ISCAS '87

89-23C

ALL NONNEGATIVE SOLUTIONS OF SETS OF LINEAR EQUATIONS AND THE LINEAR COMPLEMENTARITY PROBLEM .

Bart De Moor ; Joos Vandewalle

ESAT Laboratory - Katholieke Universiteit Leuven
Kardinaal Mercierlaan 94 / 3030 Heverlee / Belgium
tel. 016-220931 / telex 25941 elekul

Abstract: In this paper a novel geometrically inspired solution strategy is presented to three fundamental linear algebraic problems that have a lot of applications in system theory and network analysis and simulation :
- Problem 1 : - Given a nx1 vector p. Find all nonnegative nx1 vectors x that are orthogonal to p : $p^t.x = 0$.
- Problem 2 : - Given a mxn matrix P. Find all nonnegative nx1 vectors x that are orthogonal to all rows of P : P . x = 0.
- Problem 3 : - The linear complementarity problem : Given a nx1 vector r and a nxn matrix M, find all nonnegative nx1 vectors w and z such that M.z+r=w and $w^t.z = 0$ and the generalized linear complemetarity problem: Given a nx1 vector r, nxn matrices M and N, find all nonnegative numbers a and nonnegative vectors w and z such that M.z+a.r=N.w and $w^t.z=0$.
The algorithm that is presented computes all solutions to the generalized linear complemetarity problem in a numerically reliable way without imposing any a priori conditions to the matrices M and N . Moreover, the solution set is described in a geometrical way.

1. Introduction and problem statement.

Orthogonality of vectors with respect to the 'ordinary' Euclidean inner product is the key tool for the analysis of the solution of systems of linear equations. The kernel of a mxn matrix P of rank r is defined as $\ker(P) = \{ x \in R^n \mid P.x=0 \}$
Ker(P) is a n-r dimensional subspace of the n-dimensional ambient vectorspace R^n and is the orthogonal complement of the row space of the matrix P. There exist a lot of algebraic and numerical methods to compute a basis for the orthogonal complement (Gram - Schmidt orthogonalization, QR - factorization, singular value decomposition,... see [1] for a survey).
However, in a lot of applications, there is an additional constraint on the vectors of the kernel : all components of the must be nonnegative .
They may for instance represent electrical, physical, economical, chemical,... quantities that make only sense when they are nonnegative. The asymptotic stability of certain systems of differential equations [4] depends upon the existence of nonnegative solutions to a system of linear equations. The linear complementarity problem (LCP) is equivalent to the nonnegative solution of a system of linear equations with orthogonality constraints : it appears in structural analysis, linear and quadratic convex programming,...(see [2] and [7] for a survey) and in the piecewise linear modelling and analysis of VLSI circuits [3]. Typical problems in computational geometry as convex hull determination and maximization of an object function with nonnegativity constraints can be stated in terms of nonnegative solutions of linear equations [5]. It can be concluded that the problem of finding all nonnegative solutions of linear equations frequently appears in a lot of applications in signal analysis, circuit and system theory , VLSI design and image processing.
This paper is organised as follows: In section 2 we present a new geometrical solution strategy to find all nonnegative solutions to a set of linear equations. The linear complementarity problem is formulated in section 3 and a very simple geometrical solution scheme is presented.
The advantages of the approach are its simplicity and its geometrical motivation. The proofs of the results (which are omitted for brevity) will be given elsewhere together with a more detailed description of the applications.

2. Nonnegative solutions of a set of linear equations.

In this section, the set of all nonnegative vectors orthogonal to a given vector will be characterized. This result will be used in the construction of the nonnegative solution set of linear equations.
The following lemma is classical and well known.

Lemma 1: The set of vectors $x \geq 0$, satisfying $p^t.x=0$, where p is an nx1 vector, is a polyhedral convex cone.
Proof: For a proof we refer to [6].

Our first theorem describes the procedure to determine in a very straightforward way the generators of the polyhedral cone, containing all nonnegative vectors orthogonal to a given vector p.

Theorem 1: Given a nx1 vector $p \in R^n$. The generators of the polyhedral convex cone of all nonnegative vectors x satisfying $p^t.x=0$ are the columnvectors a_i of the matrix A, to be constructed in the following way :
- For each pair p_i, p_j (i<j) of components of p such that $p_i.p_j<0$ add a column to A of the form :

$$[0 \,..0\, |p_j|\, 0\, .\, .0\, |p_i|\, 0\, ..\, 0]^t$$
(with p_j at position i and p_i at position j)

- For each zero component $p_k=0$ of p, add a column to A of the form:

$$[0\, ...\, 0\, 1\, 0\, ...\, 0]$$
(with 1 at position k)

The set of all nonnegative vectors orthogonal to the given vector p is then formed by all nonnegative linear combinations of the columns of the matrix A. ∎

Let us clarify this result with the following example.

Example 1: $p^t=[\ 1\ -2\ 0\ 3\ 1]$. Find all nonnegative x such that $p^t.x=0$. The matrix A here is the 5x4 matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

All nonnegative vectors x satisfying $p^t.x=0$ are given by $x = A\ .\ v$ where v is an arbitrary nonnegative vector of R^4.

Remark 1: Let s^+, s^-, z be the number of positive ,negative and zero components of the vector p. Then the number of columns of the matrix A equals $s^+.s^- + z$.

Remark 2: The zero-structure of generators of the convex polyhedral solution cone (the columns of the matrix A), reveals their geometrical interpretation : They all lie in the ordinate planes on the boundaries of the positive orthant.

We are now ready to establish the second fundamental theorem of this work. First, let us remind the following classical lemma:

Lemma 2: The intersection of finitely many polyhedral convex sets is polyhedral.
Proof: For a proof we refer to [6].

Clearly, if it exists, the set of nonnegative solutions x to P.x=0, where P is a mxn matrix ,is the intersection of m polyhedral convex cones and hence is polyhedral. The second theorem provides a very simple constructive algorithm to find the generators of the nonnegative solution set of a set of linear equations.

Theorem 2: Given a mxn matrix P. The generators of the convex polyhedral cone of all nonnegative vectors x of the kernel of P, are the columns of the matrix $A = A_1.A_2.\ \ldots\ .A_m$ where the matrices A_i are determined as follows:
- A_1 contains the generators of the polyhedral cone of nonnegative vectors orthogonal to the first row of P. A_1 is determined as in theorem 1.
- The columns of the matrix $A_i (i\geq 2)$ are determined according to theorem 1 and are the generators of the polyhedral cone of nonnegative vectors orthogonal to the i-th row of $(P.A_1.A_2 \ldots A_{i-1})$. ∎

Let us again clarify this result with a simple example:

Example 2:

$$P = \begin{bmatrix} -3 & -1 & 3 & 0 & 1 \\ 4 & 2 & 0 & 1 & -3 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix}$$

From theorem 1, we have :

$$A_1 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 \end{bmatrix}$$

$$P.A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 12 & -5 & 6 & -1 & 1 \\ 6 & -2 & 4 & 0 & 0 \end{bmatrix}$$

From theorem 1 and 2 :

$$A_2 = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 12 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 5 & 0 & 1 \end{bmatrix}$$

$$P.A_1.A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 6 & 8 & -2 & 4 & 0 \end{bmatrix}$$

From theorem 1 and 2 :

$$A_3 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 6 & 6 & 8 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we have that:

$$A_1.A_2.A_3 = \begin{bmatrix} 60 & 12 & 20 & 4 & 0 \\ 0 & 24 & 30 & 18 & 1 \\ 30 & 6 & 10 & 2 & 0 \\ 30 & 30 & 40 & 20 & 1 \\ 90 & 42 & 60 & 24 & 1 \end{bmatrix}$$

All nonnegative solutions x to P.x=0 are now generated by nonnegative linear combinations of the columns of A.

Remark 3: When for $k \leq m$, $P.A_1 \ldots A_k = 0$, the matrix $A=A_1 \ldots A_k$ contains all the generators of the nonnegative polyhedral solution cone. When for $k = m$, $P.A_1 \ldots A_k \neq 0$, then there is no solution. Also, when for $k \leq m$, $P.A_1 \ldots A_k$ contains only strictly positive numbers, a nonnegative solution does not exist.

3. The linear complementarity problem (LCP).

The simulation of complex electronic networks has become one of the important and vital tasks in the design of large scale integrated circuits. In a lot of cases it is natural to use the linear piecewise description as a basis for modelling, whereby the behavior of electronic components is approximated by piecewise linear functions specified by a list of mappings in polyhedral regions. For an approximation of sufficient accuracy, the number of required regions may become quite large and the description is still a collection of local linear descriptions without strong interrelationship, except for continuity demands at the region boundaries. As is shown in [3] , the linear complementarity formulation offers an implicit global description of piecewise linear mappings, which avoids excessive searching and localizing the particular region in which the solution is to be found. Moreover, the LCP formulation allows for global piecewise linear modelling of components and networks, interconnecting them and adding dynamic action in an elegant and straightforward way [3].
The Linear Complemetarity Problem is the following:

For a given mx1 vector $r \in R^m$, a mxm matrix $M \in R^{mxm}$, find (or conclude there are no) nonnegative vectors $w \in R^m$ and $z \in R^m$, such that :

$$w = r + M\ .\ z \qquad w \geq 0,\ z \geq 0\ ,\ w^t.z=0 \quad (\ I\ )$$

In the literature a lot of algorithms can be found to solve this LCP [2] [3] [7] . In all cases explicit properties of the matrix M are studied, in order to guarantee uniqueness of the solutions because the algorithm only computes one solution at a time.
In this paper we consider the Generalized Linear Complemetarity Problem (GLCP):

Given a mx1 vector $r \in R^m$, mxm matrices M and $N \in R^{mxm}$.
Find (or conclude there are not) nonnegative vectors $w,z \in R^m$ and a nonnegative number $a \in R$, such that :

$$N\ .\ w = M\ .\ z + a\ .\ r \qquad w \geq 0,\ z \geq 0,\ a \geq 0\ ,\ w^t.z=0 \quad (\ II\ )$$

As can be seen, the LCP is a special case of the GLCP.

The algorithm we will derive is based on the results of section 2 and has the following desirable properties:

- One need not to invert the matrix M or N in order to reconvert the Generalized Linear Complementarity Problem into the "ordinary" LCP. In fact, in our scheme, the matrices M and N may be very ill conditionned numerically (high condition number) or may even be singular.
- All solutions are found at once . Hence there is no need to run the algorithm several times in order to compute possible other solutions when the conditions for uniqueness of the solution are not satisfied, as is the case for the existing methods.
- Also solutions for a = 0 will be found. These are solutions for the case where the vector r disappears from the problem formulation (Typically, for an electronic circuit, this vector contains the voltage and current sources. Hence r = 0 or a = 0 corresponds to a network without sources).

The algorithm for solving the GLCP is based on the following two observations:

Property 1: The solutions to the GLCP M.z + a.r = N.w for given M , N , r are nonnegative solutions to the set of linear equations:

$$\begin{bmatrix} N & -M & -r \end{bmatrix} \cdot \begin{bmatrix} w \\ z \\ a \end{bmatrix} = 0$$

where [N -M -r] is a m x (2m+1) matrix .

So, in order to solve the GLCP, one could use the algorithm of theorem 2 in section 2 . However, not all nonnegative solutions to the set of equations will satisfy the complementarity - orthogonality condition $w^t.z=0$. But the following important observation allows to reduce dramatically the number of vertices obtained in each step of the algorithm of theorem 2.

Property 2

Let $\begin{bmatrix} w_1 \\ z_1 \\ a_1 \end{bmatrix}$ and $\begin{bmatrix} w_2 \\ z_2 \\ a_2 \end{bmatrix}$ be two nonnegative vectors and

let $\beta_1 \geq 0$ and $\beta_2 \geq 0$ be two nonnegative real numbers. Define

$$\begin{bmatrix} w \\ z \\ a \end{bmatrix} = \beta_1 . \begin{bmatrix} w_1 \\ z_1 \\ a_1 \end{bmatrix} + \beta_2 . \begin{bmatrix} w_2 \\ z_2 \\ a_2 \end{bmatrix}$$

Then $w^t.z=0$ if and only if $w_1^t.z_1=0$, $w_2^t.z_2=0$, $w_1^t.z_2=0$, $w_2^t.z_1=0$.

Proof: Trivial.

This property implies that a nonnegative vector, that is a nonnegative combination of two nonnegative vectors, satisfies the orthogonality - complementarity condition if both vectors satisfy the complementarity condition and moreover, the cross-complementarity conditions $w_1.z_2=0$ and $w_2.z_1=0$ are satisfied.

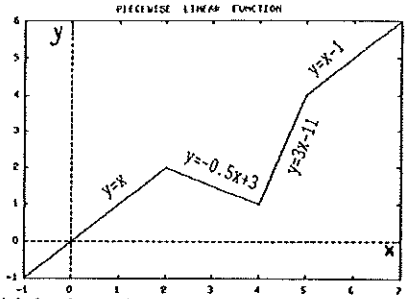
The algorithm of theorem 2 computes the subsequent products $A_1.A_2...A_i$ for i=1,m where m is the number of rows of the matrix [N -M -r].
Using property 2, it can easily be seen that column vectors in the productmatrix $A_1...A_i$ that do not satisfy themselves the complementarity condition,can be eliminated, because if they are involved in a nonnegative combination, the resulting vector will never satisfy the complementarity condition.

Hence, for every i (i=1,...m), the columnvectors of $A_1..A_i$ can be checked for the complementarity condition and the vectors not satisfying this condition can simply be omitted.
If this strategy is applied, the following phenomenon can be observed: As for low i the matrix $A_1..A_i$ is still sparse, the complementarity condition is "easily" satisfied for a lot of vectors of $A_1..A_i$. The number of columns in $A_1..A_i$ increases for low values of i. However, for increasing i, the sparsity of the productmatrix $A_1..A_i$ disappears and it becomes more and more difficult to satisfy the complementarity condition. Hence a lot of vectors in $A_1..A_i$ can be omitted and the number of columns is reduced dramatically for each step i.

4. Examples.

4.1. Example 1.

This example will be simple to show how the algorithm operates. Consider the following piecewise linear model:



A LCP model for it can be constructed (see [3] for more details):

$$\begin{bmatrix} y \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 7/2 & -2 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

Here v_1,v_2,v_3 and i_1,i_2,i_3 are the components of the vectors of the complementarity problem that have to satisfy $v_i,i_j \geq 0$ i=1,2,3 and $\Sigma\, v_i.i_j = 0$.
We now show the solution for the case x=3.

$$R = [\, N \; -M \; -r \,] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 \end{bmatrix}$$

$$A_1 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Clearly, the first column of A_1 does not satisfy the complementarity condition and can be omitted.

$$R.A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} \qquad A_1.A_2 = \left[\begin{array}{cccc} 0&0&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ \hline 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ \hline 1&0&0&0 \end{array}\right]$$

Clearly, the second column of $A_1.A_2$ does not satisfy the complemetarity condition and can be omitted.

$$R.A_1.A_2 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ -2&1&-1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1&0 \\ 2&1 \\ 0&1 \end{bmatrix} \qquad A_1.A_2.A_3 = \left[\begin{array}{cc} 0&0 \\ 1&0 \\ 2&1 \\ \hline 1&0 \\ 0&0 \\ 0&1 \\ \hline 1&0 \end{array}\right]$$

The second column of $A_1.A_2.A_3$ does not satisfy the complementarity condition and can be omitted.
The solution of the LCP is given by $\{v_1\ v_2\ v_3\}=\{0\ 1\ 2\}$, $\{i_1\ i_2\ i_3\}=\{1\ 0\ 0\}$. Hence $y = 3 - 3/2 = 3/2$.

Remark:
The algorithm can be easily generalized to compute the result for several inputs simultaneously. This extension however will not be shown here.

4.2. Example 2.

The following example illustrates the power of the new geometrical approach.

$$M = \begin{bmatrix} 0&0&1&2 \\ 2&1&-2&0 \\ 4&2&-3.9&0.2 \\ -2&-1&2.9&1.8 \end{bmatrix} \quad N = \begin{bmatrix} 1&2&4&-1 \\ 2&-1&0&0 \\ 4.1&-1.8&0.4&-0.1 \\ -1.1&2.8&3.6&-0.9 \end{bmatrix} \quad r = \begin{bmatrix} 8 \\ -22 \\ -43.2 \\ 29.2 \end{bmatrix}$$

The GLCP to be solved is : find $a,w,z \geq 0$, $w^t.z=0$ such that

$$N . w = M . z + a . r$$

Observe that both the matrices M and N are singular (and that this is the case for all possible reorderings of the columns of M and N) so that this GLCP can never be converted to an ordinary LCP.
The algorithm delivers 22 solutions:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
w	1	1	1	2	1	4	2	8	8	8	0	0	0	0	0	0	0	0	0	0	0	0
	2	0	4	8	0	8	0	38	38	0	4	22	22	8	2	22	22	4	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2	0	1	2	2	2
	5	1	8	0	0	0	0	76	0	0	6	36	44	0	0	0	0	0	4	0	0	0
z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	0	9	0	0	11	0
	0	2	0	0	4	0	4	0	0	38	0	0	0	0	0	0	0	0	0	0	0	22
	0	0	1	2	1	10	0	0	0	0	2	0	0	4	4	0	0	0	0	0	0	0
	0	0	0	8	0	0	1	0	38	0	0	0	0	6	0	18	22	0	0	4	0	0
a	0	0	0	0	0	0	0	1	1	1	0	1	1	0	0	1	1	1	0	0	1	1

This solution set can now be further studied in a geometrical way using the results of section 2. For instance, convex polyhedral cones in which all vectors are solutions satisfying the complementarity condition, can be determined. One such a polyhedral cone has the vertices 1 - 3 - 8 - 11 - 12. Any nonnegative combination of these vertices is a solution to the GLCP.

5. Conclusion.

In this paper, we have presented a very simple and geometrically inspired answer to the non-trivial linear algebraic problem of determining all nonnegative solutions to a set of linear equations. This problem is important in a lot of identification and signal analysis problems, in computational geometry and in the piecewise linear modelling of electronic circuits.
This geometrical approach resulted in a powerful algorithm for the generalized linear complementarity problem. The advantages of this algorithm are : It computes all solutions of the GLCP without imposing any a priori conditions to the matrices as is the case for all other existing algorithms. The computations are extremely simple and geometrically interpretable. There is no need to invert any matrix, even singular cases can be solved.

Future work will be done to optimize the number of computations be looking for the optimal order in which the rows of the matrix [N -M -r] are "deflated". Also the conditioning and numerical reliability of the GLCP will be further studied.

6.References.

1. Golub G.H., Van Loan C.F. Matrix Computations. North Oxford Academic, Oxford 1983.
2. Berman A., Plemmons R.J. Nonnegative matrices in the mathematical sciences. Computer Science and Applied Mathematics Series, Academic Press, New York 1979.
3. Van Bokhoven W.M.G. Piecewise linear modelling and analysis. Kluwer Technische Boeken, Deventer, 1981.
4. Kaykobad M. Positive solutions of a class of linear systems. Linear Algebra and Applications 72:97-105 (1985).
5. Preparata F.P., Shamos M.I. Computational Geometry, an introduction. Springer Verlag texts and monographs in computer science, New York, 1985.
6. Rockafellar R.T. Convex analysis. Princeton University Press, 1970.
7. B.C. Eaves, F.J. Gould, H.O. Peitgen, M.J. Todd (editors). Homotopy methods and global convergence. Nato Conference series, Series II: Systems science, Plenum Press, New York, 1983.