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IRREDUCIBILITY AND JOINT CONTROLLABILITY OBSERVABILITY IN SINGULAR SYSTEMS

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ABSTRACT

The concept of irreducibility of MFD's at infinity is introduced. By merely considering common divisors at infinity, this concept turns out to be a complement of the well-known irreducibility for MFD's. A combination of these two naturally leads to the notion of complete irreducibility. The second part of the paper is devoted to show that for a so-called pseudo-proper MFD, a block controllable singular system realization can be obtained such that (1). the order of the realized singular system is equal to the generalized determinantal degree of the denominator (GDDD) of this MFD; (2). the realization is controllable; (3). it will be observable if and only if the MFD is completely irreducible. In the final part of the paper, we show that for an arbitrary MFD, any of its nth-order singular system realization (n is GDDD) will be jointly controllable and observable if and only if the MFD is completely irreducible, which is an interesting counterpart of the case for regular systems.

1. INTRODUCTION

It is well-known that in linear systems the concept of irreducibility serves as an alternative characterization of the joint controllability and observability. The precise connection between the two concepts is as follows: starting with a transfer matrix in matrix fraction description (MFD), any of its realizations with the order equal to the degree of the determinant of the denominator matrix will be jointly controllable and observable if and only if the given MFD is irreducible (see, for example, [5]).

Our primary purpose in this paper is to extend the concept of irreducibility to include the infinity, this then leads to the introduction of the notion of complete irreducibility. As an analogous result, we establish the following connection: for a given MFD, any of its realization in singular system with the order equal to the generalized determinantal degree of the denominator matrix will be jointly controllable and observable if and only if the given MFD is completely irreducible.

The development will start by introducing the notion of coprimeness at infinity for two matrix polynomials. By considering exclusively common divisors at infinity, this concept turns out to be a natural counterpart and also a complement of the normal coprimeness which only accounts for finite common divisors. A simple and interesting theorem is also stated as a criterion to check the coprimeness at infinity. The complete coprimeness is then defined as a combination of the normal coprimeness and the newly introduced coprimeness at infinity. As far as an MFD is concerned, the complete irreducibility of the MFD is naturally defined to be the complete coprimeness between the numerator and the denominator matrices. In Section 3, a so-called block controllable realization is constructed from a pseudo-proper right MFD. A number of properties of this realization are presented. This special realization will serve as a tool in Section 4 to prove the general relation between complete irreducibility and joint controllability observability. The conclusion and remarks are given in the final section.

2. COPRIMENESS AT INFINITY AND COMPLETE COPRIMENESS FOR MATRIX POLYNOMIALS

A polynomial with matrix coefficients will be called a matrix polynomial, i.e.,

$$P(s) = P_0 s^k + P_1 s^{k-1} + \ldots + P_k \qquad (1)$$

where $P_i \in R^{n\times m}$, i=0,...,k; R denotes the real field and $R^{n\times m}$ denotes the set of all nxm matrices with entrices in R. Unless specified otherwise, the underlying field will always be assumed to be the real field.

Obviously, (1) may have finite and/or infinite zeros. Finite zeros are defined as those of the invariant polynomials; while in the literature, zeros at infinity are treated quite independently from the finite ones (see [5] and the references therein). In order to study these two kinds of zeros in a unified way, we introduce the following homogeneous form for the matrix polynomial (1)

$$P(s,r) = P_0 s^k + P_1 s^{k-1} r + \ldots + P_k r^k \qquad (2)$$

where r is simply a parameter different from s.

Intuitively, one may think that the properties of (2) at (s,r)=(1,0) will reflect the properties of (1) at infinity. If each P_i (i=0,...,k) is a square matrix (nxn in dimension), the determinant of (2) is defined. It is easily verified that this determinant is a scalar homogeneous polynomial with degree nk. Let the polynomial det P(s,r) be factored as

$$\det P(s,r) = f(s,r).r^v \qquad (3)$$

where v is a nonnegative integer; f(s,r) is some homogeneous polynomial and $f(s,0)\neq 0$. Then we shall say that P(s) has v zeros at infinity. Further, the generalized determinantal degree of (1) is defined to be the degree of det P(s,r), which is obviously equal to nk. By counting the finite zeros which are defined

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in the conventional sense, we can check easily that the number of finite and infinite zeros of a matrix polynomial (multiplicity counted) is precisely its generalized determinantal degree. It should be noted that the above definition of zeros at infinity is not identical to the conventional ones (for example, via the Smith-McMillan form after a conformal mapping, say $s=1/\lambda$). However we shall not discuss this difference here.

For nonsquare matrix polynomials, zeros can be defined as those of the invariant polynomials of (2) (a set of invariant polynomials of the homogeneous form (2) can be obtained by using the well-known formula which is based on the greatest common divisors of the minors of different orders [4, p.141]). For conciseness of the paper, however, we shall not go any further in this direction.

The homogeneous form (2) is only a special case of homogeneous polynomial matrices of which all entries are homogeneous polynomials of different degrees. If we view all such matrices as a set, the matrix operations (addition, multiplication, inversion, etc.) can be defined in the normal way. However, when performing polynomial operations, a special rule should be observed, viz., $a(s,r)+b(s,r)$ and $a(s,r)=b(s,r)$ are defined if and only if $a(s,r)$ and $b(s,r)$ are of the equal (homogeneous) degree, where $a(s,r)$ and $b(s,r)$ are two homogeneous polynomials. The underlying idea is to ensure that after operations the polynomials resulted are also homogeneous so that the properties at infinity are preserved.

Now consider two matrix polynomials $D(s)$ with dimension pxp, and $N(s)$ with dimension mxp. Their homogeneous forms are $D(s,r)$ and $N(s,r)$, respectively. It is further assumed that $D(s)$ is nonsingular (i.e., $\det D(s)\neq 0$). If $D(s,r) = \bar{D}(s,r)R(s,r)$ and $N(s,r) = \bar{N}(s,r)R(s,r)$, where $R(s,r)$ is also a square matrix (so that $R(s,r)$ must be nonsingular), then $R(s,r)$ is called a common right divisor of $D(s,r)$ and $N(s,r)$.

For nontrivial divisors, i.e., divisors which are not square nonsingular constant matrix, because of the restriction on the homogeneous polynomial addition, they will have certain degree patterns. This is to say that the homogeneous degrees of the entries appear in a regular way. The same is true for the multiple (i.e. $\bar{D}(s,r)$ with respect to the divisor $R(s,r)$). It is a routine matter to check that if $D(s,r)$ is given in (2), then the entries of $\bar{D}(s,r)$ in the same column will have the same degree, and the entries of $R(s,r)$ in the same row will have the same degree (zero is considered to have any degree). The former degree pattern is suggestively called column-homogeneous, the latter row-homogeneous.

Definition 1.

A nonsingular matrix $R(s,r)$ is called a common right divisor at infinity of matrix polynomials $D(s)$ and $N(s)$, if (i). it is a common right divisor of $D(s,r)$ and $N(s,r)$; (ii). $\det R(s,r)=c\cdot r^{v}$, where c is a constant different from zero, and v is a positive integer.

The above definition allows to introduce the notion of coprimeness at infinity.

Definition 2.

$D(s)$ and $N(s)$ are said to be right coprime at infinity if they have no common right divisor at infinity.

A simple characterization of this concept can in fact be given as a rank condition of a constant matrix constructed with the leading coefficient matrices of both $D(s)$ and $N(s)$. This characterization can also be considered as a criterion to check if the two given matrix polynomials are right coprime at infinity.

Theorem 1.

$D(s)$ and $N(s)$ are right coprime at infinity if and only if

$$\operatorname{rank}\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} = p \qquad (4)$$

where D_0 and N_0 are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

Proof.

Necessity. Suppose that $D(s)$ and $N(s)$ are right coprime at infinity, but the columns of the matrix $[D_0^T \; N_0^T]^T$ are not independent. Then a nonsingular pxp constant matrix K can be found such that the last column of the matrix

$$\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} K$$

is zero. If this K is applied to $[D^T(s,r) \; N^T(s,r)]^T$, then all the entries in the last column of the matrix

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} K \qquad (5)$$

will have the common factor r. Thus (5) can be factorized as

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} K = \begin{bmatrix} \bar{D}(s,r) \\ \bar{N}(s,r) \end{bmatrix} B(r) \qquad (6)$$

where $B(r)=\operatorname{diag}\{1,\ldots,1,r\}$, $\bar{D}(s,r)$ and $\bar{N}(s,r)$ denote the matrices $D(s,r)K$ and $N(s,r)K$ after the extraction of r from the last column.

Postmultiplying both sides of (6) by K^{-1}, and defining $B(r)K^{-1}=R(s,r)$, we obtain

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} = \begin{bmatrix} \bar{D}(s,r) \\ \bar{N}(s,r) \end{bmatrix} R(s,r) \qquad (7)$$

Since $\det R(s,r)=(\det K)^{-1}\cdot r$, $D(s)$ and $N(s)$ can not be right coprime at infinity by Definition 2. This contradiction proves the necessity.

Sufficiency. Suppose that $[D_0^T \; N_0^T]^T$ is of full column rank, but $D(s)$ and $N(s)$ are not coprime at infinity. Then a common right divisor at infinity, say $R(s,r)$, exists such that $D(s,r) = \bar{D}(s,r)R(s,r)$ and $N(s,r) = \bar{N}(s,r)R(s,r)$. Now we try to compute the rank of the matrix $[D_0^T \; N_0^T]^T$ via these two factorizations.

As noted before, since $D(s,r)$ is a homogeneous matrix polynomial, its factor $\bar{D}(s,r)$ must be column-homogeneous and $R(s,r)$ row-homogeneous. The same is true for $\bar{N}(s,r)$ and $R(s,r)$. The special degree pattern of $\bar{D}(s,r)$ enables the following decomposition

$$\bar{D}(s,r) = \bar{D}_h(s) + \bar{D}_l(s,r) \tag{8}$$

with $\bar{D}_h(s)$ being

$$\bar{D}_h(s) = \bar{D}_{h0} \begin{bmatrix} s^a & & 0 \\ & s^b & \\ & & \ddots \\ 0 & & s^c \end{bmatrix} \tag{9}$$

where $\bar{D}_{h0}$ is a constant matrix, a,b,...,c are a set of homogeneous column degrees of $\bar{D}(s,r)$; $\bar{D}_l(s,r)$ denotes the matrix containing the lower degree (with respect to s) terms. Similarly, the row-homogeneous matrix $R(s,r)$ can be decomposed as

$$R(s,r) = R_h(s) + R_l(s,r) \tag{10}$$

where

$$R_h(s) = \begin{bmatrix} s^e & & 0 \\ & s^f & \\ & & \ddots \\ 0 & & s^g \end{bmatrix} R_{h0} \tag{11}$$

In the above, R_{h0} is a constant matrix, e,f,...,g are a set of homogeneous row degrees of $R(s,r)$; and $R_l(s,r)$ contains all the lower degree (with respect to s) terms. We remind that since $\det R(s,r)$ is independent of s, the matrix R_{h0} must be singular. In view of (8) and (10), the leading coefficient matrix (with respect to s) of $D(s,r) = \bar{D}(s,r)R(s,r)$ is obviously given by $D_0 = \bar{D}_{h0} \cdot R_{h0}$. Exactly the same arguments apply to the pair $\bar{N}(s,r)$ and $R(s,r)$. Hence with the following decomposition of $\bar{N}(s,r)$

$$\bar{N}(s,r) = \bar{N}_h(s) + \bar{N}_l(s,r) \tag{12}$$

where

$$\bar{N}_h(s) = \bar{N}_{h0} \begin{bmatrix} s^q & & 0 \\ & s^d & \\ & & \ddots \\ 0 & & s^t \end{bmatrix} \tag{13}$$

and $\bar{N}_{h0}$ is a constant matrix, q,d,...,t are a set of homogeneous column degrees of $\bar{N}(s,r)$, the leading coefficient matrix (with repect to s) N_0 can be computed as $\bar{N}_{h0} \cdot R_{h0}$.

Combining the foregoing discussions, it is clear that

$$\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} = \begin{bmatrix} \bar{D}_{h0} \cdot R_{h0} \\ \bar{N}_{h0} \cdot R_{h0} \end{bmatrix} = \begin{bmatrix} \bar{D}_{h0} \\ \bar{N}_{h0} \end{bmatrix} R_{h0} \tag{14}$$

Since R_{h0} is singular, the columns of the matrix $[D_0^T \; N_0^T]^T$ can not be linearly independent. This contradicts our assumption. Hence, the theorem is proved.

The notion of coprimeness for two matrix polynomials is well-known. A good reference on this subject is [7]. Here we simply recall that two matrix polynomials are coprime if their greatest common divisor is a unimodular matrix. One characterization can also be given as a rank condition of an appropriate matrix. A combination of this concept along with the concept of coprimeness at infinity leads to the following definition.

Definition 3.

$D(s)$ and $N(s)$ are said to be completely right coprime if they are both coprime and coprime at infinity.

Based on Theorem 1 and the characterization of the coprimeness (see, for example, [7]), an interesting characterization of the complete coprimeness for two matrix polynomials can be stated.

Theorem 2.

$D(s)$ and $N(s)$ are completely right coprime if and only if

$$\text{rank} \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} = p$$

and

$$\text{rank} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = p, \text{ for all finite } s.$$

We recall that $D(s)$ is of dimension pxp, and is nonsingular, $N(s)$ mxp; D_0 and N_0 are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

Up to now, only the right coprimeness has been considered. In an algebraically dual way, the left coprimeness can also be treated. We shall, however, omit the treatment.

3. BLOCK CONTROLLABLE REALIZATION OF RIGHT MFD IN SINGULAR SYSTEM AND ITS PROPERTIES

The standard form for singular systems is the following [8], [6], [9]

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{15}$$

where x is an n-vector of internal variables, u is a p-vector of control inputs and y is an m-vector of outputs. E, A are matrices of dimension nxn, B and C are matrices of dimensions nxp and mxn, respectively. In particular, E can be singular. n will be called the order of system (15).

The fundamental concepts of controllability and observability of (15) have been studied by many authors [8], [9], [10]. A comprehensive and unifying

treatment has been presented recently by Cobb [2]. Here, we shall adopt Cobb's conventions and definitions.

Definition 4.

The singular system (15) is controllable at infinity if

$$\text{rank } [E \quad B] = n \tag{16}$$

It is finitely controllable if

$$\text{rank } [Es-A \quad B] = n, \text{ for all finite } s \tag{17}$$

It is controllable if it is both finitely controllable and controllable at infinity.

Definition 5.

The singular system (15) is observable at infinity if

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n \tag{18}$$

It is finitely observable if

$$\text{rank} \begin{bmatrix} Es - A \\ C \end{bmatrix} = n, \text{ for all finite } s \tag{19}$$

It is observable if it is both finitely observable and observable at infinity.

Let a transfer matrix G(s) of dimension mxp be given in right MFD, i.e.,

$$G(s) = N(s)D^{-1}(s) \tag{20}$$

where N(s) and D(s) are matrix polynomials of dimensions mxp and pxp, respectively. Our purpose in this section is to realize (20) in a controllable singular system (15) of a special structure, the so-called block controllable realization.

The homogeneous forms for D(s) and N(s) can be written down as

$$\begin{aligned} D(s,r) &= D_0 s^k + D_1 s^{k-1} r + \ldots + D_k r^k \\ N(s,r) &= N_0 s^g + N_1 s^{g-1} r + \ldots + N_g r^g \end{aligned} \tag{21}$$

It is observed that the matrices N_0 and D_0 in (21) may be singular or even zero. The following definition characterizes a special class of MFD's which will be considered exclusively in the ensuing development.

Definition 6.

If g=k-1 in (21), the corresponding MFD will be called pseudo-proper.

An interesting feature of a pseudo-proper MFD is indicated in the following theorem.

Theorem 3.

Any pseudo-proper right MFD can be realized in singular system (15), where

$$E = \begin{bmatrix} D_0 & 0 & \ldots & 0 \\ 0 & I_p & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & I_p \end{bmatrix} \qquad A = \begin{bmatrix} -D_1 & -D_2 & \ldots & -D_{k-1} & -D_k \\ I_p & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & I_p & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I_p \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad C = [N_0 \quad N_1 \quad \ldots \quad N_{k-1}] \tag{22}$$

In the above, I_p denotes the pxp unit matrix. The realization is in the sense that

$$N(s)D^{-1}(s) = C(Es-A)^{-1}B \tag{23}$$

Proof. By direct computation, the first p columns of the matrix $(Es-A)^{-1}$, which are obviously the matrix $(Es-A)^{-1}B$ by (22), are given by

$$\begin{bmatrix} I_p s^{k-1} \\ I_p s^{k-2} \\ \cdot \\ \cdot \\ I_p \end{bmatrix} \qquad D^{-1}(s) = (Es - A)^{-1}B \tag{24}$$

Premultiplying the matrix C on both sides of the equation (24) will lead to (23). Observe that even if the matrices D_0 and N_0 are zero, they should not be dropped in the operations.

The realization (22) is important in several aspects. The following are some of its properties relevant to our development.

Theorem 4.

The realization (22) has the properties below:

1. det D(s,r) = det (Es - Ar);

2. deg det D(s,r) = the order of the realization, where deg denotes the degree of the polynomial followed;

3. it is controllable.

Proof.

1. Let Es-Ar be partitioned into a block row matrix

$$Es - Ar = [F_1 \quad F_2 \quad \ldots \quad F_k] \tag{25}$$

where each F_i (i=1,...k) consists of p columns. Apply a transformation T to (25) such that

$$(Es - Ar)T = [\tilde{F}_1 \quad \tilde{F}_2 \quad \ldots \quad \tilde{F}_k] \tag{26}$$

where $\tilde{F}_i = F_i r^{i-1} + \tilde{F}_{i-1} s$ (i=2,...,k) and $\tilde{F}_1 = F_1$. From the special structure of Es-Ar it follows that after the transformation, the determinant can be computed directly, which is equal to

$$\det D(s,r).\det(I_p r)\ldots\det(I_p r^{k-1})$$

On the other hand, $\det[Es-Ar)T]=\det(Es-Ar).\det T$, and $\det T$ is apparently equal to $\det(I_p r)\ldots\det(I_p r^{k-1})$. We therefore conclude

$$\det(Es - Ar) = \det D(s,r) \qquad (27)$$

this proves 1.

2. Since $\deg \det(Es - Ar) = kp$, which is also the order of the realization (22), by using 1, $\deg \det D(s,r) = \deg \det (Es - Ar) = kp$.

3. It is readily checked that $\operatorname{rank}[E \;\; B] = kp$, and $\operatorname{rank}[Es-A \;\; B] = kp$, for all finite s. By Definition 4, this implies that (22) is controllable.

Because of the above properties, the realization (22) is called the block controllable realization. Note especially that the order of the realization is equal to the generalized determinantal degree (defined in Section 2) of the denominator matrix of the given MFD.

The notions of coprimeness at infinity and complete coprimeness have been introduced in the preceding section. We now use them to define the concepts of irreducibility at infinity and complete irreducibility for MFD's.

A right MFD is said to be right irreducible at infinity if its denominator matrix and numerator matrix are right coprime at infinity. Similarly, it is said to be completely right irreducible if the numerator matrix and the denominator matrix are completely right coprime. The following theorem shows an interesting connection between the complete right irreducibility of (20) and the observability of the realization (22).

Theorem 5.

The block controllable realization (22) is observable if and only if the pseudo-proper MFD (20) is completely right irreducible.

Proof. We only prove the equivalence between the observability at infinity and the right irreducibility at infinity. The finite case is left for the reader.

From (22), the columns of the matrix

$$\left[\begin{array}{c} E \\ \hline C \end{array}\right] = \left[\begin{array}{cccc|c} D_0 & 0 & \ldots & 0 & \\ 0 & I_p & \ldots & 0 & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ 0 & 0 & \ldots & I_p & \\ \hline N_0 & N_1 & \ldots & N_{k-1} & \end{array}\right]$$

are linearly independent if and only if the columns of the matrix $[D_0^T \; N_0^T]^T$ are linearly independent, which is, by Theorem 1, equivalent to the right irreducibility at infinity for (20).

4. RELATION BETWEEN COMPLETE IRREDUCIBILITY AND JOINT CONTROLLABILITY OBSERVABILITY

A singular system (15) in fact assumes a canonical decomposition given as follows [2], [10]

$$\begin{array}{ll} \dot{x}_1 = E_1 x_1 + B_1 u & E_2\dot{x}_2 = x_2 + B_2 u \\ y_1 = C_1 x_1 & y_2 = C_2 x_2 \end{array} \qquad (28)$$

where E_2 is a nilpotent, x_1 is an n_1-vector, x_2 is an n_2-vector and $n_1+n_2=n$. The first equation set in (28) corresponds to the proper part of the system, the second to the polynomial part. The output of the whole system is $y=y_1+y_2$.

The controllability and the observability of the overall system can also be stated with respect to this decomposition. We quote one result from Cobb [3].

Theorem 6.

Define $U_k(E,B) = [B \;\; EB \;\; E^2B \;\ldots\; E^{k-1}B]$ and

$$V_k(E,C) = \left[\begin{array}{c} C \\ CE \\ CE^2 \\ \cdot \\ \cdot \\ CE^{k-1} \end{array}\right]$$

Then, the singular system (15) is finitely controllable if and only if $\operatorname{rank} U_{n_1}(E_1,B_1)=n_1$; it is controllable at infinity if and only if $\operatorname{rank} U_{n_2}(E_2,B_2)=n_2$. Similarly, the singular system (15) is finitely observable if and only if $\operatorname{rank} V_{n_1}(E_1,C_1)=n_1$; it is observable at infinity if and only if $\operatorname{rank} V_{n_2}(E_2,C_2)=n_2$.

This theorem enables us to establish the following result

Theorem 7.

If a transfer matrix (20) has one controllable and observable nth-order realization, then all its nth-order realization must also be controllable and observable.

Proof. Let (28) be an nth-order realization with $n_1+n_2=n$. Its Hankel matrices

$$H_1 = \left[\begin{array}{cccc} C_1B_1 & C_1E_1B_1 & \ldots & C_1E_1^{n_1-1}B_1 \\ C_1E_1B_1 & C_1E_1^{2}B_1 & \ldots & C_1E_1^{n_1}B_1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_1E_1^{n_1-1}B_1 & C_1E_1^{n_1}B_1 & \ldots & C_1E_1^{2n_1-2}B_1 \end{array}\right]$$

$$H_2 = \left[\begin{array}{cccc} C_2B_2 & C_2E_2B_2 & \ldots & C_2E_2^{n_2-1}B_2 \\ C_2E_2B_2 & C_2E_2^{2}B_2 & \ldots & C_2E_2^{n_2}B_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_2E_2^{n_2-1}B_2 & C_2E_2^{n_2}B_2 & \ldots & C_2E_2^{2n_2-2}B_2 \end{array}\right]$$

depend only on the transfer matrix, since

$$G(s) = C_2E_2^{n_2-1}B_2s^{n_2-1} + \ldots + C_2E_2B_2s + C_2B_2 + C_1B_1 1/s + C_1E_1B_1 1/s^2 + \ldots$$

It is easily checked that

$$H_1 = V_{n_1}(E_1,C_1)U_{n_1}(E_1,B_1)$$
$$H_2 = V_{n_2}(E_2,C_2)U_{n_2}(E_2,B_2)$$

Therefore, if $\{E, A, B, C\}$ and $\{\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\}$ are two nth-order realizations of $G(s)$, using the decomposition (28) we know that

$$H_1 = V_{n_1}(E_1,C_1)U_{n_1}(E_1,B_1) = V_{n_1}(\tilde{E}_1,\tilde{C}_1)U_{n_1}(\tilde{E}_1,\tilde{B}_1) \quad (29a)$$

and

$$H_2 = V_{n_2}(E_2,C_2)U_{n_2}(E_2,B_2) = V_{n_2}(\tilde{E}_2,\tilde{C}_2)U_{n_2}(\tilde{E}_2,\tilde{B}_2) \quad (29b)$$

The notations used in (29) are self-explanatory. Now if $\{E,A,B,C\}$ is controllable and observable, then

$$\text{rank } [V_{n_1}(E_1,C_1)U_{n_1}(E_1,B_1)] = n_1 \quad (30a)$$

and

$$\text{rank } [V_{n_2}(E_2,C_2)U_{n_2}(E_2,B_2)] = n_2 \quad (30b)$$

From (29) and (30), we know that

$$n_1 \leq \min\{\text{rank } V_{n_1}(\tilde{E}_1,\tilde{C}_1), \text{ rank } U_{n_1}(\tilde{E}_1,\tilde{B}_1)\}$$
$$n_2 \leq \min\{\text{rank } V_{n_2}(\tilde{E}_2,\tilde{C}_2), \text{ rank } U_{n_2}(\tilde{E}_2,\tilde{B}_2)\}$$

Hence $\{\tilde{E},\tilde{A},\tilde{B},\tilde{C}\}$ is controllable and observable as well.

We are now in the position to state an important relation.

Theorem 8.

A pseudo-proper transfer matrix (20) is completely irreducible if and only if all its nth-order realizations are controllable and observable, where n is the generalized determinantal degree of $D(s)$, i.e., $n = \deg \det D(s,r)$.

Proof. In view of Theorem 7, it suffices to show that the theorem holds for a special nth-order realization. But Theorem 5 indicates that the block controllable realization can serve this purpose.

5. CONCLUSIONS

In this paper, we have introduced the notions of coprimeness at infinity and complete coprimeness for two matrix polynomials. For MFD's, they naturally become irreducibility at infinity and complete irreducibility, respectively. As an important application, we established the equivalence between the complete irreducibility of a given MFD on the one hand, and the joint controllability observability of the singular realizations on the other. The corresponding result is well-known for regular systems ($E=I$ in (15)).

In fact, if the minimality of a singular realization is properly defined (an example is [1]), we can show that minimality, complete irreducibility and joint controllability observability will form an equivalence circle, an exciting property appearing in singular systems.

An important concept used throughout is the pseudo-properness of MFD's. This is a natural generalization of the conventional properness, and is not as restricted as one might think. However, it does reveal the restriction imposed by the structure (15).

Another important issue is the zeros at infinity of a matrix polynomial. The difference between the definition used here and that by using Smith-McMillan form will be examined in a forthcoming paper.

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