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Inversion of Singular Systems

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Abstract—A novel method for inverting square singular systems is presented. The construction explores the flexibility of singular system representation, and provides a simple and closed form for the constructed inverse. One important property is that the minimality of the inverse (in the sense of Verghese) is equivalent to that of the original system.

I. Introduction

The dynamical systems considered in this paper is represented by

$$E\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where x is an n-vector of internal variables, u is an m-vector of inputs and y is an m-vector of outputs. E, A, B, C, D are matrices of appropriate dimensions. In particular, E may be

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singular. Such systems have recently received considerable attention [1]–[4] due to the fact that (1) arises naturally in the study of large-scale interconnected systems, noncausal systems, network analysis and socio-economic systems [3], [8], etc. There have been many terms attached to the system (1): descriptor system, generalized state-space system and singular system. In this paper, (1) is called a singular system, and the vector x is termed a descriptor vector.

Many important properties of singular systems have been found in the past few years [1]–[5]. Apart from its distinctive dynamical behaviour, its representation indeed provides a great deal of flexibility in dealing with many problems including system inversion.

The problem of inverting dynamical systems has been under investigation for some time [9]–[12]. Although there are many algorithms to be used in the inversion of time-invariant regular systems (i.e., $E = I$ in (1a)), for example, [11], [13], their common deficiency is that the resulting inverse does not remain in a standard representation. This is simply due to the fact that the state-space representation can only accommodate proper dynamical systems, while inverses of proper dynamical systems are not necessarily proper.

One of the purposes of this paper is to show that the representation of singular systems (1) provides a closed form for inverse of any dynamical system. In fact, we shall present a method to construct an inverse directly from (1). Some of the important properties of such an inverse are also stated.

The paper is organized as follows. In Section II, we give some necessary definitions from the theory of singular systems. These will serve as a background for the subsequent development. Section III is devoted to present a method of constructing an inverse of (1). We also show that the invertibility test is incorporated as a solvability test of the constructed inverse. Important properties of our inverse are stated in Section IV along with a dimension reduction procedure. We show, in particular, that the construction is both controllable and observable in the sense of Verghese *et al.* [1] if and only if the original system does. Section V contains an example to demonstrate the method. Conclusions and possible extension of the method to the left inversion of non-square systems are given in the final section.

II. Background

Let a singular system (1) be given. It is said to be solvable if $\det(Es - A)$ is not identically zero [3], [4]. In this case, the pencil $Es - A$ is called regular. We will assume throughout the paper that (1) is solvable. Also $f(=\text{rank } E)$ is defined as the *generalized order* of system (1).

The matrix

$$P(s) = \left(\begin{array}{c|c} Es - A & -B \\ \hline C & D \end{array}\right) \tag{2}$$

constructed from (1) is called the system matrix of (1) [6]. Consider the operation on (2), which is embodied in a transformation, of the form

$$\left(\begin{array}{c|c} M & O \\ \hline Q & I \end{array}\right)\left(\begin{array}{c|c} Es - A & -B \\ \hline C & D \end{array}\right)\left(\begin{array}{c|c} N & R \\ \hline O & I \end{array}\right) = \left(\begin{array}{c|c} E_1 s - A_1 & -B_1 \\ \hline C_1 & D_1 \end{array}\right) \tag{3}$$

with M, N nonsingular and $QE = 0$, $ER = 0$; and the modification of the system matrix (2) to the form

$$\left(\begin{array}{cc|c} Es - A & O & -B \\ O & I & O \\ \hline C & O & D \end{array}\right) \tag{4}$$

is termed a trivial augmentation of (2), and the reverse process, i.e., from (4) to (2), is called a trivial deflation.

Two systems are said to be strongly equivalent [1] (str. eq.) if one can be obtained from the other by the operations (3) and trivial augmentations and/or trivial deflations.

Finally, construct two matrices, namely, the observability matrix

$$\begin{pmatrix} Es - A \\ C \end{pmatrix} \tag{5a}$$

and the controllability matrix

$$[Es - A \quad -B] \tag{5b}$$

By str. eq. operations (5) can be brought to

$$\left(\begin{array}{cc} E_1 s - A_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right) \tag{6a}$$

with E_1 of full column rank and

$$\left(\begin{array}{c|c} \tilde{E}_1 s - \tilde{A}_1 & -\tilde{B}_1 \\ \tilde{A}_2 & -\tilde{B}_2 \end{array}\right) \tag{6b}$$

with $\tilde{E}_1$ of full row rank. Then the system (1) is said to be observable if (5a) is of full column rank for all finite s, and the matrix

$$\begin{pmatrix} E_1 & A_2 \\ 0 & C_2 \end{pmatrix} \tag{7a}$$

is of full column rank; it is said to be controllable if (5b) is of full row rank for all finite s and the matrix

$$\begin{pmatrix} \tilde{E}_1 & 0 \\ \tilde{A}_2 & -\tilde{B}_2 \end{pmatrix} \tag{7b}$$

is of full row rank.

III. Construction of an Inverse System

First, the square system (1) is said to be invertible if the determinant of its transfer matrix

$$G(s) = C(Es - A)^{-1}B + D \tag{8}$$

does not vanish identically. The following construction will result in a singular system. It will be shown later that if (1) is invertible, then such a construction will yield an inverse of (1).

First, (1b) is alternatively expressed as

$$y = [C \quad D - I]\begin{pmatrix} x \\ u \end{pmatrix} + u \tag{9}$$

This is equivalent to a change of descriptor vector. The effect on (1a) is obviously

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -I_m \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} B \\ I_m \end{pmatrix} u \tag{10}$$

where I_m is a unit matrix of order m, 0's are zero matrices of compatible dimensions.

Define $z^T = [x^T \quad u^T]^T$, rewrite (9) in the form

$$u = [-C \quad I - D]z + y \tag{11a}$$

substituting (11a) into (10), simple manipulations give

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\dot{z} = \begin{pmatrix} A - BC & B - BD \\ -C & -D \end{pmatrix} z + \begin{pmatrix} B \\ I \end{pmatrix} y. \tag{11b}$$

Now we show that the invertibility of system (1) is in fact equivalent to the solvability of the system (11). First we state a lemma.

Lemma 1: The singular system (1) is invertible if and only if the determinant of its system matrix (2) is not identically zero.

Remark: This lemma is a simple generalization of a result given by Wang and Davison [7].

Proof: Since $Es - A$ is regular, the determinant of $P(s)$ can be readily seen as

$$\det P(s) = \det(Es - A)\cdot\det\left[C(Es - A)^{-1}B + D\right].$$

Therefore, $\det P(s)$ is zero if and only if (1) is not invertible by definition.

The following result follows directly from Lemma 1.

Theorem 1: System (1) is invertible if and only if (11) is solvable, or equivalently, the pencil (12) is regular:

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} s - \begin{pmatrix} A - BC & B - BD \\ -C & -D \end{pmatrix}. \tag{12}$$

Proof: By adding to the first block row in (12) the second one premultiplied by $-B$, the resulting pencil is precisely (2). Note that such a transformation will not affect the determinant of (12). The theorem is proved by recalling Lemma 1.

Finally, we verify that if (1) is invertible, (11) will indeed be an inverse of it. The verification is carried out in a straightforward way.

$$\begin{aligned}
\hat{G}(s) &= [\,-C \quad I - D]\begin{pmatrix} Es - A + BC & BD - B \\ C & D \end{pmatrix}^{-1}\begin{pmatrix} B \\ I \end{pmatrix} + I \\
&= [\,-C \quad I - D]\begin{pmatrix} Es - A & -B \\ C & D \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix} + I \\
&= [\,-C \quad I - D]\begin{pmatrix} Es - A & -B \\ 0 & G(s) \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix} + I \\
&= -C(Es - A)^{-1}BG^{-1}(s) + (I - D)G^{-1}(s) + I \\
&= G^{-1}(s).
\end{aligned}$$

IV. Properties of the Inverse System and Dimension Reduction

Apart from the great simplicity, other features of the inverse system (11) are that no assumption of controllability and observability is required, also the generalized order of the inverse (11) is the same as the original system (1). Moreover, when (1) is controllable and observable (defined in Section II), (11) is also controllable and observable, thus this inverse has the minimal order whenever the original system does. This property is proved below.

Theorem 2: The inverse system (11) is controllable and observable if and only if (1) is controllable and observable.

Proof: We only give the proof for the observable part, the controllable part follows in a similar but simpler way. Start with the following observability matrix

$$\left(\begin{array}{cc} Es - A + BC & BD - B \\ \hline C & D \\ -C & I - D \end{array}\right). \tag{13}$$

Add to the first block row the second one premultiplied by $-B$ (note that this operation is a str. eq. operation), (13) is brought to

$$\left(\begin{array}{cc} Es - A & -B \\ \hline C & D \\ -C & I - D \end{array}\right) \tag{14}$$

Column-compressing E by a str. eq. operation, (14) is put into

$$\left(\begin{array}{ccc} E_1 s - A_1 & A_2 & -B \\ \hline C_1 & C_2 & D \\ -C_1 & -C_2 & I - D \end{array}\right) \tag{15}$$

of course, E_1, A_1, A_2, C_1, C_2 are the same as those in (6a). In order for (11) to be observable, (13) (or equivalently (14)) and the matrix

$$\left(\begin{array}{ccc} E_1 & A_2 & -B \\ \hline 0 & C_2 & D \\ 0 & -C_2 & I - D \end{array}\right)$$

should be of full column rank. They will be so if and only if (5a) and (7a) both have full column rank, or equivalently, (1) is observable.

Clearly, the dimension of the inverse (11) is the double of that of the original system. Fortunately, since the generalized order is more significant than the dimension of a singular system, in some cases the dimension of (11) can be reduced by applying str. eq. operations. In what follows, we shall provide a general procedure to perform the dimension reduction.

To begin with, consider the system matrix of the inverse (11)

$$\left(\begin{array}{cc|c} Es - A + BC & BD - B & -B \\ C & D & -I \\ \hline -C & I - D & I \end{array}\right). \tag{16}$$

Apply a str. eq. transformation to bring (16) to the following form

$$\left(\begin{array}{ccc|c} Is - A_{11} + \tilde{B}\tilde{C} & -A_{12} + \tilde{B}\bar{C} & \tilde{B}D - \tilde{B} & -\tilde{B} \\ -A_{21} + \bar{B}\tilde{C} & -A_{22} + \bar{B}\bar{C} & \bar{B}D - \bar{B} & -\bar{B} \\ \tilde{C} & \bar{C} & D & -I \\ \hline -\tilde{C} & -\bar{C} & I - D & I \end{array}\right)$$

where the matrices $[\tilde{C} \quad \bar{C}]$ and $[\tilde{B}^T \quad \bar{B}^T]^T$ are the proper partitionings of the matrices C, B after the transformation. The above matrix can be written more compactly as

$$\left(\begin{array}{cc|c} Is - A_{11}^* & -A_{12}^* & -\tilde{B} \\ -A_{21}^* & -A_{22}^* & -B^* \\ \hline -\tilde{C} & -C^* & I \end{array}\right) \tag{17}$$

where

$$\begin{aligned}
A_{11}^* &= A_{11} - \tilde{B}\tilde{C} \\
A_{12}^* &= \left[A_{12} - \tilde{B}\bar{C} \quad -\tilde{B}D + \tilde{B}\right] \\
A_{21}^* &= \begin{pmatrix} A_{21} - \bar{B}\tilde{C} \\ -\tilde{C} \end{pmatrix} \qquad A_{22}^* = \begin{pmatrix} A_{22} - \bar{B}\bar{C} & \bar{B} - \bar{B}D \\ -\bar{C} & -D \end{pmatrix} \\
B^* &= \left[\bar{B}^T \quad I\right]^T, \qquad C^* = [\bar{C} \quad D - I].
\end{aligned}$$

Since it is always possible to find nonsingular matrices M, N such that

$$MA_{22}^*N = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}$$

further str. eq. transformation will bring (17) into the following form:

$$\left(\begin{array}{ccc|c} Is - A_{11}^* & -\bar{A}_{12}^* & -\bar{A}_{12} & -\tilde{B} \\ -\tilde{A}_{21}^* & I & 0 & -B_1^* \\ -\bar{A}_{21}^* & 0 & 0 & -B_2^* \\ \hline -\tilde{C} & -C_1^* & -C_2^* & I \end{array}\right). \tag{18}$$

Now using the I in the second block row to eliminate the nonzero elements in the second block row and the second block column, we obtain

$$\left(\begin{array}{ccc|c} Is - A_{11}^* - \tilde{A}_{12}^*\tilde{A}_{21}^* & 0 & -\bar{A}_{12}^* & -\tilde{B} - \tilde{A}_{12}^* B_1^* \\ 0 & I & 0 & 0 \\ -\bar{A}_{21}^* & 0 & 0 & -B_2^* \\ \hline -\tilde{C} - C_1^*\tilde{A}_{21}^* & 0 & -C_2^* & I - C_1^* B_1^* \end{array}\right). \quad (19)$$

Note that the above operation is a str. eq. transformation. As the final step, a trivial deflation will put (19) into

$$\left(\begin{array}{cc|c} Is - A_{11}^* - \tilde{A}_{12}^*\tilde{A}_{21}^* & -\bar{A}_{12}^* & -\tilde{B} - \tilde{A}_{12}^* B_1^* \\ -\bar{A}_{21}^* & 0 & -B_2^* \\ \hline -\tilde{C} - C_1^*\tilde{A}_{21}^* & -C_2^* & I - C_1^* B_1^* \end{array}\right) \quad (20)$$

Equation (20) obviously has the dimension $2n - k$, where $k =$ rank A_{22}^*. In fact, it can be shown that further reduction in dimension is not possible, thus (20) actually has the minimal dimension. Moreover, since str. eq. transformation will not affect controllability and observability, Theorem 2 still holds for system (20) with the reduced dimension.

The dimension reduction procedure described here is in fact quite general. When applied to the real problems, one can fully explore the concrete structure of the problems so as to achieve considerable simplification. In the next section, we shall show this principle by an example.

III. Illustrative Example

In this section we give an example to demonstrate our inversion method.

Example: Let us consider the following 2×2 square system

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}u \quad (21a)$$

$$y = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}x + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}u \quad (21b)$$

the transfer matrix $G(s)$ of (21) is invertible,

$$G(s) = \begin{pmatrix} s+1 & s \\ s & s-1 \end{pmatrix}.$$

Modifying (21b) we have

$$y = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right)z + u. \quad (22)$$

Equation (21a) is also modified to be

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\dot{z} = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)z + \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)u. \quad (23)$$

Substitute (22) into (23)

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\dot{z} = \left(\begin{array}{cc|cc} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \end{array}\right)z + \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)y$$

$$u = \left(\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{array}\right)z + y \quad (24)$$

the transfer matrix of (24) is

$$\hat{G}(s) = \begin{pmatrix} -s+1 & s \\ s & -s-1 \end{pmatrix}.$$

Next, (24) can be transformed by str. eq. operation to the form

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\dot{q} = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)q + \left(\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ \hline 1 & -1 \\ 0 & 1 \end{array}\right)y$$

$$u = \left(\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{array}\right)q + y. \quad (25)$$

A trivial deflation will bring (25) to

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\dot{p} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}p + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}y$$

$$u = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}p + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}y. \quad (26)$$

Equation (26) obviously has the same transfer matrix as (24).

V. Conclusions

We have presented a method for inverting singular systems. Since state-space systems are special cases of singular systems, our inversion method can always be applied to the inversion of state-space systems. The principle behind the construction is that instead of generalizing the existing inversion algorithms to singular systems (one example is [14]), we explored the flexibility of the representation of singular systems. In fact, this principle was originally used by Rosenbrock [2], [6] in a construction of an inverse for a more general type of systems [6, p. 172, 15].

The extension of our construction to the left inversion of non-square singular systems is obvious. To see this, instead of (9) we simply modify (1b) into

$$y = [C \quad D - F]\begin{pmatrix} x \\ u \end{pmatrix} + Fu \quad (27)$$

with F of full column rank. Premultiplying the left inverse F^+ of F on both sides of (27), and rearranging terms we have

$$u = [-F^+C \quad I - F^+D]\begin{pmatrix} x \\ u \end{pmatrix} + F^+y. \quad (28)$$

A simple substitution of (28) into (10) will yield the desired left inverse of the system. Furthermore, the properties similar to Theorems 1 and 2 can also be established. We shall, however, omit the details.

Although finding the left inverse of a non-square system presents no additional complications, our inversion method does not seem to be easily suited for finding the right inverse.

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