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Complete decoupling of linear multivariable systems by means of linear static and differential state feedback

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The concept of complete decoupling is introduced in this paper. We show in particular that for any linear controllable system with non-singular transfer matrix, there always exists a static and differential state feedback such that the closed-loop system is completely decoupled.

1. Introduction

The problem of decoupling a linear multivariable system using state feedback has been a subject of extensive investigation for some time (see Hautus and Heymann 1980 for a concise survey). While much effort has been concentrated on the solvability of the decoupling problem using only the static state feedback, Morse and Wonham (1971) did show, by employing a geometric formulation, that if the dynamic compensation is included, the solvability of the decoupling problem (the extended decoupling problem, in their terminology) can be completely answered. The idea of dynamic compensation has further been clarified by Hautus and Heymann (1980, 1983) who classified a whole range of static and dynamic compensations into the following four categories: dynamic state feedback, static state feedback, precompensation and combined compensation.

The main objective of this paper is to show that for any linear controllable system with square and non-singular transfer matrix, there always exists a static and differential state feedback which is a special class of dynamic state feedback, such that the transfer matrix of the closed-loop system reduces to the unit matrix. This kind of decoupling will be termed the complete decoupling of the system.

The paper is organized as follows. In § 2, some preliminary definitions and conventions are presented to facilitate the ensuing development. The main result is stated and proved in § 3, which also discusses further implications of the result. The final section contains conclusions and remarks.

2. Preliminaries

Let a linear multivariable system be given by

$$\left.\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}\right\} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^m$; and A, B, C are constant matrices of compatible sizes. The transfer matrix of the system is obviously

$$H(s) = C(Is - A)^{-1}B \tag{2}$$

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which is of course a square rational matrix. It will be assumed throughout the paper that $H(s)$ is non-singular. We note that this condition implies that the matrix C has full row rank and the matrix B has full column rank.

We now apply to the system a control law of the form

$$u(t) = K\dot{x}(t) + Fx(t) + Gv(t) \tag{3}$$

where $v(t) \in R^m$ is a new input vector; K, F, G are constant matrices of appropriate dimensions, and G is non-singular. We shall call (3) a *static and differential state feedback*. The effect of such a feedback on (1) can be seen when (3) is substituted for $u(t)$ in (1), which results in

$$\left.\begin{aligned}(I - BK)\dot{x}(t) &= (A + BF)x(t) + BGv(t)\\ y(t) &= Cx(t)\end{aligned}\right\} \tag{4}$$

It then follows directly that the transfer matrix from the new input $v(t)$ to $y(t)$ is given by

$$\tilde{H}(s) = C[(I - BK)s - (A + BF)]^{-1}BG \tag{5}$$

It should be noted that some choices of matrices K, F may cause the pencil $Q(s) = (I - BK)s - (A + BF)$ in (5) to be singular. Therefore, we have to further restrict the matrices K and F to ensure that $Q(s)$ is always non-singular.

Another point we wish to mention is that after the state feedback (3), system (4) may no longer be in the state-space form, depending on whether or not the matrix $I - BK$ is non-singular. It is, however, in the so-called singular system representation (Rosenbrock 1974, Verghese *et al.* 1981). We shall not discuss the properties of singular systems in the present paper; interested readers are referred to Rosenbrock (1974) and Verghese *et al.* (1981) for details.

If system (1) is rewritten in the system matrix form (Rosenbrock 1970)

$$P(s) = \begin{bmatrix} Is - A & B \\ -C & 0 \end{bmatrix} \tag{6}$$

the effect of the static and differential state feedback (3) on (1) can be considered as the following operation applied to $P(s)$:

$$\tilde{P}(s) = \begin{bmatrix} Is - A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -Ks - F & G \end{bmatrix} = \begin{bmatrix} (I - BK)s - (A + BF) & BG \\ -C & 0 \end{bmatrix} \tag{7}$$

The corresponding system represented by $\tilde{P}(s)$ is easily seen to be (4). Moreover, in view of (7), the effect of the feedback can further be treated as a kind of column elementary operation performed on the system matrix $P(s)$, but restricted by (7). The exact nature of these operations is, of course, determined by the constant matrices K, F and G.

3. Complete decoupling by static and differential state feedback

To begin the development, we first define the notion of complete decoupling for linear systems.

Definition 1

A linear system is said to be completely decoupled if its transfer matrix is an identity matrix.

We now state the main result of the paper.

Theorem 1

For any controllable system (1) with non-singular transfer matrix, there exists a static and differential state feedback (3) such that the closed-loop system is completely decoupled.

To prove the theorem, we first need the following lemma.

Lemma 1

The transfer matrix of any controllable system (1) admits a right MFD factorization of the form

$$H(s) = N(s)D^{-1}(s) \tag{8}$$

where $D(s)$ is some non-singular polynomial matrix and $N(s)$ is a polynomial matrix defined by

$$N(s) = CT(s)$$

where

$$T^{\mathrm{T}}(s) = \text{block diag}\,\{[s^{k_i-1} \ \dots \ s \quad 1], \quad i = 1, \dots, m\}$$

k_i $(i = 1, \dots, m)$ is the ith controllability index (see, for example, Kailath 1980).

Proof

Apply a similarity transformation to bring (1) to its controller canonical form (see, for example, Kailath 1980 for a standard derivation):

$$P_c(s) = \begin{bmatrix} L & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Is - A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} Is - A_c & B_c \\ -C & 0 \end{bmatrix} \tag{9}$$

where L is a constant non-singular matrix; C remains unchanged and

$$Is - A_c = \begin{bmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ R_{21} & R_{22} & \dots & R_{2m} \\ \vdots & \vdots & & \vdots \\ R_{m1} & R_{m2} & \dots & R_{mm} \end{bmatrix} \tag{10}$$

where the R_{ii} $(i = 1, \dots, m)$ are companion matrices

$$R_{ii} = \begin{bmatrix} s + r_1 & r_2 & \dots & r_{k_i-1} & r_{k_i} \\ -1 & s & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & s \end{bmatrix}$$

and the R_{ij} $(i \neq j)$ have all rows zero except possibly the first one. While

$$B_c = \begin{bmatrix} B_{c1} \\ \vdots \\ B_{cm} \end{bmatrix} \tag{11}$$

where each B_{ci} is a $k_i \times m$ matrix $(i = 1, \dots, m)$ and

$$B_{ci} = \begin{bmatrix} b_1 & b_2 & \dots & b_m \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \begin{matrix} b_i = 1 \\ b_1 = b_2 = \dots = b_{i-1} = 0 \end{matrix}$$

The structure of B_c enables us to find a non-singular matrix G such that

$$\tilde{B}^{\mathrm{T}} = (B_c G)^{\mathrm{T}} = \text{block diag}\,\{[1 \quad 0 \quad \dots \quad 0], \quad 1 \times k_i, \quad i = 1, \dots, m\} \tag{12}$$

Construct a square polynomial matrix $\tilde{D}(s)$ such that

$$\tilde{B}\tilde{D}(s) = (Is - A_c)T(s) \tag{13}$$

It can then easily be shown, by applying elementary operations, that $\det \tilde{D}(s) = \det(Is - A_c)$. We therefore conclude that $\tilde{D}(s)$ is non-singular, so that (13) can be rearranged into (note that $\tilde{B} = B_c G$)

$$(Is - A_c)^{-1} B_c = T(s)\tilde{D}^{-1}(s)G^{-1} \tag{14}$$

Premultiplying the matrix C on both sides of (14) and defining $D(s) = G\tilde{D}(s)$, we obtain

$$C(Is - A_c)^{-1} B_c = N(s)D^{-1}(s) \tag{15}$$

The left-hand side of (15) is of course the transfer matrix $H(s)$. Hence the lemma is proved. □

It is noted that since $H(s)$ is non-singular, $N(s)$ in (8) (or (15)) must also be non-singular.

We now come to the proof of our main theorem.

Proof of Theorem 1

We shall apply a sequence of elementary operations defined by the feedback (3) to bring (1) to such a form that its transfer matrix is a unit matrix.

To begin with, (1) is put into the controller canonical form (9) by similarity transformations. Since (1) is controllable, this can always be done. For easy reference, (9) is repeated here:

$$P_c(s) = \begin{bmatrix} Is - A_c & B_c \\ -C & 0 \end{bmatrix} \tag{16 a}$$

where $Is - A_c$ is defined by (10), B_c by (11) and C is simply

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix} \tag{16 b}$$

Then we make use of the elements of value 1 in the matrix B_c in (16 *a*) to change the top row of the *i*th block row of $Is - A_c$ (see (10)) $[R_{i1} \quad R_{i2} \quad \ldots \quad R_{im}]$ such that the top row of the *i*th block row equals the *i*th row of the matrix C ($i = 1, \ldots, m$) (note that such changes should be done from the bottom to the top of the block rows of the matrix $(Is - A_c)$). Equivalently, there exist matrices K and F such that

$$\begin{bmatrix} \tilde{E}s - \tilde{A} & B_c \\ -C & 0 \end{bmatrix} = \begin{bmatrix} Is - A_c & B_c \\ -C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Ks - F & I \end{bmatrix} = \begin{bmatrix} M_1(s) \\ M_2(s) \\ \vdots \\ M_m(s) \end{bmatrix} \tag{17}$$

where $\tilde{E} = I - B_c K$, $\tilde{A} = A_c + B_c F$ and

$$M_i(s) = \begin{bmatrix} c_{i1} & c_{i2} & \ldots & c_{ir_i} & c_{i,r_i+1} & \ldots & c_{i,r_{i+1}-1} & c_{ir_{i+1}} & \ldots & c_{in} \\ & & & -1 & s & \ldots & 0 & 0 & & \\ & & & \vdots & \vdots & & \vdots & \vdots & & \\ & & & 0 & 0 & \ldots & -1 & s & & \end{bmatrix}$$

in the above, $k_0 = 1$,

$$r_i = \sum_{j=0}^{i-1} k_j$$

and the c_{ij} are elements of C defined in (16 *b*); the remaining elements not shown in $M_i(s)$ are zeros.

We now show that the transfer matrix of (17) is indeed the unit matrix.

First observe that the pencil $\tilde{E}s - \tilde{A}$ can easily be brought by elementary operations to the form

$$\begin{bmatrix} N(s) & 0 \\ 0 & I \end{bmatrix} \quad (N(s) \text{ is defined in (8)})$$

Since it follows from Lemma 1 that $N(s)$ must be non-singular, we then know that $\tilde{E}s - \tilde{A}$ must be non-singular as well.

If the matrix G in (3) is so chosen that $\tilde{B} = B_c G$ is given by (12), it is readily seen that the following equation holds:

$$\tilde{B}N(s) = (\tilde{E}s - \tilde{A})T(s) \tag{18}$$

Rearranging (18) and substituting $B_c G$ for $\tilde{B}$ yield

$$(\tilde{E}s - \tilde{A})^{-1} B_c G = T(s)N^{-1}(s) \tag{19}$$

Premultiplying both sides of (19) by C, we arrive at

$$\tilde{H}(s) = C(\tilde{E}s - \tilde{A})^{-1} B_c G = CT(s)N^{-1}(s) = N(s)N^{-1}(s) = I$$

This completes the proof. □

It is known that a static state feedback can always be implemented as a precompensator for the system (Hautus and Heymann 1983). We can check quite straightforwardly that this is also true for static and differential state feedback. From this point of view, what has been done above can in fact be considered as a

construction of the inverse of the original system, this inverse turns out to be of the form

$$H^{-1}(s) = [I - (Ks + F)(Is - A)^{-1} B]^{-1} G \tag{20}$$

Therefore, the theorem also provides an alternative way of constructing the inverse of linear systems.

4. Conclusions

In this paper, we have shown that for any controllable system with non-singular transfer matrix, there always exists a static and differential state feedback which completely decouples the system. However, this can never be achieved by using the static state feedback only. Roughly speaking, the reason for this is that, unlike the pure static state feedback, the static and differential state feedback can arbitrarily alter the poles of the system, and thus can cause the complete cancellation between zeros and poles (finite and at infinity) for controllable systems.

It should also be noted that Theorem 1 only holds for controllable systems. Our conjecture is, however, that the conclusion should remain true without the controllability assumption. A proof for this more general statement perhaps needs a different approach.

References

HAUTUS, M. L. J., and HEYMANN, M., 1980, *Proc. 4th Int. Conf. on the Analysis and Optimization of Systems*, p. 562; 1983, *I.E.E.E. Trans. autom. Control*, **28**, 823.

KAILATH, T., 1980, *Linear Systems* (Englewood Cliffs, NJ: Prentice-Hall).

MORSE, A. S., and WONHAM, W. M., 1971, *I.E.E.E. Trans. autom. Control*, **16**, 568.

ROSENBROCK, H. H., 1970, *State Space and Multivariable Theory* (London: Nelson); 1974, *Int. J. Control*, **20**, 191.

VERGHESE, G. C., LEVY, B. C., and KAILATH, T., 1981, *I.E.E.E. Trans. autom. Control*, **26**, 811.