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INVERSION OF SINGULAR SYSTEMS

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ABSTRACT

A novel method for inverting square singular systems is presented. The construction explores the flexibility of singular systems, and provides a simple and closed form for the constructed inverse. One important property is that the minimality of the inverse (in the sense of Verghese) is equivalent to that of the original system.

1. INTRODUCTION

The dynamical systems considered in this paper is represented by

$$E\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where x is an n-vector of internal variables, u is an m-vector of inputs and y is an m-vector of outputs. E, A, B, C, D are matrices of appropriate dimensions. In particular, E may be singular. Such systems have recently received considerable attention [1, 2, 3, 4] due to the fact that (1) arises naturally in the study of large-scale interconnected systems, noncausal systems, network analysis and socio-economic systems [3, 8] etc. There have been many terms attached to system (1): descriptor system, generalized state-space system and singular system. In this paper, (1) is called the singular system, and the vector x is termed the descriptor vector.

Many important properties of singular systems have been found in the past few years [1, 2, 4, 5]. Apart from its distinctive dynamical behaviours, its representation indeed provides a great deal of flexibility in dealing with many problems including system inversion.

The problem of inverting the dynamical systems has been under investigation for some time [9, 10, 11, 12]. Although there are many algorithms to be used in the inversion of time-invariant regular systems (i.e., E=I in (1a)), for example, [11, 13], their common deficiency is that the resulting inverse does not remain in a standard representation. This is simply due to the fact that the state-space repersentation can only accommodate proper dynamical systems, while inverses of proper dynamical systems are not necessarily proper.

One of the purposes of this paper is to show that the representation of singular systems (1) will provide a closed form for inverse of any dynamical system. In fact, we shall present a method to construct an inverse directly from (1). Some of the important properties of such an inverse are also stated.

The paper is orgnized as follows. In Section 2, we give some necessary definitions from the theory of singular systems. These will serve as a background for the subsequent development. Section 3 is devoted to present a method of constructing an inverse of (1). We also show that the invertibility test is incorporated as a solvability test of the constructed inverse. Important properties of our inverse are stated in Section 4. We show, in particular, that the construction is both controllable and observable in the sense of Verghese et al. [1] if the original system does. Section 5 contains two examples to demonstrate the method. Conclusions and remarks are in the final section.

2. BACKGROUND

Let a singular system (1) be given. It is said to be solvable if $\det(Es-A)$ is not identically zero [3, 4]. In this case, the pencil Es-A is called regular. We will assume throughout the paper that (1) is solvable. Also f(= rank E) is defined as the generalized order of system (1).

The matrix

$$P(s) = \left[\begin{array}{c|c} Es - A & -B \\ \hline C & D \end{array}\right] \tag{2}$$

constructed from (1) is called the system matrix of (1) [6]. Consider the operation on (2), which is embodied in a transformation, of the form

$$\left[\begin{array}{cc} M & 0 \\ Q & I \end{array}\right]\left[\begin{array}{c|c} Es-A & -B \\ \hline C & D \end{array}\right]\left[\begin{array}{cc} N & R \\ 0 & I \end{array}\right] = \left[\begin{array}{c|c} E_1s-A_1 & -B_1 \\ \hline C_1 & D_1 \end{array}\right] \tag{3}$$

with M, N nonsingular and $QE = 0$, $ER = 0$; and the modification of the system matrix (2) to the form

$$\left[\begin{array}{ccc|c} Es - A & 0 & & -B \\ 0 & I & & 0 \\ \hline C & 0 & & D \end{array}\right] \tag{4}$$

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which is termed a trivial augmentation of (2), and the reverse process, i.e., from (4) to (2), which is called a trivial deflation.

Two systems are said to be strongly equivalent [1] (str. eq.) if one can be obtained from the other by the operations (3) and trivial augmentations and/or trivial deflations.

Finally, construct two matrices, namely, the observability matrix

$$\begin{bmatrix} Es - A \\ C \end{bmatrix} \tag{5a}$$

and the controllability matrix

$$[Es - A \quad -B] \tag{5b}$$

by str. eq. operations, (5) can be brought to

$$\left[\begin{array}{cc} E_1 s - A_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right] \tag{6a}$$

with E_1 of full column rank and

$$\left[\begin{array}{c|c} \tilde{E}_1 s - \tilde{A}_1 & -B_1 \\ \tilde{A}_2 & -B_2 \end{array}\right] \tag{6b}$$

with $\tilde{E}_1$ of full row rank. Then the system (1) is said to be observable if (5a) is of full column rank for all finite s, and the matrix

$$\begin{bmatrix} E_1 & A_2 \\ 0 & C_2 \end{bmatrix} \tag{7a}$$

is of full column rank; it is said to be controllable if (5b) is of full row rank for all finite s and the matrix

$$\begin{bmatrix} \tilde{E}_1 & 0 \\ \tilde{A}_2 & -B_2 \end{bmatrix} \tag{7b}$$

is of full row rank.

3. CONSTRUCTION OF AN INVERSE SYSTEM

First, the square system (1) is said to be invertible if the determinant of its transfer matrix

$$G(s) = C(Es - A)^{-1}B + D \tag{8}$$

does not vanish identically. The following construction will result in a singular system. It will be shown later that if (1) is invertible, then such a construction will yield an inverse of (1).

As the first step, Eq.(1b) is alternatively expressed as

$$y = [C \quad D-I] \begin{bmatrix} x \\ u \end{bmatrix} + u \tag{9}$$

This is equivalent to a change of descriptor vector. The effect on (1a) is obviously

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} B \\ I_m \end{bmatrix} u \tag{10}$$

where I_m is a unit matrix of order m, 0's are zero matrices of compatible dimensions.

Define $z^T=[x^T \; u^T]^T$, rewrite (9) in the form

$$u = [-C \quad I-D]z + y \tag{11a}$$

substituting (11a) into (10), simple manipulations give

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} A-BC & B-BD \\ -C & -D \end{bmatrix} z + \begin{bmatrix} B \\ I \end{bmatrix} y \tag{11b}$$

Now we show that the invertibility of system (1) is in fact equivalent to the solvability of the system (11). First we state a lemma.

Lemma 1. The singular system (1) is invertible if and only if the determinant of its system matrix (2) is not identically zero.

Remark. This lemma is a simple generalization of a result given by Wang and Davison [7].

proof. Since Es-A is regular, the determinant of P(s) can be readily seen as

$$\det P(s) = \det (Es-A) \cdot \det [C(Es-A)^{-1}B + D]$$

Therefore, detP(s) is zero if and only if (1) is not invertible by definition.

The following result follows directly from lemma 1.

Theorem 1. System (1) is invertible if and only if (11) is solvable, or equivalently, the pencil

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} s - \begin{bmatrix} A-BC & B-BD \\ -C & -D \end{bmatrix} \tag{12}$$

is regular.

proof. Adding to the first block row in (12) the second one premultiplied by -B, the resulting pencil is precisely (2). Note that such a transformation will not affect the determinant of (12). The theorem is proved by recalling Lemma 1.

Finally we verify that if (1) is invertible, (11) will indeed be an inverse of it. The verification is carried out straightforwardly.

$$\hat{G}(s) = [-C \; I-D] \begin{bmatrix} Es-A+BC & BD-B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} B \\ I \end{bmatrix} + I$$

$$= \lfloor -C \quad I-D \rfloor \begin{pmatrix} Es-A & -B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} + I$$

$$= \lfloor -C \quad I-D \rfloor \begin{pmatrix} Es-A & -B \\ 0 & G(s) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} + I$$

$$= -C(Es-A)^{-1}BG^{-1}(s) + (I-D)G^{-1}(s) + I$$

$$= G^{-1}(s)$$

This is what we have to show.

4. PROPERTIES OF THE INVERSE SYSTEM

Apart from the great simplicity, other features of the inverse system (11) are that no assumption of controllability and observability is required, also the generalized order of the inverse (11) is the same as the original system (1). Moreover, when (1) is controllable and observable (defined in section 2), (11) is also controllable and observable, thus this inverse has the minimal order whenever the original system does. This property is proved below.

Theorem 2. The inverse system (11) is controllable and observable if and only if (1) is controllable and observable.

proof. We only give the proof for the observable part, the controllable part follows in the same but simpler way.

Start with the following observability matrix

$$\left(\begin{array}{cc} Es-A+BC & BD-B \\ C & D \\ \hline -C & I-D \end{array}\right) \tag{13}$$

Add to the first block row the second one premultiplied by -B (note that this operation is a str. eq. operation), (13) is brought to

$$\left(\begin{array}{cc} Es-A & -B \\ C & D \\ \hline -C & I-D \end{array}\right) \tag{14}$$

Column-compressing E by a str. eq. operation, (14) is put into

$$\left(\begin{array}{ccc} E_1s-A_1 & A_2 & -B \\ C_1 & C_2 & D \\ \hline -C_1 & -C_2 & I-D \end{array}\right) \tag{15}$$

of course, E_1, A_1, A_2, C_1, C_2 are the same as those in (6a).

For (11) to be observable, (13) (or equivalently (14)) and the matrix

$$\left(\begin{array}{ccc} E_1 & A_2 & -B \\ 0 & C_2 & D \\ \hline 0 & -C_2 & I-D \end{array}\right)$$

are of full column rank. They will be so if and only if (5a) and (7a) both have full column rank, or equivalently, (1) is observable.

Since it is the generalized order rather than the dimension of a singular system that is of significance, for some cases the dimension of (11) can be reduced by applying str. eq. operations. In the next section we shall show by an example how this can be done.

5. ILLUSTRATIVE EXAMPLES

In this section we give two examples to demonstrate our inversion method.

Example 1. Let a system be given by

$$\dot{x} = -x + u \tag{16a}$$

$$y = x \tag{16b}$$

which is of course a single-variable system. Its transfer function $G(s)=1/(s+1)$. Modify (16b) in the following way

$$y = \lfloor 1 \quad -1 \rfloor z + u \tag{17}$$

where $z^T = \lfloor x \; u \rfloor^T$, and the corresponding modification of (16a) is clearly

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{z} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} z + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \tag{18}$$

the substitution of (17) into (18) yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \dot{z} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 1 \end{pmatrix} y$$

$$u = \lfloor -1 \quad 1 \rfloor z + y \tag{19}$$

one may show that the transfer function of (19) is $s+1$.

Example 2. Now we consider the following 2x2 square system

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u \tag{20a}$$

$$y = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} u \tag{20b}$$

the transfer matrix of (20) is

$$G(s) = \begin{pmatrix} s+1 & s \\ s & s-1 \end{pmatrix}$$

which is invertible.

Again modifying (20b) we have

$$y = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right) z + u \tag{21}$$

(20a) is also modified to be

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \dot{z} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] z + \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u \tag{22}$$

substitute (21) into (22)

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \dot{z} = \left[\begin{array}{cc|cc} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \end{array}\right] z + \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] y$$

$$u = \left[\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{array}\right] z + y \tag{23}$$

the transfer matrix of (23) is

$$\hat{G}(s) = \begin{bmatrix} -s+1 & s \\ s & -s-1 \end{bmatrix}$$

Next, (23) can be transformed by str. eq. operation to the form

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \dot{q} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] q + \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right] y$$

$$u = \left[\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{array}\right] q + y \tag{24}$$

A trivial deflation will bring (24) to

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{p} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} p + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} y$$

$$u = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} p + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} y \tag{25}$$

(25) obviously has the same transfer matrix as (23).

6. CONCLUSIONS

We have presented a method for inverting singular systems. The principle behind the construction is that instead of generalizing the existing inversion algorithms to singular systems (one example is [14]), we explored the flexibility of the representation of singular systems. In fact, this principle was originally used by Rosenbrock [2, 6] in a construction of an inverse for a more general type of systems [6, p.172, 15].

The extension of our construction to arbitrary singular systems (i.e., nonsquare case) seems possible and is currently under consideration by the authors.

REFERENCES

[1]. G. C. Verghese, B. C. Levy and T. Kailath, "A generalized state-space for singular systems", IEEE Trans. Autom. Control, Vol.AC-26, pp.811-831, 1981.

[2]. H. H. Rosenbrock, "Structural properties of linear dynamical systems", Int. J. Control, Vol.20, pp.191-202, 1974.

[3]. D. G. Luenberger, "Dynamic equations in descriptor form", IEEE Trans. Autom. Control, Vol.AC-22, pp.312-321, 1977.

[4]. E. L. Yip and R. F. Sincovec, "Solvability, controllability and observability of continuous descriptor systems", IEEE Trans. Autom. Control, Vol.AC-26, pp.702-707, 1981.

[5]. D. Cobb, "Controllability, observability and duality in singular systems", IEEE Trans. Autom. Control, Vol.AC-29, pp.1076-1082, 1984.

[6]. H. H. Rosenbrock, "State-space and multivariable theory", New York: Wiely, 1970.

[7]. S. H. Wang and E. J. Davison, "A new invertibility criterion for linear multivariable systems", IEEE Trans. Autom. Control, Vol.AC-18, pp.220-225, 1973.

[8]. A. Dervisoglu and C. A. Desoer, "Degenerate networks and minimal differential equations", IEEE Trans. Circ. Syst., Vol.CAS-22, pp.769-775, 1975.

[9]. M. K. Sain and J. L. Massey, "Invertibility of linear time-invariant dynamical systems", IEEE Trans. Autom. Control, Vol.AC-14, pp.141-148, 1969.

[10]. J.L.Massey and M. K. Sain, "Inverse of linear sequential circuits", IEEE Trans. Comput. Vol. C-17, 1968.

[11]. L. M. Silverman, "Inversion of multivariable linear systems", IEEE Trans. Autom. Control, Vol.AC-14, pp.270-276, 1969.

[12]. E. Emre, O. Huseyin and K. Abdullah, "On the inversion of rational matrices", IEEE Trans. Circ. Syst., Vol.CAS-21, pp.8-11, 1974.

[13]. P.J. Moylan, "Stable inversion of linear systems", IEEE Trans. Autom. control, Vol.AC-22, pp.74-78, 1977.

[14]. F. L. Lewis, "Inversion of descriptor systems", Proc. 1983 American Control Conference, pp.1153-1158, 1983.

[15]. H. H. Rosenbrock and A. J. J. Van der Weiden, "Inverse systems", Int. J. Control, Vol.25, pp.389-592, 1977.