new canonical form (we shall call it the generalized rational canonical form in this paper) for regular pencils of matrices. In the present paper, by connecting this canonical form with the theory of generalized invariant polynomials, a formal theoretical basis for the form is built. Within this framework, questions such as uniqueness can be answered easily and satisfactorily.

The paper is organized as follows. First the notion of generalized invariant polynomials is introduced in the next section. Then in § 3 a criterion for strict equivalence of two regular pencils is stated. This criterion is crucial and serves as the basis for the development of the new canonical form. A full account of the construction of the canonical form follows this criterion. Two examples are also used to demonstrate the theorem. Section 4 contains conclusions and remarks.

2. Generalized invariant polynomials

Let $Es - A$ be an $n \times n$ regular pencil of matrices, i.e. $\det(Es - A) \neq 0$ for almost all s. For simplicity of the discussion, it is further assumed that all entries of the matrices in this paper are real.

In order to preserve the possible infinite elementary divisors, a parameter t is added, so that the given pencil is put in the homogeneous form $Es - At$ (Gantmacher 1960). It is easy to see that the minors of order k ($k = 1, \ldots, n$) are homogeneous polynomials of parameters s and t. Denoting by $D_k(s, t)$ the monic greatest common divisor of all minors of order k of the pencil $Es - At$, and setting $D_0(s, t) = 1$, we obtain a sequence of homogeneous polynomials $\{D_k(s, t)\}$ ($k = 0, \ldots, n$). Since each $D_{k-1}(s, t)$ divides $D_k(s, t)$ ($k = 1, \ldots, n$), $D_k(s, t)/D_{k-1}(s, t)$ is also a homogeneous polynomial. We have the following definition.

Definition 1

For a regular pencil of matrices $Es - A$, the homogeneous polynomials

$$i_1(s, t) = \frac{D_n(s, t)}{D_{n-1}(s, t)}, \quad i_2(s, t) = \frac{D_{n-1}(s, t)}{D_{n-2}(s, t)}, \quad \ldots, \quad i_n(s, t) = \frac{D_1(s, t)}{D_0(s, t)} \tag{1}$$

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are called the *generalized invariant polynomials*. In (1), $D_k(s, t)$ $(k = 1, ..., n)$ is the monic greatest common divisor of all minors of order k of the pencil $Es - At$, and $D_0(s, t)$ is defined to be 1.

Two immediate consequences of the above definition are that none of the generalized invariant polynomials can be identically zero and they cannot all be 1 either. Moreover, the following interesting property is possessed by these polynomials.

Lemma

In the sequence of generalized invariant polynomials $i_1(s, t), ..., i_n(s, t)$, each polynomial $i_{j+1}(s, t)$ divides the preceding one, $i_j(s, t)$ $(j = 1, ..., n - 1)$.

Proof

Let us note two simple facts. First, if s is set to 1, we get

$$p_k(t) = i_k(1, t) \quad (k = 1, ..., n) \tag{2}$$

a set of invariant polynomials of the pencil $E - At$. Hence, by examining the basic properties of the invariant polynomials of a polynomial matrix, the sequence $\{p_k(t), k = 1, ..., n\}$ will be such that from $p_2(t)$ onwards each divides the preceding one (see e.g. Barnett 1971).

Secondly, an analogous property will hold if, instead of s, t is set equal to 1.

Now choose arbitrarily $i_k(s, t)$ and $i_p(s, t)$ in (1), with $k < p$. Split them into powers of irreducible homogeneous polynomials; then

$$\begin{aligned} i_k(s, t) &= f_1^a(s, t) \dots f_u^b(s, t) \\ i_p(s, t) &= g_1^c(s, t) \dots g_v^d(s, t) \end{aligned} \tag{3}$$

where $a, ..., b, c, ..., d, u, v$ are non-negative integers. This can always be done as far as the homogeneous polynomials are concerned. However, it may not be possible with general polynomials of two variables. We claim that each $g_r(s, t)$ $(r = 1, ..., v)$ must be the same as one of the $f_j(s, t)$ $(j = 1, ..., u)$. If not, by setting $s = 1$ (or $t = 1$), the first (or second) observation given above will be contradicted. Thus $i_p(s, t)$ can in fact be split into

$$i_p(s, t) = f_1^e(s, t) \dots f_u^h(s, t) \tag{4}$$

where $e, ..., h$ are non-negative integers.

We must also have $a \geqslant e, ..., b \geqslant h$. Otherwise, setting $s = 1$ (or $t = 1$) will again lead to a contradiction. The lemma is proven. □

To end this section, we remark that generalized invariant polynomials are defined as a set of homogeneous polynomials in Definition 1. In fact, there is a natural correspondence between homogeneous polynomials and polynomials with one variable (possibly of zero leading coefficient). Thus the use of the parameter t can be avoided, provided that an adequate set of operations is defined to properly handle polynomials with zero leading coefficient.

3. Construction of the canonical form

Given two pencils $Es - A$ and $E_1 s - A_1$, each of dimension $n \times n$, they are said to be strictly equivalent if there exist constant square non-singular matrices M and N such that (see e.g. Gantmacher 1960)

$$M(Es - A)N = E_1 s - A_1 \tag{5}$$

When $Es - A$ and $E_1 s - A_1$ are both regular we can state the following criterion for their strict equivalence.

Theorem 1

Two regular pencils $Es - A$ and $E_1 s - A_1$ are strictly equivalent if and only if they have the same set of generalized invariant polynomials.

Proof

Sufficiency. Suppose that these two pencils have the same set of homogeneous invariant polynomials. Then they have the same set of (finite and infinite) elementary divisors. Hence they are strictly equivalent.

Necessity. If they are strictly equivalent then there exist two non-singular matrices M and N such that

$$M(Es - At)N = E_1 s - A_1 t \tag{6}$$

By applying the Binet–Cauchy theorem to relate the minors of both $Es - At$ and $E_1 s - A_1 t$ via (6), it follows easily that

$$D_k(s, t) = \tilde{D}_k(s, t) \quad (k = 1, \ldots, n)$$

where $D_k(s, t)$ is the monic greatest common divisor of all minors of order k of $Es - At$, and $\tilde{D}_k(s, t)$ is that of $E_1 s - A_1 t$. Hence, by Definition 1, these two pencils have the same set of generalized invariant polynomials. The theorem is proved. □

The reader may observe the difference between this theorem and that given in Gantmacher (1960). Although the theorem is important in its own right, here, we shall use it as a basis for the subsequent derivation of the canonical form.

Select arbitrarily from (1) a generalized invariant polynomial $i_p(s, t) \neq 1$ $(1 \leqslant p \leqslant n)$. Then $i_p(s, t)$ can be represented in the standard form

$$i_p(s, t) = a_0 s^k + a_1 s^{k-1} t + \ldots + a_k t^k \tag{7}$$

where not all the a_j $(j = 0, \ldots, k)$ are zero, but a_0 can be zero. Consider the $k \times k$ pencil

$$Cs - L = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & a_0 \end{bmatrix} s - \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_k \\ 1 & 0 & \ldots & 0 & -a_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_1 \end{bmatrix} \tag{8}$$

It is readily checked that

$$\det (Cs - Lt) = a_0 s^k + a_1 s^{k-1} t + \ldots + a_k t^k = i_p(s, t)$$

Since a_0 can be zero, C may be a singular matrix. Moreover, from

$$Cs - Lt = \begin{bmatrix} s & 0 & \dots & 0 & a_k t \\ -t & s & \dots & 0 & a_{k-1} t \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -t & a_0 s + a_1 t \end{bmatrix}$$

it can be verified that the minor of $a_k t$ is $(-1)^{k-1} t^{k-1}$ and that of $a_0 s + a_1 t$ is s^{k-1}. Hence $D_{k-1}(s, t) = 1$. The set of generalized invariant polynomials of (8) will be

$$\tilde{i}_1(s, t) = \frac{D_k(s, t)}{D_{k-1}(s, t)} = D_k(s, t) = a_0 s^k + a_1 s^{k-1} t + \dots + a_k t^k$$

and $\tilde{i}_2(s, t) = \dots = \tilde{i}_k(s, t) = 1$. Thus $Cs - L$ only has a single generalized invariant polynomial different from 1, say $\tilde{i}_1(s, t) = i_p(s, t) = \det(Cs - Lt)$. For obvious reasons, we shall call (8) the *generalized companion form* with respect to $i_p(s, t)$.

Let $Es - A$ be an arbitrary regular pencil with a set of generalized invariant polynomials

$$i_1(s, t), \quad i_2(s, t), \quad \dots, \quad i_e(s, t), \quad i_{e+1}(s, t) = 1, \quad \dots, \quad i_n(s, t) = 1 \tag{9}$$

Then the generalized companion forms with respect to $i_d(s, t)$ $(d = 1, \dots, e)$ (only those polynomials different from (1) can be constructed in the way explained above. Denoting them by $C_1 s - L_1$, $C_2 s - L_2, \dots, C_e s - L_e$, a quasi-diagonal pencil of dimension $n \times n$ can be built up using these as diagonal blocks:

$$\bar{C}s - \bar{L} = \text{block diag}\,\{C_1 s - L_1, C_2 s - L_2, \dots, C_e s - L_e\}$$

$$= \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & C_e \end{bmatrix} s - \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & L_e \end{bmatrix} \tag{10}$$

$\bar{C}s - \bar{L}$ will be called the *generalized rational canonical form.* This term is justified by the following theorem.

Theorem 2

Every regular pencil is strictly equivalent to a generalized rational canonical form (10). This form has the following properties:

(i) it is quasi-diagonal;

(ii) each of its diagonal blocks is a generalized companion form;

(iii) the dimensions of these diagonal blocks are in descending order.

Proof

According to Theorem 1, the only thing we have to do is to check if pencils $Es - A$ and $\bar{C}s - \bar{L}$ (as in (10)) have the same set of generalized invariant polynomials. This is obvious, since both the structure of (10) with (8) as diagonal blocks and the properties of generalized invariant polynomials indicated in the lemma allow easy and direct

computation of $\bar{D}_k(s, t)$ $(k = 1, \ldots, n)$, which turns out to be

$$\bar{D}_k(s, t) = i_{n-k+1}(s, t) \ldots i_n(s, t) \tag{11}$$

□

It is important to note that the form (10) is determined entirely by the set of generalized invariant polynomials. Since for a regular pencil the set of generalized invariant polynomials is unique, (10) is also unique. This explains why (10) is called a canonical form. Another interesting feature is that when E is non-singular for a pencil $Es - A$, the canonical form (10) is actually the rational canonical form for the matrix $E^{-1}A$.

To conclude this section, we shall work out two illustrative examples. These examples are taken from Gantmacher (1960, p. 26).

Example 1

A regular pencil is given as

$$Es - A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} s - \begin{bmatrix} -2 & -1 & -3 \\ -3 & -2 & -5 \\ -3 & -2 & -6 \end{bmatrix} \tag{12}$$

The set of generalized invariant polynomials can be computed as $i_1(s, t) = st^2 + t^3$, $i_2(s, t) = i_3(s, t) = 1$. The generalized rational canonical form for (12) is

$$\bar{C}s - \bar{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} s - \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{13}$$

Example 2

The second pencil is

$$Es - A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} s - \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -1 \end{bmatrix} \tag{14}$$

The set of generalized invariant polynomials is then $i_1(s, t) = st + t^2$, $i_2(s, t) = t$, $i_3(s, t) = 1$. Thus the generalized rational canonical form for (14) is

$$\bar{C}s - \bar{L} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] s - \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & -1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \tag{15}$$

4. Conclusions

The main contribution of this paper is the elaboration of the construction of the canonical form for regular pencils of matrices (Tan and Vandewalle 1986 a). The whole theory is based on the concept of generalized invariant polynomials, through which the set of generalized companion matrices can be built. Although the method

here is not a constructive one (compared with that in Tan and Vandewalle 1986 a), it does provide further insight into the structure of a regular pencil.

Like the Weierstrass canonical form (Gantmacher 1960), wider applications of this canonical form in system theory can also be expected. In fact, based on the structures (10) and (8), some interesting canonical forms for singular systems have been obtained (Tan and Vandewalle 1986 b).

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