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A UNIFIED APPROACH FOR ANALYSIS AND CONTROL OF MULTIVARIABLE NON-CAUSAL SYSTEMS

Promotor :
Prof. Dr. ir. J. VANDEWALLE

Proefschrift voorgedragen tot
het behalen van het doctoraat
in de toegepaste wetenschappen

door

Shaohua TAN

October 1987

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U.D.C. 681.511.2

October 1987

Preface

I would first like to thank my promotor Prof. J. Vandewalle. It is a great honour and pleasure to work for him. His stringent scientific style, broad research interests and knowledge have influenced me to a great extent. His patient instructions over 5 years have made me realized my personal potential and gradually built up my ability to carry out the scientific research.

I am also grateful to Prof. J. Peperstraete and Prof. A. Bultheel. They have, as members of my thesis reading committee, carefully reviewed the manuscript. The comments which they have provided have improved the quality of the thesis considerably.

Several people have made important contributions and expressed great interests to the research work reported in this thesis. Among them, I feel particularly indebted to Dr. P. Van Dooren from Philips Research Laboratory. He has constantly provided me with the valuable references which enabled me to gain a great deal of insights. I would also like to thank Prof. F. Callier from Facultes Universitaires de Namur. His great interests to this research work are highly appreciated.

A deciding factor which has supported me to complete this work is undoubtedly the cheering atmosphere created in our group SISTA in ESAT. Dr. A. Barbe, Dr. S. Van Huffel, B. De Moor, D. Callaerts, J. Chen, J. Vanderschoot and M. Moonen have provided me with friendly helps and encouraged me to carry on to the end. I would particularly like to mention B. De Moor. His ingenuity, creativity and enthusiasm have influenced me very much. I wish to thank him for generous helps he has offered to me.

I also wish to acknowledge the financial assistance I have received for many years from Katholieke Universiteit Leuven. Without it, my study would be impossible.

Finally, I feel greatly indebted to my wife Wei Wei. Her endless patience and encouragement have been vital to the success of this work.

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Chapter 1

Introduction

In this thesis we are mainly concerned with the so-called infinite dynamical structure of linear multivariable systems. Under this very broad heading, we shall try to develop an algebraic theory which is capable of providing an explicit and clear description of the redundancy and cancellation at infinity, and deriving a complete realization and control scheme. But before attempting to present our theory, it is of importance to give a short historical view of the problem, and motivate our interests in such a problem with examples.

1.1 A Historical Prospective and the Motivation of the Thesis

The interests to the infinite dynamical structure of linear systems were apparently initiated by Rosenbrock (1974), who took the first step away from the state-space system theory (a term which is used in this thesis to refer to the well-known theory dealing with the causal systems only), and considered systems of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.1}$$

where E, A, B, C are matrices of dimensions $n \times n, n \times n, n \times p, m \times n$, respectively. In particular, E may be singular. We shall, moreover, call x the *descriptor vector*. It is clear from (1.1) that this system representation is more general and contains the familiar state-space representation as a special case (i.e., $E = I$). Moreover, we can easily check that the system represented by (1.1) can be non-proper. Rosenbrock (1974) also gave two

reasons for considering such systems. First, the representation (1.1) naturally arises when dealing with an interconnection of subsystems; secondly, (1.1) sets up a very convenient framework for further exploration of the so-called structural properties at infinity of the underlying non-proper system. The concepts of infinite decoupling zeros have marked the beginning of a new research domain.

Shortly after the work of Rosenbrock (1974), Luenberger (1977,1978) independently raised the same system which he termed *descriptor system*. The reasons he supplied to back up the consideration of such systems include the practical modelling of large-scale interconnected systems and non-causal systems, the analysis of singular perturbations, etc. (we shall highlight some of these points by examples in the later sections). There is, however, an important difference in this later work in the sense that the structural properties at infinity of (1.1) were not observed.

Rosenbrock's line of thought has been effectively pursued by many researchers. In 1981, Yip and Sincovec extended the familiar concept of reachable states from the state-space system to include (1.1). This important extension has served as a basis for the definition of reachability for (1.1) (or C-controllability in their terminology). Interestingly enough, a simple frequency domain characterization of reachability turns out to coincide with the condition given by Rosenbrock (1974), which guarantees the absence of the finite and infinite decoupling zeros. This important connection has offered a somewhat surprising dynamical characterization of Rosenbrock's theory of (finite and infinite) decoupling zeros. The latter was however purely motivated by Kronecker's theory of infinite elementary divisors (see, Gantmacher 1960 for a concise exposition of the theory) without any explicit reference to the time domain and related dynamical significance.

Another important development has been made by Verghese et al. in a series of publications (Verghese 1978, 1980, Verghese et al. 1981, 1979a, 1979b). Their major contributions are not merely restricted to systems represented by (1.1), more importantly they have also considered non-proper systems represented by matrix fractions (the so-called *polynomial matrix form* first defined by Rosenbrock 1970), i.e.,

$$\begin{aligned} T(s)z(s) &= U(s)u(s) \\ y(s) &= V(s)z(s) + W(s)u(s) \end{aligned} \tag{1.2}$$

where $T(s), U(s), V(s), W(s)$ are polynomial matrices of appropriate dimensions; further, $T(s)$ is non-singular. Their approach to (1.1) differs considerably from the earlier ones although the underlying idea is still to reveal the structural properties at infinity, or more specifically to characterize dynamically the infinite modes and possible redundancy at infinity of the system.

One of the crucial observations made in Verghese et al. 1981 (see also Verghese 1978) is that the system represented by (1.1) which they call *singular system* may be capable of impulsive motion, i.e., the free-response of (1.1) under *arbitrary* initial conditions may contain impulses. These impulses have been shown to be linked to the infinite zero structure of $Es - A$ (defined via the Smith-McMillan form). This result parallels, in an interesting way, the well-known connection between the free exponential modes and the finite zero structure of $Is - A$ in the state-space case. This special type of modes has enabled them to define the concept of controllability (or modal controllability, as in their terminology). Although Verghese's notion of controllability has largely been motivated by the dynamical considerations, it differs from reachability as defined by Ronsenbrock and Yip et al. These results also formed a basis for them to gain important insight into a number of problems including composite systems, degenerate networks (see, for example, Dervisoglu and Desoer 1975 for basic definitions), and singularly perturbed systems.

Since (1.1) is only one of the possible representations for non-causal systems, it is natural to shift attention to some more general representations, such as (1.2). In fact, extensive efforts have also been made in Verghese 1978, Verghese and Kailath 1979 to study the infinite structure of the systems represented by (1.2). By utilizing the so-called normalized form of (1.2), which can be seen as an inverse of (1.2), they were able to define the concepts of infinite modes, controllability and so on. The way which they followed is essentially the same as the one in handling (1.1).

Quite recently, Cobb (1984) took a formal theoretical approach in an attempt to unify the aforementioned two treatments to (1.1). By resorting to the theory of distributions, he was able to describe precisely (in the analytical sense) the jump behaviour of dynamical systems represented by (1.1). The description subsequently led him to introduce the concept of impulsive controllability which is proven to be consistent with controllability at infinity defined by Verghese et al. (1981), and serves as an alternative characterization of controllability at infinity in the time domain. In Cobb's term, controllability at infinity, although not directly related to the reachable sets, is equivalent to the ability to generate a maximal class of impulses using only non-impulsive controls. It is easy to see, however, from within Cobb's framework, that when applied to the same system, the reachability and controllability are two completely different concepts, each of which deals with a different dynamical aspect of the system. From this point of view, their unification seems out of the question.

The above fruitful results, although mainly centered around control-

lability, reachability and their algebraic dual concepts of reconstructibility and observability, have indeed paved the way for further researches in other aspects related to (1.1). Cobb (1981) made the first attempt in an effort to solve the pole placement problem for (1.1) by applying a feedback of the form $u = Fx$. Since Cobb's approach is based on the canonical decomposition of (1.1), which is a simple application of the classical Weierstrass canonical form for regular pencils of matrices, it in essence considers separately the feedback for finite modes and the one for infinite modes of the system. These two types of feedbacks are termed *fast* and *slow* feedback by Cobb (1981). There are a number of interesting consequences derived in this work, namely, within the framework of Rosenbrock, all the finite modes of a reachable system can be shifted to arbitrary finite locations by the slow feedback; the fast feedback can bring some (certainly not all) infinite modes of the system to arbitrary finite locations; whereas within the framework of Verghese, all the infinite modes of a controllable system can be placed to arbitrary finite locations. The approach taken by Cobb, however, has ruled out the possibility of considering the arbitrary linear feedback.

The optimal regulation of (1.1) with quadratic cost function also attracted some interests. The question was raised by Pandolfi (1981) who also obtained a feedback solution.

As was pointed out by Cobb (1983), Pandolfi's formulation of the problem is not satisfactory since it essentially ignores all infinite modes of (1.1), thereby reduces to the regular, but reformulated, LQ problem. Aimed at rectifying the above deficiency, Cobb (1983) provided yet another solution within the framework of Verghese. It is quite conceivable that the optimal solution is a constant descriptor vector feedback, and the feedback gain can be obtained by solving a Riccati equation (see Cobb 1983 for further details).

Recent researches in the area have been extensive and diversifying, with more emphasis on systems represented by (1.1) (the so-called singular system). This more general type of linear systems apparently has opened a new research domain and has attracted not only control and system engineers, but mathematicians, ecomonists, circuit theorists as well.

The topics covered by the so-called theory of singular system have been mainly extensions of the well-established topics for state-space system. These include observer design (El-Tohami et al. 1983, 1984), realization (Bosgra and Van der weiden 1981, Cullen 1986, Christodoulou and Mertzios 1985, Christodoulou 1987), decoupling by the descriptor vector feedback (Christodoulou 1986), inversion (El-Tohami et al. 1985, Lewis 1983) and so on. Meanwhile, different characterizations and discussions of controllability and reachability, and their algebraic duals, pole placement, LQ control are still among the problems which attract the major attention (Lewis 1985,

Lewis and Ozcaldiran 1984, Van der weiden and Bosgra 1980, Armentano 1986, Zhou et al. 1987, Fletcher et al. 1986, Kucera 1986, Lovass-Nagy et al. 1986). We can certainly expect that the problems like Kalman filtering, disturbance decoupling of singular system and multidimensional singular system will soon be touched upon.

Despite the fact that the recent researches are active, and the results are enormous in this area, confusions and misunderstandings have also arisen. In fact, a number of fundamental issues still need to be clarified. The following is a list of some of these issues from our point of view.

(a). As has been made clear by Verghese et al. (1981), the impulsive motion and redundancy at infinity comprise the nature of non-causal systems, which is distinct from causal systems. Therefore, any attempt to exclude these concepts from the discussion when dealing with (1.1) or (1.2) should not be considered relevant to the generalized theory. A number of recent research papers assume, even as the starting point, that the singular system at hand is devoid of impulsive motions. This assumption is nothing more than saying that the underlying system is in fact a state-space system, but represented in a singular system fashion (such as (1.1)), which is, of course, possible. Therefore, their theories developed simply reformulate and solve the problems whose solutions are well-known. Further, the approaches based on first transforming (1.1) to a state-space system then trying to infer some properties of (1.1) from the state-space system appears to be quite unnatural.

(b). The concepts of reachability, controllablity and their algebraic duals were originally introduced in order to characterize the possible redundancy at infinity for non-causal system in much the same way as the well-known concepts of controllability and its algebraic dual were introduced in order to characterize the finite redundancy for causal systems.

It is thus easily understood that when the mode (finite and infinite) related operations are considered, such as mode placement, mode observation, etc., these concepts must be included to guarantee the absence of the redundancy. There are also some authors who have claimed to be able to construct observers without observability assumption. Their results are therefore really questionable.

(c). There are essentially two frameworks in dealing with (1.1), namely the framework of Rosenbrock and that of Verghese. It may seem difficult to decide in which framework the problem should be solved. For example, when designing an observer for a singular system should we choose to use observability (in Rosenbrock's framework), or reconstructibility (in

Verghese's framework)?

A tendency of recent research papers seems to be in favour of Rosenbrock's framework, because it allows easy generalizations of existing results from state-space systems. The fundamental reason for this, however, is not clear at all.

(d). The ultimate purpose of studying non-causal systems should be to provide adequate tools to analyze and synthesize the practical systems which are beyond the scope covered by the techniques for causal systems. Unfortunately, the recent investigations in this area seem disappointing in the sense that they do not suggest much of the practical implications of the theory.

In view of the these issues, we realize that there is a *need* to construct a *unified and solid basis* which substantiates the theory developed so far, and to closely examine the practical significance of the theory.

The primary purpose of this thesis therefore is to provide such a basis on which the first three issues raised are clarified and differences are reconciled satisfactorily. Although it is not our intention to discuss the practical aspects of the theory, we shall nevertheless point out in the next section some typical situations where the systems can be modelled by (1.1) or (1.2), and provide some examples to show the applications of the theory. The exposition of the contribution of the thesis along with a detailed review of the content will be deferred till Sec.1.3.

1.2 Practical Examples

In this section we provide several examples which show the need to consider the more general type of the systems like (1.1), (1.2).

Example 1.1 *Asymptotic behaviour of singularly perturbed systems*

Consider a system modelled by

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1u \\ \epsilon\dot{z} &= A_{21}x + A_{22}z + B_2u \end{aligned} \tag{1.3}$$

where ϵ is a small parameter. Such a system arises in many practical situations where ϵ may represent certain negligible physical parameters (Kokotovic et al. 1976). One of the most natural problems is to obtain the asymptotic solution of (1.3), i.e.,

$$\begin{aligned} \bar{x}(t) &= \lim_{\epsilon\to 0} x(t,\epsilon) \\ \bar{z}(t) &= \lim_{\epsilon\to 0} z(t,\epsilon) \end{aligned} \tag{1.4}$$

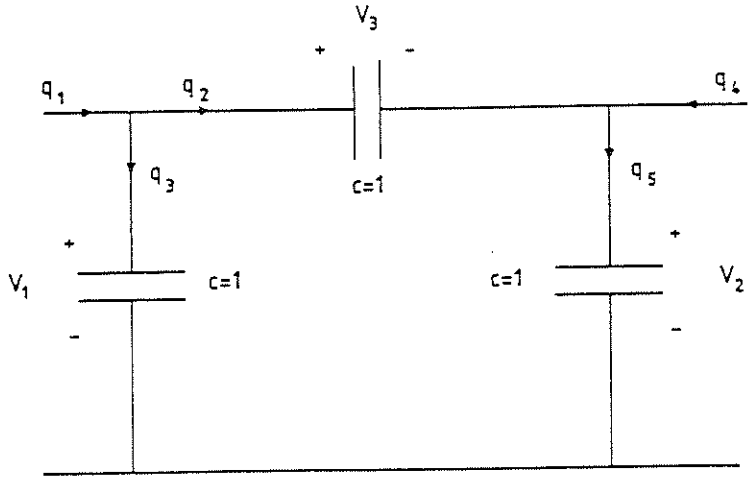


Figure 1.1: A simple degenerate circuit

where $x(t,\epsilon), z(t,\epsilon)$ are some normal solutions under the desired initial conditions, which depend, of course, on ϵ . It is certainly of interest to know whether the solution (1.4) is actually the one for the reduced system

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1u \\ 0 &= A_{21}x + A_{22}z + B_2u \end{aligned}$$

or written in a sightly different (but equivalent) way

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{1.5}$$

We can see that (1.5) is in the form of (1.1). For the special case where A_{22} is non-singular, the answer is known and in fact constitutes the essential part of the singular perturbation theory. However, it is quite recently that the attention has been turned to the more general case where the non-singularity of A_{22} is not required (see Francis 1981 and some related references therein). It can be expected that a complete theory for (1.1) can serve to develop this more general perturbation theory for the so-called *singular singularly perturbed systems.* □

Example 1.2 *A simple electrical circuit*

The circuit in FIG. 1.1 was originally analyzed by Rosenbrock et al. (1974). It is also briefly discussed in Verghese et al. (1981) more in the vein

of the theory developed by them. Here we shall use it as a simple example to motivate the reader.

This circuit is somehow degenerate in the classical sense. However, we shall see that it fits into the general framework of (1.1), thus can be analyzed using the general theory.

Let us assume that q_1, q_4 are the two inputs, and v_1, v_2 are the outputs of the circuit. By way of direct inspection, we can readily set up the following equation

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_5 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \\ q_5 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_4 \end{bmatrix} \qquad (1.6)$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \\ q_5 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Now the circuit in FIG. 1.1 is represented in the form (1.1) and ready to be analyzed. The general theory is especially significant since it can recover from (1.4) the structure of the infinite modes associated with this circuit. These properties, however, can never be seen from the viewpoint of the state-space theory. □

Example 1.3 *Switched-capacitor networks*

It is shown in Vandewalle et al. 1981 that a linear T-periodic SC network (with only ideal elements) can be described in the time domain by

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} v_{k+p} \\ i_{k+p} \end{bmatrix} = \begin{bmatrix} E_k v_{k+p-1} \\ 0 \end{bmatrix} + \begin{bmatrix} w_{k+p} \\ u_{k+p} \end{bmatrix} \qquad (1.7)$$

and

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} v^*(t) \\ i^*(t) \end{bmatrix} = \begin{bmatrix} w^*(t) \\ u^*(t) \end{bmatrix} \qquad (1.8)$$

where

$$\begin{aligned} v_{k+p} &= v(t^-_{k+p+1}) \\ v^*(t) &= v(t) - v_{k+p} \\ &\quad \text{for } t \in \Delta_{k+p} \end{aligned}$$

and analogous definitions and notations apply to $i^*(t), i_{k+p}, w^*(t), w_{k+p}$, $u^*(t), u_{k+p}$; $v(t)$ (resp. $u(t)$) is the vector of the voltage responses at the nodes (resp. voltage sources in some selected branches), and $i(t)$ (resp. $w(t)$) is the vector of the currents flown in some selected branches (resp. injected by current sources in the nodes) between t^-_{k+p} and t for $t \in \Delta_{k+p}$

Equation (1.7) is a mixture of difference and algebraic equations and easily be rearranged into

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} v_{k+p} \\ i_{k+p} \end{bmatrix} = \begin{bmatrix} E_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{k+p-1} \\ i_{k+p-1} \end{bmatrix} + \begin{bmatrix} w_{k+p} \\ u_{k+p} \end{bmatrix} \qquad (1.9)$$

which again fits with (1.1). It is quite natural to expect that the suitable model for SC networks has to be singular system, because such networks involve frequent change of states, and their impulsive behaviour is therefore essential. The reader may note that the classical solution to (1.9) is to separate first the difference equations from the algebraic ones, it therefore restricts the attention to the regular behaviour of the network. □

From the above examples, It is readily seen that singular systems represent an even broader class of linear systems than the state-space systems. More importantly, the study of this more general type of systems will certainly bring about deeper understanding and better control of the physical world. To illustrate this point, let us take the so-called impulsive behaviour of the systems as the first example.

Consider a system composed of a number of components or subsystems. It can be imagined intuitively that any component failure will cause abrupt change of the configuration of the system, thus the reconfigured system will start up assuming the states just before the failure as the initial conditions. As a result, the system will exhibit impulsive responses even if being excited only by non-impulsive inputs. Similarly, for a system involving internally frequent switching (the SC networks are a typical example), exactly the same situation described above will happen, although this time the frequent change of the system configuration is the desired feature. The point is how we can explain this impulsive behaviour. Since it is known already that the exponential behaviour of linear systems are linked to the eigenstructure of the matrix A, we certainly want to know if a similar link can be established for the impulsive counterpart. The answers to these problems are useful since if the impulsive behaviour is well understood, we can always find ways

to eliminate them. To achieve such an understanding is one of the tasks of studying singular systems.

As the second example, when physical systems are modelled as (1.1) or (1.2), it is of practical importance to try to find ways to control them. Apparently, without a deep insight into the structures of the systems, it is impossible to generate sensible control schemes. Moreover, without such an insight, we are even not sure about the qualitative nature of the system, such as whether or not the inputs could influence all the internal variables of the system, whether or not all the internal variables can actually be observed from the outputs, and so on.

There are still a great deal of practical situations where the models (1.1) and (1.2) enter naturally. Therefore, gaining a deep understanding of such systems is of great practical significance.

1.3 A General Overview of the Thesis

Having presented the practical usefulness and *the state of the art* of the theory, we can now provide an overview of the thesis.

As explained above, the main objective in this thesis is to set up a new algebraic framework within which we can provide satisfying answers to the questions posed at the end of Sec.1.1.

Aimed at providing a more complete theory for dealing with the infinite structure of linear system, we take an entirely different approach to the problem. The underlying idea of our approach is to consider some very fundamental mathematical concepts like polynomials and rational functions, and to try to build up a theory based on these simple concepts.

The treatment presented in this thesis is somewhat axiomatic in the sense that the infinity is introduced not by setting s to ∞ but rather by accounting for zero leading coefficient of polynomials. This simple viewpoint effectively puts all finite points and infinity into an equal framework. The treatment of the infinity, therefore, is carried out in much the same way as its finite counterparts.

Once these few guidelines are laid down, the development can proceed quite straightforwardly. We can easily define infinite zeros and infinite common factors for polynomials. These concepts can be characterized precisely within the ring of polynomials without resorting to rational functions. By defining a rational function to be the ratio of two polynomials, the concepts of infinite poles and zeros, irreducibility at infinity, etc. are just easy extensions from polynomials.

The main emphasis of the theory is, as can be expected, on polynomial and rational matrices where we follow the same line of development as

we do for polynomials and rational functions. Although polynomial and rational matrices are more involved we can still define, in an relatively easy way, the concepts of infinite poles and zeros, infinite common factor, and coprimeness, irreducibility at infinity.

The above concepts have indeed set up a new basis on which we can develop further the theory of realization, discuss the unification between the two major approaches in singular systems and so on.

We believe that the main contributions of the thesis are as follows:

(1). We provide a systematic treatment of the polynomials with zero leading coefficient. Even though the concept of infinite zero of this more general type of polynomials seems known (perhaps not so widespread, indeed, it eludes even some famous works and textbooks in algebra, such as Bocher 1907, Maclane and Birhoff 1967, Van der waerden 1953), the concept of infinite common divisor and coprimeness at infinity are believed to be new.

(2). The treatment mentioned in (1) has led to a completely new framework within which the infinite structure of polynomials, rational functions, polynomial matrices and rational matrices can be characterized and interpreted. This framework is believed to be another important contribution of the thesis.

(3). The classification of rational matrices into *pseudo-proper* and *general* is made for the first time. Such a classification enables to clarify and resolve the difference between Rosenbrock's and Verghese's approaches to singular systems. Following our analysis, it should be clear that the same system is viewed differently in these two approaches. In fact, this is the only difference between these two approaches. Apart from it, both reachability at infinity and controllability at infinity address the absence of cancellation at infinity. Therefore, a unification of these concepts is achieved in a purely algebraic way. Our treatment also provides a better understanding of these fundamental concepts.

(4). A realization theory is thought to be an important contribution of the thesis. We can now start with an MFD (pseudo-proper or general) and realize it into a singular system quite easily and without loss of structural information (finite and infinite). Based on our realization theory, we can easily establish an equivalence circle among complete irreducibility, joint reachability and observability (controllability and reconstructibility), minimality in dimension (order) in Rosenbrock's (Verghese's) framework.

(5). The development of canonical form for both pseudo-proper and gen-

eral singular systems is also regarded as one of the technical contributions of the thesis. These canonical forms open the doors for investigations of the descriptor vector feedback and observation of the descriptor vector. We have also demonstrated how the canonical forms can be used in the pole placement design and the construction of observers.

(6). Other contributions of the thesis include the development of a canonical form for regular pencils of matrices, an inversion technique for singular systems and a result on decoupling.

1.4 Chapter by Chapter Overview

Much effort has been made in order to organize the content of the thesis in a coherent way so as to ease the reading. By simply reading the chapters, however, the main line of the thesis, some results in each chapter and their connections to other results may still not be clear to the reader. Therefore, in this last section a chapter by chapter overview is provided in order to give a general scope of the results in each chapter.

Chapter 2 starts with the discussion of polynomials with zero leading coefficient. The treatment presented is less formal and rigorous but comprehensive and constructive. The development gradually carries the reader through rational functions, and finally ends up with polynomial and rational matrices. In the process, some important concepts such as homogeneous form, infinite zero, common factor at infinity for two polynomials and polynomial matrices, coprimeness at infinity, pseudo-proper and general MFD, complete irreducibility for MFD, etc. are introduced. Some characterizations of the coprimeness at infinity are obtained. It is shown that our result can provide a satisfying interpretation for the degenerate case of the Sylvester resultant. Based on the concept of homogeneous form, we have also derived a new canonical form for regular pencils of matrices.

The attention in Chapter 3 is focused on the pseudo-proper MFD and its realization into singular system. As will be seen, this actually fits into the framework set up by Rosenbrock. The most important results obtained are the procedures to realize pseudo-proper MFD's, and a pleasing equivalence circle among the complete irreducibility of pseudo-proper MFD, joint reachability and observability of the realization, and the minimality in dimension of the realization. We show at the end the chapter that the realization can also lead to a general analog-computer simulation scheme.

Chapter 4 is dedicated purely to the pseudo-proper singular system representation. We discuss, among others, the development of canonical forms, and two related topics, namely, the pole placement and the observer

design.

The general MFD and its realizations are discussed in Chapter 5. This chapter parallels the development in Chapter 3 and 4, except that neither the pole placement nor the observer design problems are considered. A similar equivalence circle also appears, which links the complete irreducibility of the general MFD, the joint controllability and reconstructibility of the realization, and the minimality in order of the realization. Fig.1.2 summarizes the equivalence circles for both the pseudo-proper MFD and the general MFD.

Chapter 6 is devoted to the application of the theory developed in Chapter 5. Two problems, namely, the singular system inversion and the decoupling, are solved within the framework we set up in the previous chapter.

We conclude the whole thesis in chapter 7 by mentioning a number of open problems and suggest the ways for further research.

As a final remark of this chapter, the terminology for singular system adopted in this thesis largely follows that of Lewis and Ozcaldiran (1984). Most of the terminology for MFD is standard, but wherever new terminology is needed, appropriate and suggestive names will be used.

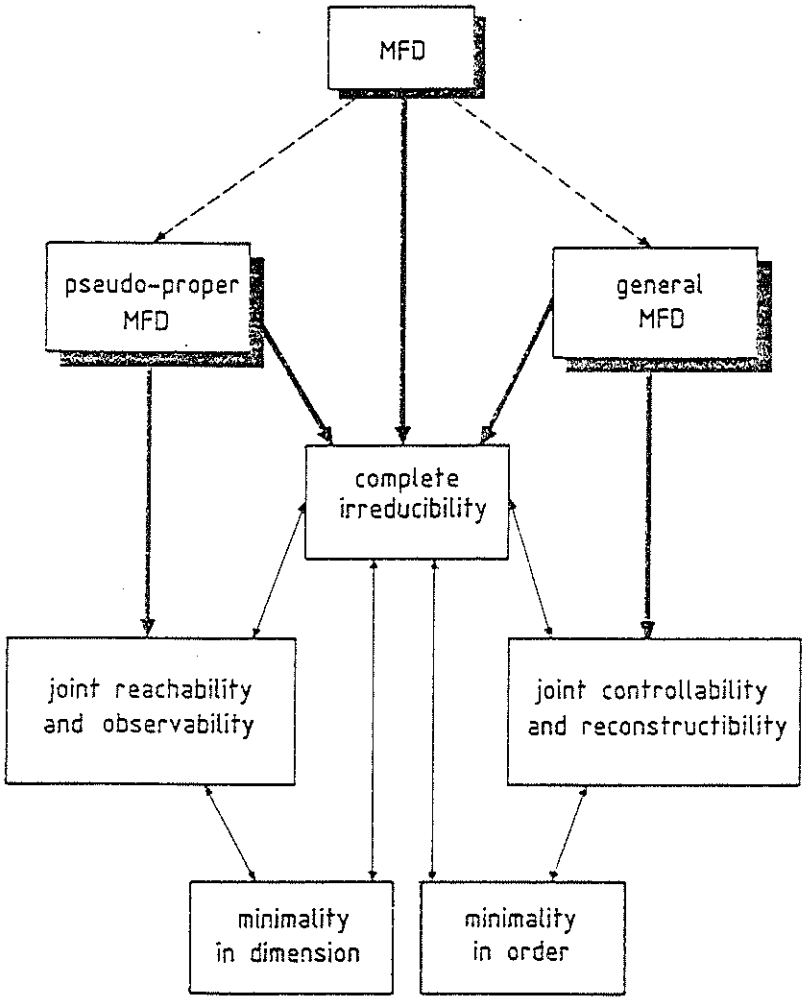


Figure 1.2: Equivalence circles for pseudo-proper and general MFD

Chapter 2

Polynomials and Rational Matrices at Infinity

2.1 Introduction

We begin the investigation by first examining the algebraic properties at infinity of polynomial and rational matrices. In the current literature such a topic is mainly treated using a conformal mapping and/or the well-known Smith-McMillan form (see, for example, Pugh 1976, Pugh and Ratcliffe 1979, Verghese and Kailath 1981, etc., and the references therein). Our primary treatment however is based on the concept of divisor at infinity for polynomial matrices. It will be shown that this new type of divisors arises quite naturally when the scope tends to include the polynomials with, possibly, zero leading coefficient, and the polynomial matrices whose entries belong to this class of polynomials.

The emphasis of the discussion will be on the so-called coprimeness at infinity for two polynomial matrices. As it is in the finite case, a common divisor at infinity can always be regarded as a measure of the amount of overlap at infinity between two polynomial matrices. The absence of such an overlap is really crucial in a number of situations (for example, for a class of dynamical systems known as singular systems which we are going to discuss in a considerable length in the ensuing chapters), and will be termed the coprimeness at infinity. The most important results derived in this chapter include the simple characterizations of the coprimeness at infinity for three classes of polynomial matrices. As a direct consequence, new interpretations for the degenerate case of the famous Sylvester resultant are provided satisfactorily.

Once the headway is made for polynomial matrices, it is relatively easy to proceed to rational matrices. In fact, any rational matrix can always be factored as a ratio of two (or more) polynomial matrices which fall into some specific classes. Similar to the finite case, this kind of factorization, or more generally, any polynomial matrix based factorizations may induce redundancy at infinity between the numerator(s) and the denominator(s). The concept of coprimeness at infinity therefore enters naturally into the context to describe the irreducibility at infinity between the numerator(s) and the denominator(s) of the factorized rational matrix. The concepts of general MFD and pseudo-proper MFD are also introduced in the development. We shall see that any rational matrix belongs to one of the two classes.

Our treatment of rational matrices is of special significance, since it settles the long-standing separation between the finite and the infinite structures, and puts the two into a unified framework. We shall show in Chapter 5 that our theory is consistent with that based on the Smith-McMillan form.

The key to a systematic presentation and derivation is the introduction of the homogeneous form for polynomials and polynomial matrices. The fundamental importance of the homogeneous form lies in the fact that it provides a natural mechanism to trace both the finite and the infinite structures for the corresponding polynomial matrix. It is especially advantageous in defining the divisor at infinity for polynomial matrices and in performing the algebraic operations involving polynomials with zero leading coefficient. We also show in this chapter that the homogeneous form can provide a neat and compact derivation of a canonical form for the regular pencils of matrices.

The development of this chapter reflects and extends the results in four of our early papers (Tan and Vandewalle 1986a, 1987a, 1986b, 1987b), and is divided into two parts. In the first two sections which follow, our attention is focused on presenting the basic theory of polynomials with zero leading coefficient, and polynomial matrices of finite degree. We introduce, among other things, the notions of homogeneous form, zeros, divisor at infinity for a polynomial and a polynomial matrix, common divisor at infinity, coprimeness at infinity for two polynomials and polynomial matrices. Key characterizations for some of these concepts are stated.

Section 2.4 puts emphasis on the pencils of matrices (which are matrix polynomials of degree one). The most interesting result derived is a new canonical form for regular pencils of matrices.

Section 2.5 is devoted to rational functions and rational matrices. The concept developed in the preceding sections are shown to be extremely useful in introducing and characterizing the irreducibility at infinity, and the complete irreducibility for rational matrices. A number of interesting

examples are also provided to demonstrate our theory and to clarify some confusions regarding rational matrices. The last section contains remarks and conclusions.

2.2 Polynomials and Divisor at Infinity

2.2.1 Algebraic Operations, Homogeneous Form and Zeros at Infinity

Let $R_n[s]$ denote the set of all polynomials $p(s)$ of the form

$$p(s) = p_0 s^n + p_1 s^{n-1} + \cdots + p_n \tag{2.1}$$

where each coefficient p_i $(i = 0, \ldots, n)$ belongs to the real field R; or more compactly, $R_n[s] = \{p(s) : p(s) = \sum_{i=0}^{n} p_i s^{n-i},\ p_i \in R\}$. The restriction to R is purely for simplicity of discussion. The idea, however, extends easily to other fields. It should be emphasized that $p_0 \neq 0$ is not required. Hence, essentially all polynomials of the form (2.1) are in $R_n[s]$. n will be called the *degree* of the polynomial (2.1), and p_0 the *leading coefficient* of $p(s)$. The reader may have realized that the familiar notion "degree" has been slightly generalized. Further, if $p_i = 0$ $(i = 0, \ldots, n)$, then $p(s)$ is said to be a zero polynomial, and is considered to have *any* degree. Another special case is $n = 0$, where we consider $R_0[s] = R$, the real field.

Since polynomials with zero leading coefficient are involved, a modification to the normal algebraic operations becomes necessary. To define the new set of algebraic operations for polynomials of either the same or different degrees, it is convenient to introduce the homogeneous form for each polynomial $p(s)$ in $R_n[s]$ (we refer to Tan and Vandewalle (1986d) for other approaches where the homogeneous form is not needed).

Given a polynomial in $R_n[s]$, say $p(s)$ (2.1), the following (two-variable) homogeneous polynomial, denoted by $p(s, r)$, can be uniquely constructed

$$p(s, r) = p_0 s^n + p_1 s^{n-1} r + \cdots + p_n r^n \tag{2.2}$$

and conversely, each homogeneous polynomial $p(s, r)$ corresponds uniquely to a polynomial $p(s)$. We shall call $p(s, r)$ the *homogeneous form* for $p(s)$. Note that a polynomial of zero degree, i.e., an element in R, is considered to be its own homogeneous form.

One of the most obvious advantages of the homogeneous form for a polynomial is that it carries both the degree and the set of coefficients in a way independent of the value of p_0. For example, if $p_0 = 0$, the term $p_0 s^n$ in (2.2) could be removed without affecting the polynomial itself, since the

missing term can easily be recovered from the degree preserved by the other terms in (2.2). However, the removal of p_0 ($= 0$) in $p(s)$ itself would mean an irreversible loss of the original degree, unless the degree is kept separately. Thus, in what follows when the homogeneous forms for polynomials are concerned, all the leading zero terms will be dropped.

Other advantages of the homogeneous form emerge when the algebraic operations among polynomials are defined upon them.

Definition 2.1 *Let $a(s) \in R_n[s], b(s) \in R_m[s], c(s) \in R_k[s], \alpha \in R, \alpha \neq 0$; and $a(s,r), b(s,r), c(s,r)$ be the homogeneous forms for $a(s), b(s), c(s)$ respectively. Then*

Equivalence $a(s) = b(s)$ *if* $a(s,r) = b(s,r)$;

Addition and substraction $a(s)+b(s) = c(s)$ *if* $a(s,r)+b(s,r) = c(s,r)$;

Scalar multiplication $\alpha \cdot a(s) = b(s)$ *if* $\alpha \cdot a(s,r) = b(s,r)$;

Polynomial multiplication $a(s) \cdot b(s) = c(s)$ *if* $a(s,r) \cdot b(s,r) = c(s,r)$.

□

We note that since the algebraic operations for two-variable homogeneous polynomials are well-defined, the above definition is meaningful.

Two remarks are in order concerning the operations defined above. First, the equivalence, addition and substraction are defined only for two polynomials of the *same* degree. This is due to the fact that two homogeneous polynomials of different degrees can never be equivalent, and the addition and/or substraction can not lead to a homogeneous polynomial. Secondly, since the properties of algebraic operations for homogeneous polynomials have been well-established, and since the operations of the polynomials are defined as those of their corresponding homogeneous forms, we can deduce all properties of the algebraic operations, such as the commutativity, associativity and distributivity, etc. It would be tedious to produce the proofs for all these properties from Definition 2.1. Nevertheless, the reader should convince himself that all kinds of manipulations found for the monic polynomials are valid here, provided that the resulting polynomial is homogeneous.

Now we proceed to define zeros of the polynomial.

Definition 2.2 *Let $a(s) \in R_n[s]$ and $a(s,r)$ be its corresponding homogeneous polynomial. Further, let $a(s,r)$ be factorized into*

$$a(s,r) = \tilde{a}(s,r) \cdot r^k \tag{2.3}$$

where k is a non-negative integer, $\tilde{a}(s,0) \neq 0$. Then $a(s)$ is said to have k zeros at infinity. All finite zeros and their multiplicities are defined to be those for the polynomial $\tilde{a}(s,1)$. □

Note that $p(s,1)$ is simply obtained by setting the parameter r to 1 in $p(s,r)$. This definition in fact tells that the number of the zero leading coefficients of a polynomial determines the number of the infinite zeros of that polynomial. More precisely, if a given polynomial $p(s)$ is such that $p_0 = p_1 = \cdots = p_{k-1} = 0$, but $p_k \neq 0$, then the number of the infinite zeros of $p(s)$ will be k. Furthermore, an interesting theorem follows directly from the above definition.

Theorem 2.1 *A polynomial $p(s) \in R_n[s]$ has n zeros (finite and infinite), multiplicity counted.* □

This theorem can easily be proved by noting that the number of finite zeros of $p(s,1)$ (defined by the factorization (2.3)) is precisely $n-k$.

As a final remark of this subsection, we observe that the concept of poles is not defined for polynomials. Therefore polynomials are no longer regarded as a special class of rational functions. We shall see in Section 2.5 that this viewpoint is crucial in clarifying a number of confusions.

To conclude this subsection, we provide a simple example to clarify the ideas made.

Example 2.1 The polynomial is given by

$$p(s) = 0 \cdot s^3 + 0 \cdot s^2 + 2s + 1 \tag{2.4}$$

which is clearly a polynomial in $R_3[s]$. The homogeneous form of $p(s)$ is easily found to be

$$p(s,r) = 2sr^2 + r^3 \tag{2.5}$$

(2.5) can obviously be factorized into

$$p(s,r) = (2s + r)r^2 \tag{2.6}$$

From Definition 2.2, we know immediately that $p(s)$ has 3 zeros, i.e., a finite zero $s_0 = -1/2$ and two zeros at infinity. □

2.2.2 Common divisors and coprimeness at infinity

Given two polynomials $p(s) \in R_n[s]$ and $q(s) \in R_m[s]$, denote their corresponding homogeneous forms by $p(s,r)$ and $q(s,r)$, respectively. If $p(s,r) = \bar{p}(s,r)d(s,r)$ and $q(s,r) = \bar{q}(s,r)d(s,r)$, then $d(s,r)$ is called a common divisor of $p(s)$ and $q(s)$. We are particularly interested in the case where $d(s,r)$ is not a constant and is independent of s. If this is true, $d(r)(\equiv d(s,r))$ will be called a *common divisor at infinity* for $p(s)$ and $q(s)$.

Definition 2.3 *Two polynomials $p(s)$ and $q(s)$ are said to be coprime at infinity if they have no common divisor at infinity.* □

Conventionally, the leading terms with zero coefficient of a polynomial are often discarded. What is left then will be called the finite part of the polynomial. Given two polynomials, their finite parts may have non-trivial common divisors. If no such divisors exist, these two polynomials are said to be coprime. This concept is in fact well-established.

Let us combine the familiar coprimeness with the concept of coprimeness at infinity. What we arrive at is a complete description of the relative primeness between two polynomials whose leading coefficient may be zero.

Definition 2.4 *Two polynomials $p(s)$ and $q(s)$ are said to be completely coprime if they are both coprime and coprime at infinity.* □

Since there have been a number of characterizations for the well-known coprimeness, to check whether two polynomials are coprime or not is therefore a simple task. However, to check the coprimeness at infinity is even simpler, as shown in the following characterization.

Theorem 2.2 *Two polynomials $p(s) \in R_n[s]$ and $q(s) \in R_k[s]$ are coprime at infinity if and only if p_0 and q_0 are not simultaneously zero; where p_0, q_0 are the leading coefficients of $p(s)$ and $q(s)$, respectively.*

proof Let $p(s,r), q(s,r)$ be the homogeneous forms for $p(s)$ and $q(s)$, respectively. Clearly if $p_0 = q_0 = 0$, $p(s,r)$ and $q(s,r)$ can both be factorized into $p(s,r) = \bar{p}(s,r)\cdot r$ and $q(s,r) = \bar{q}(s,r)\cdot r$, hence they cannot be coprime at infinity. Conversely, suppose that either p_0 or q_0 is non-zero, say $p_0 \neq 0$. Then, because of the leading term $p_0 s^n$, $p(s,r)$ can never have a factor at infinity. This is equivalent to saying that $p(s)$ and $q(s)$ are coprime at infinity. □

Given two polynomials $p(s)$ and $q(s)$ of degrees n and k, a so-called Sylvester matrix (Bocher 1907, Jury 1974) can be built upon the coefficients of the two polynomials as the following $(n+k)\times(n+k)$ matrix

$$S(p,q) = S\begin{pmatrix} p_0 & p_1 & \cdots & p_n \\ q_0 & q_1 & \cdots & q_k \end{pmatrix}$$

$$= \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & \cdots & p_n & & & \\ & p_0 & p_1 & \cdots & \cdots & p_{n-1} & p_n & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & p_{n-1} & p_n \\ & & & \cdots & q_0 & q_1 & \cdots & q_{k-1} & q_k \\ & & & q_0 & q_1 & \cdots & \cdots & q_k & \\ & & \cdots & \cdots & \cdots & \cdots & & & \\ q_0 & q_1 & \cdots & \cdots & q_k & & & & \end{bmatrix} \quad (2.7)$$

The determinant of the above matrix is called the *Sylvester resultant*. It can be used to check if $p(s)$ and $q(s)$ are coprime. As a standard result (see, for example, Bocher 1907, p.195) if $p_0 \neq 0$ or $q_0 \neq 0$, the following is true:

Lemma 2.1 *The Sylvester resultant is nonzero if and only if $p(s)$ and $q(s)$ are coprime.* □

It is easy to observe that confined to the conventional concept of coprimeness, the case when $p_0 = q_0 = 0$ can only be considered as a degenerate one. However, in the present framework, this case is no longer degenerate, since it simply signals the presence of a totally different type of divisors — divisors at infinity — between two polynomials. If we combine Lemma 2.1 with Theorem 2.2 and Definition 2.4, it is then quite natural to obtain the following result.

Theorem 2.3 *Two polynomials $p(s)$ and $q(s)$ are completely coprime if and only if the Sylvester resultant $\det S(p,q) \neq 0$ (or equivalently, the Sylvester matrix is non-singular).*

proof It is easily seen that $det\, S(p,q) = 0$ if and only if $p(s)$ and $q(s)$ are not coprime or $p_0 = q_0 = 0$. The latter can happen if and only if $p(s)$ and $q(s)$ are not coprime at infinity by Theorem 2.2. □

The importance of Theorem 2.3 lies in that it provides a new interpretation for the well-known Sylvester resultant. We note that without the introduction of the coprimeness at infinity for polynomials, it would be difficult to conceive why the case $p_0 = q_0 = 0$ is such a special one.

2.3 Polynomial Matrices and Common Divisors at Infinity

2.3.1 Homogeneous Form, Algebraic Operations and Zeros at Infinity

A polynomial matrix is simply a matrix with polynomial entries. These entries can certainly be polynomials with zero leading coefficient. We note that it is no longer possible to change any entry of a polynomial matrix by adding some high-power terms with zero coefficient, since such an expansion may lead to a complete change of the polynomial matrix. This rule also implies that in general a polynomial matrix cannot necessarily be regarded as a polynomial with constant matrix coefficients (compare this argument with that in Gantmacher 1960a, p.76, 130).

It has been made clear in the previous section that every polynomial has its own homogeneous form. Therefore, if all entries of a polynomial matrix $P(s)$ are replaced by their corresponding homogeneous forms, the resulting matrix (denoted by $P(s,r)$) will only have homogeneous entries and will be called the *homogeneous form* of $P(s)$.

The algebraic operations among polynomials have been defined to be the same operations among their corresponding homogeneous forms provided that the outcomes of the operations remain homogeneous. Analogously, the algebraic operations among polynomial matrices will be defined as the same operations among their corresponding homogeneous forms provided that the outcomes of the operations are matrices with homogeneous entries. We provide a simple example to show that this definition excludes some seemingly possible operations.

Example 2.2 Let two polynomial matrices $P(s)$ and $Q(s)$ be given by

$$P(s) = \begin{bmatrix} 0\cdot s^3 + 0\cdot s^2 + s + 0 & s^2 + 0\cdot s + 0 \\ 0\cdot s^3 + 0\cdot s^2 + 0\cdot s + 1 & 0\cdot s^2 + 0\cdot s + 1 \end{bmatrix} \tag{2.8}$$

and

$$Q(s) = \begin{bmatrix} s^2 + s + 1 & 0\cdot s^2 + s + 1 \\ 1 & 1 \end{bmatrix} \tag{2.9}$$

The homogeneous forms for $P(s)$ and $Q(s)$ are obviously found to be

$$P(s,r) = \begin{bmatrix} sr^2 & s^2 \\ r^3 & r^2 \end{bmatrix} \tag{2.10}$$

and

$$Q(s,r) = \begin{bmatrix} s^2 + sr + r^2 & sr + r^2 \\ 1 & 1 \end{bmatrix} \tag{2.11}$$

Then we see that $Q(s,r)\cdot P(s,r)$, or equivalently, $Q(s)\cdot P(s)$ is well defined, since all elementwise operations are defined. The product turns out to be

$$\begin{aligned} R(s,r) &= Q(s,r)\cdot P(s,r) \\ &= \begin{bmatrix} s^3r^2 + s^2r^3 + 2sr^4 + r^5 & s^4 + s^3r + s^2r^2 + sr^3 + r^4 \\ sr^2 + r^3 & s^2 + r^2 \end{bmatrix} \end{aligned}$$

$R(s,r)$ is actually the homogeneous form of $R(s)$ which can easily be read out from $R(s,r)$

$$R(s) = \begin{bmatrix} 0\cdot s^5 + 0\cdot s^4 + s^3 + s^2 + 2s + 1 & s^4 + s^3 + s^2 + s + 1 \\ 0\cdot s^3 + 0\cdot s^2 + s + 1 & s^2 + 0\cdot s + 1 \end{bmatrix}$$

However, neither $P(s) + Q(s)$ nor $P(s)\cdot Q(s)$ is defined since they all involve some undefined operations. □

For the more general case, it is readily checked that the addition of two polynomial matrices $P(s)$ $(m \times n)$, $Q(s)$ $(m \times n)$ is defined if and only if $deg\ p_{ij}(s) = deg\ q_{ij}(s)$, where $p_{ij}(s), q_{ij}(s)$ are the ijth element in $P(s)$ and $Q(s)$, respectively. Similarly, the multiplication of two polynomial matrices $U(s)$ $(m \times n), V(s)$ $(n \times p)$ is defined if and only if for any fixed pair of i, j, the degree of $u_{ik}(s)\cdot v_{kj}(s)$ is the same for all k, where $u_{ij}(s), v_{ij}(s)$ are ijth element in $U(s)$ and $V(s)$, respectively.

To proceed, let us define the determinant of a square polynomial matrix. The determinant of a square matrix $P(s)$ $(m \times m)$ is defined to be $det\ P(s,r)$, where $P(s,r)$ is the homogeneous form of $P(s)$. It is easy to see that not every square polynomial matrix has a determinant because of the operational constraints. In our last example, if the element q_{21} in $Q(s)$ were replaced by s, $Q(s)$ would have no determinant. Those square polynomial matrices whose determinant is not defined will be called *abnormal at infinity*, and will not be considered in this thesis. It is easy to see that the determinant of a polynomial matrix, if defined, will always be a homogeneous polynomial (recall that a constant scalar is also considered to be a homogeneous polynomial).

From the conventional definition of determinant for square matrices (see, for example, Rosenbrock 1970, p.2), we can state the necessary and sufficient condition for the existence of the determinant for a square polynomial matrix as follows: a square polynomial matrix $P(s)$ $(n \times n)$ has the determinant if and only if P_{ij} is defined, and $\sum_{j=1}^{n} p_{ij}P_{ij}$ is defined; where

p_{ij} is the ijth element of $P(s)$, and P_{ij} is the cofactor of p_{ij}. The above recursive condition is meaningful since a polynomial matrix of dimension 1×1 is defined to be its own homogeneous form.

Let the determinant of a polynomial matrix $P(s)$ be defined. If this determinant is zero, $P(s)$ is said to be singular. Otherwise, it is non-singular. We now can define zeros of a square polynomial matrix.

Definition 2.5 *The zeros (finite and infinite) of a non-singular square matrix $P(s)$ are defined as those of the determinant of $P(s)$.* □

The determinant of a non-singular polynomial matrix can be a non-zero constant. We shall call this type of polynomial matrices *unimodular*. A unimodular polynomial matrix apparently has no zeros. Thus multiplication with a unimodular will not change zeros of polynomial matrices. Moreover, the inverse of a unimodular polynomial matrix is also a unimodular polynomial matrix.

Zeros can also be defined for singular, or even non-square polynomial matrices. As can be expected, the definition is not as straightforward as for the non-singular ones, owing to the fact that the determinant is either zero, or not defined in these cases. It is possible, however, to proceed by resorting to the notion of generalized invariant polynomials. We shall not go any further in this direction since our development will not involve zeros of singular, or non-square polynomial matrices.

2.3.2 Three Basic Classes of Polynomial Matrices

The preceding subsection has only been concerned with arbitrary polynomial matrices. As we shall see in Sec.2.5, only three special classes of polynomial matrices especially deserve our attention, namely, *matrix polynomials, column-based polynomial matrices and row-based polynomial matrices*. In this subsection, we shall describe these three classes of polynomial matrices in detail.

Matrix Polynomials If all entries of a polynomial matrix $P(s)$ $(p \times m)$ have the same degree, then $P(s)$ will be called a *matrix polynomial*. It is evident that the homogeneous form $P(s,r)$ of $P(s)$ can be written as

$$P(s,r) = P_0 s^n + P_1 s^{n-1} r + \cdots + P_n r^n \tag{2.12}$$

where n is the degree which is common for all entries of $P(s)$; P_i $(i = 0, \ldots, n)$ are $p \times m$ constant matrices. On setting r to 1, (2.12) becomes

$$P(s,1) = P_0 s^n + P_1 s^{n-1} + \cdots + P_n \tag{2.13}$$

We note that (2.13) can actually be looked upon as a polynomial with constant matrix coefficients. This also explains the term "matrix polynomial". P_0 in (2.12) or (2.13) will be called the *leading coefficient matrix* of $P(s)$, and n will be called the degree of $P(s)$.

Column-based Polynomial Matrices If the degrees of entries of a polynomial matrix $Q(s)$ $(p \times m)$ in the same column are the same, then $Q(s)$ will be called a *column-based polynomial matrix*. For such a matrix, the ith column $(i = 1, \ldots, m)$ of its homogeneous form $Q(s,r)$ is representable in the following form

$$\alpha_0^i s^{k_i} + \alpha_1^i s^{k_i - 1} r + \cdots + \alpha_{k_i}^i r^{k_i} \tag{2.14}$$

where in (2.14), each column vector α_j^i is a constant p-vector; k_i is the degree for all entries in the ith column, and will be called the ith *column degree*.

The homogeneous form of a column-based polynomial matrix can always be decomposed into the form

$$Q(s,r) = Q_{hc} S_Q(s) + Q_{lc} T_Q(s,r) \tag{2.15}$$

where

$$\begin{aligned} Q_{hc} &= [\alpha_0^1 \ \alpha_0^2 \ \cdots \ \alpha_0^m] \\ S_Q(s) &= diag\{s^{k_1}, s^{k_2}, \ldots, s^{k_m}\} \\ T_Q{}^T(s,r) &= block\,diag\{[s^{k_i-1} r \ \ s^{k_i-2} r^2 \ \ \cdots \ \ r^{k_i}], \quad i = 1, \ldots, m\} \end{aligned}$$

and Q_{lc} is an appropriate constant matrix containing all coefficients of lower powers with respect to s. Q_{hc} will be called the *leading coefficient matrix* of $Q(s)$. We shall see later that Q_{hc} will play an important role in characterizing some specific properties of the matrix.

Example 2.3 As an example, consider the polynomial matrix $P(s)$ given in (2.8). Since the elements in the first column all have the degree 3, and those in the second column have 2, it is clearly a column-based polynomial matrix. Further, $P(s,r)$ can always be decomposed into (2.15) with

$$P_{hc} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$S_P(s) = diag\ \{s^3, s^2\}$$

$$T_P(s,r) = \begin{bmatrix} s^2r & 0 \\ sr^2 & 0 \\ r^3 & 0 \\ 0 & sr \\ 0 & r^2 \end{bmatrix}$$

$$Q_{lc} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

□

Row-based Polynomial Matrices If the degrees of the entries of a polynomial matrix $W(s)$ $(p \times m)$ in the same row are the same, $W(s)$ is called a *row-based polynomial matrix*. It is easy to see that a row-based polynomial matrix is the transpose of a column-based one. This fact enables to decompose the homogeneous form of $W(s)$ as follows

$$W(s,r) = S_W(s)W_{hc} + T_W{}^T(s,r)W_{lc} \tag{2.16}$$

which can be regarded as the transpose of (2.15). W_{hc} is similarly defined to be the *leading coefficient matrix* of $W(s)$. We note that k_i in $S_W(s)$ or $T_W{}^T(s,r)$ is the ith row degree of W(s).

The relationships among these three classes of polynomial matrices can easily be established. It turns out that a matrix polynomial is a special column (or row)-based polynomial matrix, while a row-based polynomial matrix is the transpose of a column-based one. Therefore, in most cases, it suffices to consider only column-based polynomial matrices.

An important feature associated with these three classes of matrices is that if they are square their determinant is well defined. Moreover, let a square non-singular polynomial matrix $P(s)$ in one of these three classes be given, and denote by $deg\ detP(s,r)$ the determinantal degree of $P(s)$, and by $k_i = deg\ p_i(s)$ (where $p_i(s)$ is the ith column (row) of $P(s)$) $(i = 1, \ldots, m)$ the ith column (row) degree, the following relationship must always hold

$$deg\ detP(s,r) = \sum_{i=1}^{m} k_i \tag{2.17}$$

The reader may wish to draw a comparison between Eq. (2.17) and the well-known inequality given, for example, in Kailath 1980, p.384. We can now interpret that inequality as follows. The conventional definition of the determinant of a polynomial matrix only takes the finite zeros into

consideration, whereas the sum of column degrees reflects both the number of finite and the number of infinite zeros of a polynomial matrix.

Other features of these three classes of polynomial matrices can also be seen when they undergo algebraic operations. The following two lemmas present some interesting factorizations of these polynomial matrices.

Lemma 2.2 *If a matrix polynomial $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s) \tag{2.18}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ must be row-based. □

Lemma 2.3 *If a column-based polynomial matrix $P(s)$ can be factorized into*

$$P(s) = Q(s)R(s) \tag{2.19}$$

where $R(s)$ is a square matrix, then $Q(s)$ must be column-based and $R(s)$ is an arbitrary polynomial matrix whose determinant is defined. □

The reader should have no difficulty in carrying over Lemma 2.3 to row-based polynomial matrices. These lemmas follow directly from the definitions of operations. To show, for example, Lemma 2.3 , we simply recall the relation $detP(s,r) = det\,Q(s,r) \cdot detR(s,r)$. Since both $detP(s,r)$ and $det\,Q(s,r)$ are homogeneous, $detR(s,r)$ must also be homogeneous, which implies that $R(s)$ has the determinant.

2.3.3 Common Divisors and Coprimeness at Infinity

Consider two polynomial matrices $D(s)$ of dimension $p \times p$, and $N(s)$ of dimension $m \times p$. The homogeneous forms of the two matrices are $D(s,r)$ and $N(s,r)$, respectively. If $D(s,r) = \bar{D}(s,r)R(s,r)$ and $N(s,r) = \bar{N}(s,r)R(s,r)$, where $R(s,r)$ is a square non-singular polynomial matrix, then $R(s,r)$ is called a *common right divisor* of $D(s)$ and $N(s)$.

If we are only considering column-based polynomial matrices, then the determinant of a right common divisor for such matrices exists (recall Lemma 2.3). We shall call such a divisor *non-trivial*, if its determinant is not a constant, or equivalently, it is not unimodular. The development which follows *only* applies to matrix polynomials and column-based polynomial matrices. Therefore the scope covered by the term "polynomial matrix" should be clearly understood.

Definition 2.6 *A non-singular matrix $R(s,r)$ is called a common right divisor at infinity for polynomial matrices $D(s)$ and $N(s)$, if*

1. *it is a common right divisor of* $D(s)$ *and* $N(s)$;
2. *det* $R(s,r) = \alpha \cdot r^{\sigma}$, *where* α *is a constant scalar different from zero, and* σ *is a positive integer.* □

The above definition allows to introduce the notion of *coprimeness at infinity* for two polynomial matrices.

Definition 2.7 *Two polynomial matrices with the same number of columns are said to be right coprime at infinity if they have no common right divisors at infinity.* □

Exactly as for the scalar polynomial case, a characterization of this concept can be stated as a rank condition for a constant matrix constructed with the leading coefficient matrices of both $D(s)$ and $N(s)$. This characterization can also be regarded as a criterion to check if two given polynomial matrices are right coprime at infinity.

Theorem 2.4 *Let* $D(s)$ *and* $N(s)$ *be two column-based polynomial matrices of dimensions* $p \times p$ *and* $m \times p$, *respectively. Then* $D(s)$ *and* $N(s)$ *are right coprime at infinity if and only if*

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p \tag{2.20}$$

where D_{hc} *and* N_{hc} *are the leading coefficient matrices of* $D(s)$ *and* $N(s)$, *respectively.*

proof Without loss of generality we shall assume that both $D(s)$ and $N(s)$ are such that their column degrees are arranged in ascending order.

Necessity Suppose that $D(s)$ and $N(s)$ are right coprime at infinity but the columns of the matrix $[D_{hc}{}^T \quad N_{hc}{}^T]^T$ are not linearly independent. Then a non-singular $p \times p$ unimodular matrix $U(s,r)$ can be found such that the last column of the matrix

$$\begin{bmatrix} D_{hc}S_D(s) \\ N_{hc}S_N(s) \end{bmatrix} U(s,r)$$

is zero (where $S_D(s)$ and $S_N(s)$ is defined in (2.15)). If this $U(s,r)$ is applied to $[D^T(s,r) \quad N^T(s,r)]^T$, all the entries in the last column of the matrix

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} U(s,r) \tag{2.21}$$

will have the common factor r. Thus (2.21) can be factorized as

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} U(s,r) = \begin{bmatrix} \bar{D}(s,r) \\ \bar{N}(s,r) \end{bmatrix} B(r) \tag{2.22}$$

where $B(r) = diag\{1, \ldots, 1, r\}$, $\bar{D}(s,r)$ and $\bar{N}(s,r)$ denote matrices $D(s,r) \cdot U(s,r)$ and $N(s,r) \cdot U(s,r)$ after the extraction of r from the last column.

Postmultiplying both sides of (2.22) by $U^{-1}(s,r)$, and defining $R(s,r) = B(r)U^{-1}(s,r)$ we obtain

$$\begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} = \begin{bmatrix} \bar{D}(s,r) \\ \bar{N}(s,r) \end{bmatrix} R(s,r) \tag{2.23}$$

Since *det* $R(s,r) = (det\ U^{-1}(s,r)) \cdot r$, $D(s)$ and $N(s)$ can not be right coprime at infinity by Definition 2.7. This contradiction proves the necessity.

Sufficiency Suppose that $[D_{hc}{}^T \quad N_{hc}{}^T]^T$ is of full column rank, but $D(s)$ and $N(s)$ are not right coprime at infinity. Then a common right divisor at infinity, say $R(s,r)$, exists such that $D(s,r) = \bar{D}(s,r)R(s,r)$ and $N(s,r) = \bar{N}(s,r)R(s,r)$ (remind that *det* $R(s,r) = \alpha \cdot r^{\beta}$, with $\alpha \neq 0$ and β a positive integer). We now try to compute *rank* $[D_{hc}{}^T \quad N_{hc}{}^T]^T$ via these two factorizations.

As noted in Lemma 2.3, since $D(s,r)$ is a column-based polynomial matrix, its factor $\bar{D}(s,r)$ must be a column-based polynomial matrix and $R(s,r)$ is such that the determinant is defined. The same is true for $\bar{N}(s,r)$ and $R(s,r)$. Therefore, we can always write (see the discussion in the previous subsection)

$$\bar{D}(s,r) = \bar{D}_{hc}S_{\bar{D}}(s) + \bar{D}_l(s,r) \tag{2.24}$$

where $S_{\bar{D}}(s) = diag\{s^{k_1}, s^{k_2}, \ldots, s^{k_p}\}$, k_i $(i = 1, \ldots, p)$ is the ith column degree of $\bar{D}(s,r)$; $\bar{D}_{hc}$ is the leading coefficient matrix of $\bar{D}(s,r)$; $\bar{D}_l(s,r)$ denotes the matrix containing the lower degree (with respect to s) terms.

To proceed , we note that

$$\begin{aligned} D(s,r) &= \bar{D}(s,r)R(s,r) = [\bar{D}_{hc}S_{\bar{D}}(s) + \bar{D}_l(s,r)]R(s,r) \\ &= \bar{D}_{hc}W(s,r) + \bar{D}_l(s,r)R(s,r) \end{aligned} \tag{2.25}$$

where $W(s,r) = S_{\bar{D}}(s)R(s,r)$. Observe that $W(s,r)$ must also be column-based (otherwise $D(s,r)$ would not be column-based). Therefore $W(s,r)$ can be decomposed into

$$W(s,r) = W_{hc}\ diag\{s^{\alpha_1}, s^{\alpha_2}, \ldots, s^{\alpha_p}\} + W_l(s,r) \tag{2.26}$$

where the α_i $(i = 1, \ldots, p)$ is the set of column degrees of $W(s,r)$. Since $det R(s,r)$ is only a homogeneous function of r, we must always have

$$\sum_{i=1}^{p} \alpha_i > \sum_{i=1}^{p} k_i$$

On the other hand, since $detW(s,r) = (detW_{hc})s^{\alpha_1+\alpha_2+\cdots+\alpha_p} + \cdots$, we must have $detW_{hc} = 0$. Hence W_{hc} is singular. In exactly the same way, we can write

$$\begin{aligned} N(s,r) &= \bar{N}(s,r)R(s,r) = [\bar{N}_{hc}S_N(s) + \bar{N}_l(s,r)]R(s,r) \\ &= \bar{N}_{hc}V(s,r) + \bar{N}_l(s,r)R(s,r) \end{aligned} \tag{2.27}$$

where $V(s,r) = S_N(s,r)R(s,r)$. Then $V(s,r)$ can be decomposed into

$$V(s,r) = W_{hc}\, diag\{s^{\beta_1}, s^{\beta_2}, \ldots, s^{\beta_p}\} + V_l(s,r) \tag{2.28}$$

in which the β_i $(i = 1, \ldots, p)$ is the set of column degrees of $V(s,r)$. Note that since $S_{\bar{D}}(s)R(s,r)$ and $S_{\bar{N}}(s)R(s,r)$ are of the same structure, they will have the same leading coefficient matrix W_{hc}.

Substitute (2.26), (2.28) into (2.25), (2.27) respectively, we obtain

$$D(s,r) = \bar{D}_{hc}W_{hc}\, diag\{s^{\alpha_1}, s^{\alpha_2}, \ldots, s^{\alpha_p}\} + D_l(s,r) \tag{2.29}$$

and

$$N(s,r) = \bar{N}_{hc}W_{hc}\, diag\{s^{\beta_1}, s^{\beta_2}, \ldots, s^{\beta_p}\} + N_l(s,r) \tag{2.30}$$

Where $D_l(s,r)$ and $N_l(s,r)$ are the appropriate polynomial matrices. We see clearly from (2.29) and (2.30) that

$$\begin{aligned} D_{hc} &= \bar{D}_{hc}W_{hc} \\ N_{hc} &= \bar{N}_{hc}W_{hc} \end{aligned} \tag{2.31}$$

Combining the foregoing discussions, it is clear that

$$\begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = \begin{bmatrix} \bar{D}_{hc} \cdot W_{hc} \\ \bar{N}_{hc} \cdot W_{hc} \end{bmatrix} = \begin{bmatrix} \bar{D}_{hc} \\ \bar{N}_{hc} \end{bmatrix} W_{hc}$$

since W_{hc} is singular, the columns of the matrix $[D_{hc}{}^T \;\; N_{hc}{}^T]^T$ can not be linearly independent. This contradicts the assumption. Hence, the theorem is proved. □

The notion of (finite) coprimeness for two polynomial matrices is well-known (A good reference book on this subject is Rosenbrock 1970). Here we simply recall that two polynomial matrices are coprime if their greatest common divisor is a unimodular matrix (it is assumed that common divisors are always non-singular). One characterization of this concept can be given as a rank condition of an appropriate matrix. A combination of this concept together with the concept of copimeness at infinity leads to the following definition.

Definition 2.8 *Two polynomial matrices are said to be completely right coprime if they are both right coprime and right coprime at infinity.* □

Based on Theorem 2.4 and the characterization of the right coprimeness (see, for example, Rosenbrock 1970), an interesting characterization of the complete coprimeness can be stated.

Theorem 2.5 *Let $D(s)$ and $N(s)$ be two column-based polynomial matrices. Then $D(s)$ and $N(s)$ are completely right coprime if and only if*

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p$$

and

$$rank \begin{bmatrix} D(s,r) \\ N(s,r) \end{bmatrix} = p, \quad \textit{for all finite } s.$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. □

The generalized Sylvester matrix for matrix polynomials has been introduced in Rowe 1972 (see also Bitmead et al. 1978). Similar to Theorem 2.2, we can use Theorem 2.4 to interpret the degenerate case of the Sylvester matrix. It turns out that this case simply reflects the existence of the common divisor at infinity. In fact, it is easy to show that the non-singularity of the Sylvester matrix is equivalent to the complete coprimeness of two matrix polynomials.

So far we have only treated the column-based polynomial matrices in considerable detail. Since matrix polynomials are a special case of column-based polynomial matrices, the foregoing results apply without change to matrix polynomials as well. It can, however, not be extended to row-based polynomial matrices. In fact, it is more natural to consider *left* common divisors and *left* coprimeness, instead of the right ones, for row-based polynomial matrices. The treatment can therefore be proceeded by recalling the algebraic duality of the discussions for column-based polynomial matrices. We shall omit the details.

2.4 Generalized Invariant Polynomials and Generalized Rational Canonical Forms for Regular Pencils of Matrices

In this section, we shall concentrate our attention to one special type of polynomial matrices. We show that when specialized to matrix polynomials

of dimension $n \times n$ and degree 1, some more specific results can be obtained. Such first degree square matrix polynomials are termed square *pencils of matrices*.

We apply the homogeneous form for regular pencils of matrices as a basis to define the set of *generalized invariant polynomials* for a pencil of matrices. We subsequently show that the *strict equivalence* of two pencils can be characterized by the set of generalized invariant polynomials. To carry the idea further, it is shown that for each generalized invariant polynomial a so-called generalized companion form can be built up. This then naturally leads to the generalized rational canonical form for regular pencils of matrices.

2.4.1 Generalized Invariant Polynomials and Strict Equivalence for Regular Pencils of Matrices

Let $Es - A$ be an $n \times n$ regular pencil of matrices, i.e., $det(Es - A) \neq 0$ for almost all s. As usual, the homogeneous form for such a pencil is $Es - Ar$. We note that this homogeneous form has been used in Gantmacher 1960b. It is easy to observe that the minors of order k $(k = 1, \ldots, n)$ of $Es - Ar$ are homogeneous polynomials of parameters s and r. Denoting by $d_k(s, r)$ the monic greatest common divisor of all minors of order k for the pencil $Es - Ar$, and setting $d_0(s, r) = 1$, we obtain a sequence of homogeneous polynomials $\{d_k(s, r)\}$ $(k = 0, \ldots, n)$. Since each $d_{k-1}(s, r)$ divides $d_k(s, r)$ $(k = 1, \ldots, n)$, $d_k(s, r)/d_{k-1}(s, r)$ is also a homogeneous polynomial. We thus can have the following definition.

Definition 2.9 *For a regular pencil of matrices Es-A, the set of homogeneous polynomials*

$$i_1(s, r) = \frac{d_n(s, r)}{d_{n-1}(s, r)}, i_2(s, r) = \frac{d_{n-1}(s, r)}{d_{n-2}(s, r)}, \cdots, i_n(s, r) = \frac{d_1(s, r)}{d_0(s, r)} \quad (2.32)$$

are called the generalized invariant polynomials. □

Two results follow directly from the above definition, namely, none of the $i_k(s, r)$ $(k = 1, \ldots, n)$ will be identically zero and not all $i_k(s, r)$ $(k = 1, \ldots, n)$ will equal 1. Moreover, the sequence $\{i_k(s, r), \ (i = 1, \ldots, n)\}$ possesses an interesting property stated below.

Lemma 2.4 *In the sequence of generalized invariant polynomials* $i_1(s, r)$, $\cdots, i_n(s, r)$, *each polynomial* $i_{j+1}(s, r)$ *divides the preceding one* $i_j(s, r)$ $(j = 1, \ldots, n - 1)$.

proof Let us note two simple facts. First, if s is set to 1, we have

$$p_k(r) = i_k(1, r) \quad (k = 1, \ldots, n) \quad (2.33)$$

a set invariant polynomials of the pencil $E - Ar$. Hence, by recalling the basic properties of the invariant polynomials of a pencil of matrices, the sequence $\{p_k(r), \ k = 1, \ldots, n\}$ will be such that from $p_2(r)$ onwards each divides the preceding one (see, for example, Barnett 1971). Secondly, an analogous property will hold if, instead of s, r is set to 1.

Now choose arbitrarily $i_\alpha(s, r)$ and $i_\beta(s, r)$ in (2.32), with $\alpha < \beta$. Factorize these two polynomials into powers of irreducible homogeneous polynomials, then

$$\begin{aligned} i_\alpha(s, r) &= f_1^{a_1}(s, r) \cdots f_u^{a_u}(s, r) \\ i_\beta(s, r) &= g_1^{b_1}(s, r) \cdots g_v^{b_v}(s, r) \end{aligned} \quad (2.34)$$

where $a_1, \cdots, a_u, b_1, \cdots, b_v$, are non-negative integers, and u, v are positive integers. Note that such factorizations can always be performed as far as the homogeneous polynomials are concerned, since two-variable homogeneous polynomials essentially have the same nature as the one-variable counterparts. This factorization is clearly invalid for two-variable non-homogeneous polynomials.

We claim that each $g_k(s, r)$ $(k = 1, \ldots, v)$ must be the same as one of the $f_m(s, r)$ $(m = 1, \ldots, u)$. Otherwise, by setting $s = 1$ (or $r = 1$), the first (or the second) observation given above will be contradicted. Therefore, $i_\beta(s, r)$ can actually be split into

$$i_\beta(s, r) = f_1^{c_1}(s, r) \cdots f_u^{c_u}(s, r) \quad (2.35)$$

where $c_1, \ldots, c_u$ are non-negative integers. Moreover, we must also have $a_k > c_k$ $(k = 1, \ldots, n)$. Otherwise, by setting $s = 1$ (or $r = 1$), it will again contradict the preceding observations. The lemma is proven. □

In what follows, we show that the concept of strict equivalence for two pencils can be easily characterized using the generalized invariant polynomials.

First, two pencils $Es - A$ and $E_1 s - A_1$, each of dimension $n \times n$, are said to be *strictly equivalent* if there exist constant square non-singular matrices M and N, such that (Gantmacher 1960b)

$$M(Es - A)N = E_1 s - A_1 \quad (2.36)$$

In case $Es - A$ and $E_1 s - A_1$ are both regular, we can state the following criterion for the strict equivalence.

Theorem 2.6 *Two regular pencils $Es - A$ and $E_1 s - A_1$ are strictly equivalent if and only if they have the same set of generalized invariant polynomials.*

proof

Sufficiency Suppose that these two pencils have the same set of generalized invariant polynomials. Then they have the same set of (finite and infinite) elementary divisors (e.g., Gantmacher 1960b, p.27). Hence they are strictly equivalent.

Necessity For if they are strictly equivalent, then there exist two non-singular matrices M and N such that

$$M(Es - Ar)N = E_1 s - A_1 r \tag{2.37}$$

By applying the Binet-Cauchy theorem to relate the minors of both $Es - Ar$ and $E_1 s - A_1 r$ via (2.37), it follows readily that

$$d_k(s,r) = \bar{d}_k(s,r) \quad (k = 1, \ldots, n)$$

where $d_k(s,r)$ is the monic greatest common divisor of all minors of order k of $Es - Ar$, $\bar{d}_k(s,r)$ is that of $E_1 s - A_1 r$. Hence, by Definition 2.9 these two pencils have the same set of generalized invariant polynomials. The theorem is proved. □

The reader may observe that Theorem 2.6 can be seen as a complement to the theory of regular pencils of matrices found in Gantmacher 1960b. It states that, just as for pencils of the form $Is - A$ where both the invariant polynomials and the elementary divisors can be used to characterize the similarity for two such pencils, both the generalized invariant polynomials and the (finite and infinite) elementary divisors can also be used to characterize the strict equivalence for two regular pencils of the form $Es - A$.

2.4.2 Generalized Companion Form and the Generalized Rational Canonical Form

Given a regular pencil $Es - A$, its generalized invariant polynomials, say $i_1(s,r), \cdots, i_n(s,r)$ can be computed by (2.32). Now, select arbitrarily one generalized invariant polynomial $i_k(s,r) \neq 1 \quad (1 \leq k \leq n)$, then $i_k(s,r)$ can be represented in the following standard form (suppose that $deg\ i_k(s,r) = m$)

$$i_k(s,r) = a_0 s^m + a_1 s^{m-1} r + \cdots + a_m r^m \tag{2.38}$$

where not all $a_j \quad (j = 0, \ldots, m)$ are zero, but a_0 can be zero.

Consider the $m \times m$ pencil

$$Cs - L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & a_0 \end{bmatrix} s - \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_m \\ 1 & 0 & \cdots & 0 & -a_{m-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \tag{2.39}$$

It is readily checked that

$$det(Cs - Lr) = a_0 s^m + a_1 s^{m-1} r + \cdots + a_m r^m = i_k(s,r)$$

Since a_0 can be zero, C may be a singular matrix. Moreover, from

$$Cs - Lr = \begin{bmatrix} s & 0 & \cdots & 0 & a_m r \\ -r & s & \cdots & 0 & a_{m-1} r \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -r & a_0 s + a_1 r \end{bmatrix}$$

it can be verified that the minor of $a_m r$ is $(-1)^{m-1} r^{m-1}$ and that of $a_0 s + a_1 r$ is s^{m-1}. Hence $d_{m-1}(s,r) = 1$. The set of generalized invariant polynomials of (2.39) will therefore be

$$\bar{i}_1(s,r) = \frac{d_m(s,r)}{d_{m-1}(s,r)} = d_m(s,r) = a_0 s^m + a_1 s^{m-1} r + \cdots + a_m r^m$$

and $\bar{i}_2(s,r) = \cdots = \bar{i}_m(s,r) = 1$. Thus, $Cs - L$ has only one generalized invariant polynomial different from 1, say $\bar{i}_1(s,r) = i_k(s,r) = det(Cs - Lr)$.

For obvious reasons, we shall call (2.39) the *generalized companion form* with respect to $i_k(s,r)$.

Now we are ready to construct the *generalized rational canonical form* for $Es - A$. Let

$$i_1(s,r), i_2(s,r), \cdots, i_\sigma(s,r), i_{\sigma+1}(s,r) = 1, \cdots, i_n(s,r) = 1 \tag{2.40}$$

be the complete set of generalized invariant polynomials for $Es - A$. Then the generalized companion forms with respect to $i_k(s,r) \quad (k = 1, \ldots, \sigma)$ (i.e., only those polynomials different form 1) can be constructed in the way explained above. Denoting them by $C_1 s - L_1, C_2 s - L_2, \cdots, C_\sigma s - L_\sigma$, a quasi-diagonal pencil of dimension $n \times n$ can be built up using these

companion forms as diagonal blocks, i.e.,

$$\bar{C}s - \bar{L} = block\ diag\{C_1 s - L_1, C_2 s - L_2, \cdots, C_\sigma s - L_\sigma\}$$

$$= \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_\sigma \end{bmatrix} s - \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & L_\sigma \end{bmatrix} \tag{2.41}$$

$\bar{C}s - \bar{L}$ will be called the *generalized rational canonical form* for $Es - A$. This term is justified by the following theorem.

Theorem 2.7 *Every regular pencil is strictly equivalent to a generalized rational canonical form (2.41). This form has the following three properties:*

1. *it is a quasi-diagonal pencil;*
2. *each diagonal block is in the generalized companion form (2.39);*
3. *the dimensions of the diagonal blocks are in the descending order.*

proof According to Theorem 2.6, the only thing we have to show is that pencils $Es - A$ and $\bar{C}s - \bar{L}$ (as in (2.41)) have the same set of generalized invariant polynomials. However, this is obvious, since both the structure of (2.41) with (2.39) as the diagonal blocks and the properties of generalized invariant polynomials given in Lemma 2.4 allow to compute quite directly and easily $\bar{d}_k(s,r)$ $(k = 1, \ldots, n)$ for $\bar{C}s - \bar{L}r$, which turns out to be

$$\bar{d}_k(s,r) = i_{n-k-1}(s,r) \cdots i_n(s,r)$$

□

It is of importance to observe that the generalized rational canonical form is entirely determined by the set of generalized invariant polynomials. Since for a regular pencil, the set of generalized invariant polynomials is unique, thus the generalized rational canonical form is also unique. This explains why we call (2.41) a canonical form. Another interesting feature is that when E is non-singular for a pencil $Es - A$, the generalized rational canonical form reduces to the normal rational canonical form for the matrix $E^{-1}A$.

To conclude this section, we provide two examples to demonstrate the new canonical form. These examples are taken from Gantmacher 1960b, p.26.

Example 2.4 A regular pencil is given by

$$Es - A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} s - \begin{bmatrix} -2 & -1 & -3 \\ -3 & -2 & -5 \\ -3 & -2 & -6 \end{bmatrix} \tag{2.42}$$

The set of generalized invariant polynomials can be computed as $i_1(s,r) = sr^2 + r^3, i_2(s,r) = i_3(s,r) = 1$. Therefore, the generalized rational canonical form for (2.42) is

$$\bar{C}s - \bar{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} s - \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.43}$$

□

Example 2.5 The second regular pencil is

$$Es - A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} s - \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -1 \end{bmatrix} \tag{2.44}$$

The set of generalized invariant polynomials is easily calculated to be $i_1(s,r) = sr + r^2, i_2(s,r) = r, i_3(s,r) = 1$. Thus the generalized rational canonical form for (2.44) is

$$\bar{C}s - \bar{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s - \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.45}$$

□

The reader may find from these two examples that converting an arbitrary regular pencil to the canonical form via generalized invariant polynomials is computationally prohibitive, especially when a pencil of larger dimension is involved. However, we have developed a constructive scheme to perform the task. The interested reader is refered to Tan and Vandewalle 1986b for details.

2.5 Rational Functions and Rational Matrices at Infinity

In this section, we shall apply the ideas developed in the preceding sections to rational functions and matrices. It will be seen through the discussion

that our theory provides a simple and direct way to identify the structures (finite and infinite) of rational matrices.

2.5.1 Rational Functions and Complete Irreducibility

There are essentially two types of rational functions as defined in the following definitions.

Definition 2.10 *A general rational function $g(s)$ is defined to be $b(s)/a(s)$, where $a(s), b(s)$ are polynomials satisfying deg $a(s) =$ deg $b(s)$, and $a(s) \neq 0$. The poles of $g(s)$ (finite and infinite) are the zeros (finite and infinite) of $a(s)$; the zeros of $g(s)$ (finite and infinite) are the zeros (finite and infinite) of $b(s)$.* □

Definition 2.11 *A pseudo-proper rational function $f(s)$ is defined to be $b(s)/a(s)$, where $a(s), b(s)$ are polynomials satisfying deg $a(s) =$ deg $b(s) + 1$; $a(s) \neq 0$. The poles (finite and infinite) of $f(s)$ are the zeros (finite and infinite) of $a(s)$; the zeros of $f(s)$ (finite and infinite) are the zeros (finite and infinite) of $b(s)$.* □

Our definitions of rational functions are basically guided by two simple facts concerning the conventional rational functions. First, when taking infinity into account, any rational function will have the same number of poles and zeros. The fact is reflected by the degree constraint in Definition 2.10. Secondly, there are circumstances under which it is more appropriate to consider (strictly) proper rational functions in which the degree of the denominator is higher than that of the numerator. Definition 2.11 in fact generalizes these considerations. We shall provide two examples to further clarify these ideas.

Example 2.6 Let a rational function (in the conventional sense) be given by

$$g(s) = \frac{s^3 + s^2 + s + 1}{s + 1} \tag{2.46}$$

It obviously has three finite zeros and one finite pole. It also has two poles at infinity, which can be revealed by using the conformal mapping $s \longrightarrow 1/\lambda$. In accordance with Definition 2.10, $g(s)$ should be written as

$$g(s) = \frac{s^3 + s^2 + s + 1}{0 \cdot s^3 + 0 \cdot s^2 + s + 1} \tag{2.47}$$

We can see that (2.47) shows more explicitly all the poles and zeros discussed above without applying the conformal mapping. □

Example 2.7 A general expression for the (strictly) proper rational function is given by

$$g(s) = \frac{b_0 s^{n-1} + b_1 s^{n-2} + \cdots + b_{n-1}}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \tag{2.48}$$

Conventionally a_0 is required to be non-zero, and when b_0 is zero, the term $b_0 s^{n-1}$ can be dropped. These two points are clearly inconsistent with our treatment in the foregoing sections. Definition 2.11 is therefore designed to remove them. More specifically, in our context a_0 is allowed to be zero; the term $0 \cdot s^{n-1}$ should not be removed. It is easy to see that with these two modifications a rational function of the form (2.48) is not necessarily proper in the conventional sense. This is also why the term *pseudo-proper* is used. □

We note that pseudo-proper rational function is not a specialization of the general rational function. This fact somewhat contradicts the common intuition. However, it will become natural when we reach the discussion in the next chapter.

It should also be noted that a polynomial is not considered to be a rational function. We should not confuse a polynomial $p(s) = a_0 s^n + \cdots + a_n$ with the rational function $g(s) = (a_0 s^n + \cdots + a_n)/(0 \cdot s^n + \cdots + 1)$.

In what follows, the term *rational function* will designate both of the two types of rational functions.

Now we proceed to introduce the concepts of irreducibility at infinity and completely irreducibility for rational functions.

If we are given an arbitrary rational function, it is likely that there is a common divisor at infinity between the numerator and the denominator polynomials. We are particularly interested in the case where no such divisors exist.

Definition 2.12 *A (general or pseudo-proper) rational function is said to be irreducible at infinity if its denominator and numerator polynomials are coprime at infinity.* □

A simple characterization of this concept follows directly from Theorem 2.2.

Theorem 2.8 *A (general or pseudo-proper) rational function $g(s) = b(s)/a(s)$ is irreducible at infinity if and only if a_0, b_0 are not simultaneously zero, where a_0, b_0 are the leading coefficients of $a(s)$ and $b(s)$, respectively.*

□

Taking into consideration the finite irreducibility we can state the following definition.

Definition 2.13 *A (general and pseudo-proper) rational function is said to be completely irreducible if its denominator and numerator polynomials are completely coprime.* □

A characterization of the complete irreducibility follows from Theorem 2.3. We shall not repeat it here.

2.5.2 Rational Matrices and Complete Irreducibility

A rational matrix $G(s)$ of dimension $m \times p$ is simply a matrix whose entries are rational functions. Since there are two classes of rational functions, we shall more specifically have *general rational matrices* with entries of general rational functions, and *pseudo-proper rational matrices* with entries of pseudo-proper rational functions.

Let a rational matrix (general or pseudo-proper) $G(s)$ $(m \times p)$ be given. We shall show that it can be represented alternatively by the matrix fraction $G(s) = N(s)D^{-1}(s)$, where $N(s), D(s)$ are column-based polynomial matrices. This representation will be called *right matrix fraction description* (MFD), and $D(s), N(s)$ will be called the denominator matrix and the numerator matrix of $G(s)$, respectively.

To begin with, for the ith column $g_i(s)$ $(i = 1, \ldots, p)$ of $G(s)$, we bring each element in that column on the same denominator, which is the least common multiple of the denominators of all the entries in that column. Then $G(s)$ can obviously be factorized as

$$G(s) = N(s)D^{-1}(s)$$

where $N(s)$ contains all the numerators of entries of $G(s)$, and $D(s) = diag\{d_1(s), d_2(s), \cdots, d_p(s)\}$, in which each $d_i(s)$ $(i = 1, \ldots, p)$ is the least multiple of all denominators of the entries in the ith column.

Conversely, if a rational matrix $G(s)$ is given by $G(s) = N(s)D^{-1}(s)$, where $N(s), D(s)$ are column-based polynomial matrices, and moreover, let $d_i(s), n_i(s)$ be ith columns of $D(s), N(s)$, respectively, with $deg\ d_i(s) = deg\ n_i(s)$ (or $deg\ d_i(s) = deg\ n_i(s) + 1$), then $G(s)$ is a general (pseudo-proper) rational matrix. Therefore, we can also define general (pseudo-proper) rational matrices in terms of MFD. Moreover, since the polynomial matrices involved in MFD may fall into some specific classes, MFD's are further classified as shown in the following definition.

Definition 2.14 *A right MFD $G(s) = N(s)D^{-1}(s)$ is called a general (pseudo-proper) column-based MFD if $D(s)$ and $N(s)$ are column-based*

polynomial matrices, and if each of the column degrees of $D(s)$ is equal to (one greater than) the corresponding column degree of $N(s)$; $G(s)$ is called a general (pseudo-proper) matrix polynomial MFD if $D(s)$ and $N(s)$ are matrix polynomials and if the degree of $D(s)$ is equal to (one greater than) that of $N(s)$. □

A rational matrix defined in terms of MFD is more appropriate for introducing poles and zeros of rational matrices.

Definition 2.15 *Let a rational matrix $G(s)$ be given by*

$$G(s) = N(s)D^{-1}(s)$$

Then its poles (finite and infinite) are defined to be the zeros of $detD(s)$.

□

If $N(s)$ is a square matrix, then the zeros of $G(s)$ are defined to be those of $detN(s, r)$. If it is a non-square, then the definition of zeros will involve the concept of generalized invariant polynomial. We shall, however, not go any further in this direction.

The above has only been concerned with the rational matrix represented in right MFD. In fact, similar statements and definitions can easily be stated for left MFD's. Since the algebraic treatment for left MFD's is exactly the same as for the right ones, we shall omit the treatment for left MFD's in order to be concise. The reader should have no difficulties in producing the details for left MFD's when the treatment for right MFD's is available.

When a rational matrix (general or pseudo-proper) $G(s)$ is represented in right MFD, say $N(s)D^{-1}(s)$, the numerator $N(s)$ and the denominator $D(s)$ may have common right factors at infinity. The absence of such factors is termed *irreducibility at infinity*. Moreover, the absence of any common divisor (finite and infinite) is called *complete irreducibility*. We now state the characterizations for each of the two concepts.

Theorem 2.9 *Let an $m \times p$ rational matrix (general or pseudo-proper) $G(s)$ be given by*

$$G(s) = N(s)D^{-1}(s) \tag{2.49}$$

where $N(s), D(s)$ are column-based polynomial matrices. Then $G(s)$ is irreducible at infinity if and only if

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p \tag{2.50}$$

in which D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. □

Theorem 2.10 *Let $G(s)$ be given by (2.49). It is completely irreducible if and only if in addition to (2.50), the following is also true*

$$rank \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = p, \quad for\ all\ finite\ s$$

□

To demonstrate the power of our approach, we provide a few illustrative examples.

Example 2.8 Let us first analyze a simple rational matrix given by

$$G(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{2.51}$$

it has been considered elsewhere (see, for example, Kailath 1980, Van Dooren 1978) as a unimodular polynomial matrix, and found to have a pole, a zero at infinity by using the Smith-McMillan form.

Since the element s in $G(s)$ is considered as a rational function, we prefer to rewrite it down more explicitly as

$$G(s) = \begin{bmatrix} 1 & \frac{s}{0 \cdot s+1} \\ 0 & 1 \end{bmatrix} \tag{2.52}$$

By bringing the second column on its least common denominator, we have

$$G(s) = \begin{bmatrix} 1 & \frac{s}{0 \cdot s+1} \\ 0 & \frac{0 \cdot s+1}{0 \cdot s+1} \end{bmatrix}$$

or in right MFD $G(s) = N(s)D^{-1}(s)$, where

$$N(s) = \begin{bmatrix} 1 & s \\ 0 & 0 \cdot s+1 \end{bmatrix}, \ D(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \cdot s+1 \end{bmatrix}$$

This MFD is completely irreducible since

$$rank \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = 2$$

and

$$rank \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 0 & 0 \cdot s+1 \\ 1 & s \\ 0 & 0 \cdot s+1 \end{bmatrix} = 2, \text{ for all finite } s.$$

The poles of $G(s)$ can be found from $detD(s,r) = r$, the zeros of $G(s)$ from $detN(s,r) = r$, which means that $G(s)$ only has one pole and one zero at infinity. The result agrees with the conventional analysis. However, the reader can see that our analysis is much simpler, compared to the conventional one which may require the computation of the inverse.

In fact, we can bring on the least common denominator of all entries of (2.52), i.e.,

$$G(s) = \begin{bmatrix} \frac{0 \cdot s+1}{0 \cdot s+1} & \frac{s}{0 \cdot s+1} \\ 0 & \frac{0 \cdot s+1}{0 \cdot s+1} \end{bmatrix}$$

Now the numerator and the demoninator of the MFD can be seen as

$$N(s) = \begin{bmatrix} 0 \cdot s+1 & s \\ 0 & 0 \cdot s+1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 0 \cdot s+1 & 0 \\ 0 & 0 \cdot s+1 \end{bmatrix}$$

The poles of the MFD can be found from $detD(s,r) = r^2$; the zeros from $detN(s,r) = r^2$, which would suggest two poles and two zeros at infinity. However, it can easily be checked that this MFD is not completely irreducible. Actually an infinite pole will be cancelled by an infinite zero. After the cancellation, the MFD will be completely irreducible. The number of poles and zeros thereby agrees with the preceding discussion. □

The foregoing analysis seems to suggest that the McMillan degree of a general rational matrix only reflects the irreducible part of the matrices. This is indeed true in general as we shall show in Chapter 5.

Example 2.9 It is well recognized that for a given pencil, its infinite structure seen via the infinite Smith-McMillan form is quite different from the one seen via the infinite elementary divisors (Van Dooren 1979, Vardulakis and Karcanias 1983). Although the detailed connections between the two have been worked out, it is still not clear as to why these two descriptions of the infinity are essentially different.

We can, however, provide an easy answer to the question in the light of our theory. Note first that infinite elementary divisors can only apply to

polynomial matrices, not rational matrices; while the Smith-McMillan form at infinity can only apply to rational matrices not polynomial matrices.

Let a pencil $Es - A$ be given. If it is regarded as a polynomial matrix, then its elementary divisors at infinity can be found via the homogeneous form $Es - Ar$. However, if it is regarded as a rational matrix, each of its elements is considered to have the denominator $0 \cdot s + 1$, then we have to first rewrite it as follows

$$(Es - A)D^{-1}(s) \tag{2.53}$$

where $D(s) = I \cdot (0 \cdot s + 1)$, or $D(s, r) = I \cdot r$. We can see that all infinite elementary divisors are modified by the factor r^{-1}. The precise poles and zeros can be obtained by first cancelling the common factors in (2.53) to make it irreducible. We can see that the result coincides with those in Van Dooren 1979, Vardulakis and Karcanias 1983. □

2.6 Conclusions and Remarks

In this chapter, we have presented a novel approach in handling the properties at infinity of polynomial and rational matrices. The crucial concepts introduced in the course of the development are the coprimeness at infinity for two polynomial matrices , complete irreducibility and the generalized rational canonical form for regular pencils of matrices. The characterizations for these concepts are also obtained. These results will serve as the basis in the subsequent investigation of properties of linear multivariable systems at infinity.

In fact, the notion of divisor at infinity will enable an entirely algebraic characterization of the so-called redundancy at infinity in singular systems, which is dynamically identified as infinitely uncontrollable and/or unobservable parts of the system. From this viewpoint, the divisor at infinity is a conceptual complement of the finite one which is already quite familiar. The generalized rational canonical form, on the other hand, suggests the structures of canonical forms for singular systems, which are a very convenient starting point to devise the analysis and design methods.

The development in this chapter has also extended the results reported in our papers Tan and Vandewalle 1986a, 1987a. These papers only considered matrix polynomials and matrix polynomial MFD's. As we have seen, these cases are clearly not the general ones, since a cancellation will reduce a matrix polynomial MFD to a column (or row)-based MFD. Thus the treatment to the column (row)-based polynomial matrices represents a significant improvement over the early results.

Chapter 3

Pseudo-proper Matrix Fraction Descriptions and Singular System Realizations

3.1 Introduction

The primary objective of this chapter is to introduce the realizations for rational matrices in pseudo-proper matrix fraction descriptions. Since only the pseudo-proper MFD will be of concern in this chapter, we will frequently drop the term *pseudo-proper* for ease of discussion. It should cause no confusion.

Since neither the properness nor the column (row)-reducedness is likely to be satisfied by the MFD, the natural realizations for them are no longer the familiar state-space system, but the singular system instead. The latter realization preserves not only the finite structure of the MFD, but the structure at infinity as well.

It has been seen in the last section of the preceding chapter that there are basically two types of pseudo-proper *right* MFD's, i.e., matrix polynomial MFD, and column-based MFD's. The same is true for the *left* MFD, except that the second type is the row-based MFD. We shall present methods to realize these two types of right MFD's into singular systems. It is expected that the realization for the first type of the MFD is considerably simpler than the second.

As an important result, we establish the following connections. For a given column-based right MFD, any of its realizations as a singular system with the dimension equal to the determinantal degree of the denominator matrix will be jointly reachable and observable if and only if the given MFD is completely irreducible. On the other hand, the joint reachability and observability of the singular system realizations for the MFD are in turn equivalent to the minimality in dimension of the realizations. Such an equivalence circle is completely analogous to that found in the state-space case. It also provides new algebraic characterizations and interpretations for the concept of joint reachability and observability for singular systems, which are dynamical concepts.

The algorithm which realizes the pseudo-proper column (row)-based right (left) MFD into a reachable (observable) singular system elucidates the fact that when the given rational matrix is neither proper nor column (row)-reduced, a so-called differential feedback may be present. It is this feedback that brings a state-space system into a singular system. Our realization algorithm can also be rephrased in terms of analog-computer simulation. Consequently, an analog-computer simulation scheme for simulating pseudo-proper rational matrices can be devised.

This chapter is structured as follows. The development will start in the next section by reviewing a number of fundamental concepts in singular systems. The notions of reachability and observability for singular systems are introduced. We can see that the standard form for singular systems can always be regarded as a so-called two-sided pseudo-proper MFD. This viewpoint, however, is not the unique one as we shall discuss in Chapter 5. For the sake of distinction, we shall call the singular systems viewed this way *pseudo-proper singular systems*. Since in this and the next chapter, we shall be exclusively concerned with pseudo-proper singular systems, the adjective "pseudo-proper" will be dropped for convenience of notation. It will also be shown in this section that the special feature of the polynomial matrix MFD allows some simple realizations to be obtained.

The realization for column-based right MFD's is relatively less easy. Section 3.3 is devoted to a full presentation of an algorithm which realizes this type of MFD's. The connection is established claiming that the given MFD is irreducible at infinity if and only if the realization is jointly reachable and observable at infinity.

Section 3.4 establishes the general equivalence circle, i.e., the given MFD is completely irreducible if and only if any realization with the dimension equal to the determinantal degree of the denominator matrix is jointly reachable and observable. We also show that the minimality of the realization is equivalent to the joint reachability and observability of the realizations, thus completing the entire equivalence circle.

An analog-computer simulation scheme which is based on the realization theory is described in detail in Section 3.5. Although the development in this section and Sec.3.3 closely follows two of our papers Tan and Vandewalle 1987c, 1987d, a number of new results and interpretations are also included. The conclusions and remarks are contained in Section 3.6.

3.2 Simple Realization in Singular System for Pseudo-proper Right MFD's

The standard form for singular systems is given by (Rosenbrock 1974, Luenberger 1977, and Verghese et al. 1981)

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.1}$$

where x is an n-vector of internal variables called *descriptor vector*, u is a p-vector of control inputs and y is an m-vector of outputs. E, A are matrices of dimension $n \times n$, B and C are matrices of dimensions $n \times p$ and $m \times n$, respectively. In particular, E can be singular. n will be called the dimension of the system (3.1).

The transfer matrix of singular system (3.1) is obviously

$$G(s) = C(Es - A)^{-1}B \tag{3.2}$$

We see from (3.2) that the invertibility of $Es - A$ is necessary to guarantee the existence of the transfer matrix. This condition is actually called *solvability* of the system (3.1) by Luenberger (1977), and is a condition for the unique solution of (3.1) given any input u. We shall always assume, throughout the thesis, that all singular systems encountered are solvable.

The matrices in (3.2), say $Es - A, B, C$ can be regarded as matrix polynomials with degrees 1, 0, 0 respectively. In this respect, (3.2) can be seen as a pseudo-proper matrix polynomial MFD. This type of MFD, however, slightly differs from those in the preceding chapter and is termed *two-sided* MFD (Callier and Desoer 1982).

For a solvable pseudo-proper singular system (3.1), we shall define $\psi(s, r) = det(Es - Ar)$ to be the *characteristic polynomial* of the system. Furthermore, the poles (finite and infinite) of (3.1) are the zeros (finite and infinite) of its characteristic polynomial. Since the dimensions of E, A are $n \times n$, the degree of $\psi(s, r)$ is always n. Therefore, (3.1) has precisely n poles (finite and infinite).

The fundamental concepts of reachability and observability have been studied by many authors (Rosenbrock 1974, Verghese et al. 1981, and Yip

and Sincovec 1981 are representative). The comprehensive and unifying treatments for both concepts have been presented by Cobb (1984) and Lewis et al. (1984). Here we shall adopt Lewis' nomenclature and conventions.

Definition 3.1 *A singular system (3.1) is reachable at infinity if*

$$rank\,[E \quad B] = n \tag{3.3}$$

It is finitely reachable if

$$rank\,[Es - A \quad B] = n, \text{ for all finite } s \tag{3.4}$$

It is reachable if it is both finitely reachable and reachable at infinity. □

Definition 3.2 *A singular system (3.1) is observable at infinity if*

$$rank \begin{bmatrix} E \\ C \end{bmatrix} = n \tag{3.5}$$

It is finitely observable if

$$rank \begin{bmatrix} Es - A \\ C \end{bmatrix} = n, \quad \text{for all finite } s \tag{3.6}$$

It is observable if it is both finitely observable and observable at infinity. □

Let a rational matrix $G(s)$ of dimension $m \times p$ be given in pseudo-proper right MFD, i.e.,

$$G(s) = N(s)D^{-1}(s) \tag{3.7}$$

According to our convention, $D(s,r), N(s,r)$ will denote the homogeneous forms for $D(s), N(s)$, respectively. A singular system (3.1) is said to realize $G(s)$ if

$$N(s,r)D^{-1}(s,r) = C(Es - Ar)^{-1}B \tag{3.8}$$

When a singular system realizes $G(s)$, it will be called a *realization* of $G(s)$.

We now start the development of a realization theory by first discussing a very simple realization for matrix polynomial MFD's. The following theorem shows that such MFD's enable a straightforward realization.

Theorem 3.1 *Let a pseudo-proper matrix polynomial MFD be given by (3.7), where $D(s), N(s)$ have the dimensions $p \times p, m \times p$, respectively; and*

$$D(s,r) = D_0 s^\alpha + D_1 s^{\alpha-1} r + \cdots + D_\alpha r^\alpha$$
$$N(s,r) = N_0 s^{\alpha-1} + N_1 s^{\alpha-2} r + \cdots + N_{\alpha-1} r^{\alpha-1}$$

Then it can be realized in singular system (3.1), where

$$E = \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ 0 & I_p & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_p \end{bmatrix}, A = \begin{bmatrix} -D_1 & -D_2 & \cdots & -D_{\alpha-1} & -D_\alpha \\ I_p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C = [N_0\ N_1\ \cdots\ N_{\alpha-1}] \tag{3.9}$$

in which the I_p denotes the $p \times p$ unit matrix.

proof By direct computation, the first p columns of the matrix $(Es - Ar)^{-1}$, which are obviously the matrix $(Es - Ar)^{-1}B$ by (3.9), turn out to be

$$\begin{bmatrix} I_p s^{\alpha-1} \\ I_p s^{\alpha-2} r \\ \vdots \\ I_p r^{\alpha-1} \end{bmatrix} D^{-1}(s,r) = (Es - Ar)^{-1}B \tag{3.10}$$

Premultiplying the matrix C on both sides of Eq. (3.10) proves the theorem.

□

The realization (3.9) possesses many important properties. The following theorem summarizes some of these which are relevant to our development.

Theorem 3.2 *The realization (3.9) has the properties below*

1. *$det D(s,r) = det(Es - Ar)$;*
2. *$\deg det D(s,r) =$ the dimension of the realization;*
3. *it is reachable.*

proof

1. Let $Es - Ar$ be partitioned into a block row matrix

$$Es - Ar = [F_1\ F_2\ \cdots\ F_\alpha] \tag{3.11}$$

where each F_i $(i = 1, \ldots, \alpha)$ consists of p columns. Apply a transformation K to (3.11) such that

$$(Es - Ar) \cdot K = [\tilde{F}_1\ \tilde{F}_2\ \cdots\ \tilde{F}_\alpha] \tag{3.12}$$

where $\bar{F}_i = F_i r^{i-1} + \bar{F}_{i-1} s$ $(i = 2, \ldots, \alpha)$ and $\bar{F}_1 = F_1$. The precise form of K can easily be seen to be

$$K = \begin{bmatrix} I_p & I_p s & \cdots & I_p s^{\alpha-1} \\ 0 & I_p r & \cdots & I_p s^{\alpha-2} r \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_p r^{\alpha-1} \end{bmatrix}$$

From the special structure of $Es - Ar$, it follows that after the transformation, the determinant of the matrix on the right hand side of (3.12) can be computed directly, which is equal to $detD(s,r) \cdot det(I_p r) \cdots det(I_p r^{\alpha-1})$.

On the other hand, $det[(Es - Ar)K] = det(Es - Ar) \cdot det\, K$, and $det K$ is apparently equal to $det(I_p r) \cdots det(I_p r^{\alpha-1})$. We therefore conclude

$$det(Es - Ar) = detD(s,r) \tag{3.13}$$

Thus 1 is proved.

2. Since $deg\, det(Es - Ar) = \alpha \cdot p$, which is also the dimension of the realization (3.9), by using 1., $deg\, detD(s,r) = deg\, det(Es - Ar) = \alpha \cdot p$.

3. It is readily checked that $rank[E \;\; A] = rank[Es - A \;\; B] = \alpha \cdot p$, for all finite s. By Definition 3.1, the singular system (3.9) is reachable. □

Because of the above properties and the particular structure of (3.9), we shall call the realization (3.9) *block reachable realization*. Note especially that the dimension of the realization is equal to the determinantal degree of the denominator matrix of the given MFD.

The concept of complete right irreducibility has been introduced in the preceding chapter. The following theorem shows an interesting connection between the complete right irreducibility of a polynomial matrix MFD (3.7) and its block reachable realization (3.9).

Theorem 3.3 *The block reachable realization (3.9) is observable if and only if the MFD (3.7) is completely irreducible.*

proof We only prove the equivalence between the observability at infinity of the realization and the irreducibility at infinity of the MFD. A proof for the finite case can be found, for example, in Kailath 1980.

From (3.9), the columns of the matrix

$$\begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ 0 & I_p & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I_p \\ N_0 & N_1 & \cdots & N_{\alpha-1} \end{bmatrix}$$

are linearly independent if and only if the columns of the matrix $[D_0^T \;\; N_0^T]^T$ are linearly independent, which, by Theorem 2.9, is equivalent to the irreducibility at infinity of the MFD. □

3.3 Realization for Column-based MFD's

The realization procedure presented in the preceding section only applies to matrix polynomial MFD's. As we have seen in the last chapter that this is certainly not the most general case. In this section we relax the constraint by considering realization for column-based right MFD's. It is of interest to know that even if the starting point is a polynomial matrix right MFD, it will reduce to a column-based right MFD upon cancelling a common right divisor. To be precise, recall first that a right divisor of matrix polynomials is always a row-based polynomial matrix, while a left multiple of them can only be column-based polynomial matrix (Chapter 2, Lemma 2.2). Therefore, if a right common divisor of a given right MFD is cancelled out, what is left is a right MFD with the numerator and the denominator being column-based. We note in particular that when the given MFD is pseudo-proper, after the cancellation of a common right divisor the ith column degree of the denominator will still be greater than the ith column degree of the numerator. In another word, the pseudo-properness is invariant with respect to the cancellation.

The realization of column-based right MFD is considerably more involved than that of matrix polynomial right MFD. As will become clear in the ensuing development, this realization will reveal the fundamental nature of pseudo-proper MFD's.

3.3.1 Realization Algorithm

Let a column-based right MFD be given by $G(s) = N(s)D^{-1}(s)$, with $N(s)$ and $D(s)$ of dimensions $(m \times p), (p \times p)$ respectively. Denote by k_i the ith column degree of $D(s)$ (by the pseudo-properness assumption, the ith column degree of $N(s)$ should be $k_i - 1$). The aim is to find matrices

E, A, B, C such that $N(s,r)D^{-1}(s,r) = C(Es - Ar)^{-1}B$. We note that no conditions like column-reducedness or properness are required.

First decompose $D(s,r)$ in the following form (we remind that $D(s,r)$ is the homogeneous form of $D(s)$)

$$D(s,r) = D_h\Phi(s,r) + D_l\Psi(s,r)\cdot r \tag{3.14}$$

where D_h, D_l denote two appropriate constant matrices; and $\Phi(s,r)$, $\Psi(s,r)$ are two column-based polynomial matrices defined by

$$\begin{array}{rcl} \Phi^T(s,r) & = & block\ diag\{[s^{k_i}\ s^{k_i-1}r\ \cdots\ sr^{k_i-1}],\ \ i=1,\ldots,p\} \\ \Psi^T(s,r) & = & block\ diag\{[s^{k_i-1}\ s^{k_i-2}r\ \cdots\ r^{k_i-1}],\ \ i=1,\ldots,p\} \end{array} \tag{3.15}$$

In the above, the superscript T denotes the transpose. By the construction, it follows that $\Phi(s,r) = \Psi(s,r)\cdot s$. We should also note that D_h and D_l in the decomposition (3.14) are not unique.

By the assumption of pseudo-properness and the structure of $\Psi(s,r)$, the numerator $N(s,r)$ can also be decomposed into the form

$$N(s,r) = N_c\Psi(s,r) \tag{3.16}$$

We first construct a so-called *core realization* which is based on the set of column degrees of $D(s)$. This can be done as follows

$$E_c = I_n, \quad C_c = I_n$$

$$A_c = block\ diag\{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},\ k_i\times k_i,\ \ i=1,\ldots,p\} \tag{3.17}$$

$$B_c^T = block\ diag\{[1\ 0\ \cdots\ 0],\ \ k_i\times 1,\ \ i=1,\ldots m\}$$

where $n\ (=\sum_{i=1}^{p} k_i)$ is the determinantal degree of $D(s)$. With the structures in (3.17), a singular system realization of the given MFD can be obtained as

$$\begin{array}{rcl} E & = & E_c - B_cB_c^T + B_cD_h = I_n - B_cB_c^T + B_cD_h \\ A & = & A_c - B_cD_l \\ B & = & B_c \\ C & = & N_cC_c = N_c \end{array} \tag{3.18}$$

It may be noted that the construction of the matrices A, B, C are exactly the same as the construction given, for example, by Wolovich 1973 (see

also Kailath 1980, p.403). Moreover, the matrix D_h enters naturally, as a differential feedback gain, to form the matrix E.

We shall show in the next subsection that the realization (3.18) possesses some important properties which are quite similar to those of the block reachable realization presented in Sec.3.2, and to those of the well-known *Wolovich realization algorithm* (Wolovich 1973).

3.3.2 Proof of the Realization Algorithm and Properties

We now show that (3.18) is indeed a realization of the given right MFD.

By the definition of realization, if (3.18) realizes the right MFD the following must be true

$$C(Es - Ar)^{-1}B = N(s,r)D^{-1}(s,r) \tag{3.19}$$

(We still have to show that $Es - Ar$ is non-singular. However, a proof for this will be deferred for a while). From (3.19), it is clear that if we can prove

$$(Es - Ar)^{-1}B = \Psi(s,r)D^{-1}(s,r) \tag{3.20}$$

then (3.19) will follow immediately. In fact, this can be done quite easily as follows.

First multiplying both sides of (3.20) on the left by $Es - Ar$ and on the right by $D(s)$

$$BD(s,r) = (Es - Ar)\Psi(s,r) \tag{3.21}$$

substituting (3.18) and (3.14) into (3.21) yields

$$B[D_h\Phi(s,r) + D_l\Psi(s,r)\cdot r] = [(I - BB^T + BD_h)s - A_cr + BD_lr]\Psi(s,r) \tag{3.22}$$

noting $\Phi(s,r) = \Psi(s,r)s$, (3.22) can be simplified as

$$\Phi(s,r) - BB^T\Phi(s,r) = A_c\Psi(s,r)\cdot r \tag{3.23}$$

Now (3.23) can be readily checked from the structures of the matrices $\Phi(s,r), \Psi(s,r), B, A_c$. It follows that (3.20) holds, hence (3.18) indeed realizes the given right MFD.

Two important properties of Wolovich's algorithm are that the realization is controllable and there is a direct relationship between the determinant of the denominator and the characteristic polynomial of the realization. Now we show that these two properties are also preserved (in a more general sense, of course) by the realization just developed.

Theorem 3.4 *The realization (3.18) has the properties below*

1. *it is reachable;*

2. $det(Es - Ar) = detD(s,r)$;

3. $deg\, detD(s,r) =$ *the dimension of the realization.*

proof

1. From the structures of the matrices E_c, A_c, B_c in (3.18), it is readily checked that

$$\begin{aligned} rank[E_c \;\; B_c] &= n \\ rank[E_c s - A_c \;\; B_c] &= n \end{aligned} \tag{3.24}$$

As a consequence of (3.24)

$$\begin{aligned} rank[E \;\; B] &= rank[E_c + B_c(D_h - B_c^T) \;\; B_c] \\ &= rank[E_c \;\; B_c] \begin{bmatrix} I & 0 \\ D_h - B_c^T & I \end{bmatrix} \\ &= rank[E_c \;\; B_c] = n \end{aligned} \tag{3.25}$$

and for all finite s

$$\begin{aligned} & rank[Es - A \;\; B] \\ = \;& rank[(E_c - B_c B_c^T + B_c D_h)s - A_c + B_c D_l \;\; B_c] \\ = \;& rank[E_c s - A_c \;\; B_c] \begin{bmatrix} I & 0 \\ (D_h - B_c^T)s + D_l & I \end{bmatrix} \\ = \;& rank[E_c s - A_c \;\; B_c] = n \end{aligned} \tag{3.26}$$

This proves 1.

2. We note that since the matrix $D(s)$ is non-singular, this property implies that the matrix $Es - A$ is non-singular as well. This assertion is needed in the proof of the realization.

We shall first show that the two polynomials $det(Es - Ar)$ and $detD(s,r)$ have the same degree, then we skip all leading zero coefficient terms and establish the equation in the normal sense. Obviously, a combination of these two parts will prove the entire result.

Since every polynomial in $Es - Ar$ has degree one, the degree of the polynomial $det(Es - Ar)$ is equal to n $(= \sum_{i=1}^{p} k_i)$. Therefore, the polynomial has the same degree as $detD(s,r)$ which is also n.

The equation can now be proved without being concerned with the degree. This is done as follows.

Observe that

$$\{\Psi(s,1), D(s,1)\} \text{ are right coprime} \tag{3.27}$$

$$\{Es - A, B\} \text{ are left coprime} \tag{3.28}$$

The relation (3.27) follows from the fact that

$$rank \begin{bmatrix} \Psi(s,1) \\ D(s,1) \end{bmatrix} = p$$

while the relation (3.28) is a direct consequence of (3.26).

It then follows from (3.27) and (3.28) that there exist polynomial matrices $\{X(s), Y(s)\}$ and $\{\bar{X}(s), \bar{Y}(s)\}$ such that

$$(Es - A)X(s) + BY(s) = I \tag{3.29}$$

$$\bar{X}(s)\Psi(s,1) + \bar{Y}(s)D(s,1) = I \tag{3.30}$$

where the I have appropriate dimensions. (3.29) and (3.30) can be combined together with (3.21) in matrix form

$$\begin{bmatrix} Es - A & B \\ -\bar{X}(s) & \bar{Y}(s) \end{bmatrix} \begin{bmatrix} X(s) & -\Psi(s,1) \\ Y(s) & D(s,1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ Q(s) & I \end{bmatrix} \tag{3.31}$$

where $Q(s)$ is an appropriate polynomial matrix.

We see that both matrices on the left hand side of (3.31) are square. Hence, by taking determinants of both sides of (3.31) and noting that the determinant of the matrix on the right hand side is 1, it follows that the polynomial

$$det \begin{bmatrix} X(s) & -\Psi(s,1) \\ Y(s) & D(s,1) \end{bmatrix}$$

is independent of s (or in the well established terminology, the matrix is a unimodular).

Next, equation (3.29) and (3.21) allow to write down

$$\begin{bmatrix} Es - A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X(s) & -\Psi(s,1) \\ Y(s) & D(s,1) \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y(s) & D(s,1) \end{bmatrix} \tag{3.32}$$

Taking the determinants of the matrices on both sides of (3.32) and noting that the second matrix on the left hand side is a unimodular matrix, it yields

$$det(Es - A) \cdot c = detD(s,1) \tag{3.33}$$

where c is a constant. Since no scaling is made in the realization algorithm, c must be equal to 1. Equation (3.33) along with the degree assertion proved in the first half of the proof ensures the property 2.

3. This property can obviously be seen from the construction (3.18). □

To conclude this subsection, we apply the algorithm to realize a given right MFD.

Example 3.1 A right MFD is given by $G(s) = N(s)D^{-1}(s)$, where

$$N(s) = \begin{bmatrix} s^2+1 & 0 \\ 0\cdot s^2+s+0 & 1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} s^3+0\cdot s^2+s+0 & s+2 \\ 0\cdot s^3+s^2+s+1 & 0\cdot s+1 \end{bmatrix} \tag{3.34}$$

Note that (3.34) is a pseudo-proper column-based right MFD. First decompose $D(s,r)$ and $N(s,r)$ into

$$\begin{aligned} D(s,r) &= D_h\Phi(s,r) + D_l\Psi(s,r)\cdot r \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s^3 & 0 \\ s^2r & 0 \\ sr^2 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} s^2r & 0 \\ sr^2 & 0 \\ r^3 & 0 \\ 0 & r \end{bmatrix} \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} N(s,r) &= N_c\Psi(s,r) \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ sr & 0 \\ r^2 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{3.36}$$

Next, construct the core realization

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Finally, the realization has the following structure

$$\begin{aligned} E &= I - B_cB_c^T + B_cD_h \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A &= A_c - B_cD_l \\ &= \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} \\ B &= B_c \\ C &= N_c \end{aligned}$$

□

In the above example, we have taken an advantage of a nice property of our realization formulae (3.18) to gain the computation efficiency, i.e., for the construction of E, we simply replace the first, (k_1+1)th,$\cdots$, $(k_1+\cdots+k_{p-1}+1)$th rows of the identity matrix with the first, $\cdots$, the last row of the matrix D_h, and so on. A similar construction applies to the matrix A.

As far as the observability of the realization (3.18) is concerned, a theorem similar to Theorem 3.3 can also be established. As we have seen in the previous subsection, the matrices D_h, D_l are not unique. Therefore, for simplicity of the discussion we shall choose a specific pair to show the theorem. In the next section we shall show, by following a different route, that the conclusion holds for any choice of the two matrices.

The pair of matrices we select is the following. D_h is such that the first, the (k_1+1)th, $\cdots$, the $(k_1+\cdots+k_{p-1}+1)$th column is the same as the first, $\cdots$, the last column of D_{hc}, all the other columns are zero; D_l satisfies Eq. (3.14) along with the chosen D_h. This pair of matrices will be denoted by D_h^* and D_l^*.

Theorem 3.5 *The realization (3.18) with D_h^*, D_l^* is observable if and only if the right column-based MFD is completely irreducible.*

proof Again, we shall only prove the equivalence between the observability at infinity and the irreducibility at infinity. The proof for the finite part can be found, for example, in Kailath 1980.

From (3.18) and the specific choice of matrices D_h^*, D_l^*, the matrix $[E^T \quad C^T]^T$ can be brought, by matrix equivalence transformations, to the following form

$$\begin{bmatrix} D_{hc} & 0 \\ N_{hc} & 0 \\ 0 & I \end{bmatrix}$$

The columns of the above matrix are linearly independent if and only if the columns of the matrix $[D_{hc}^T \quad N_{hc}^T]^T$ are linearly independent. The

latter condition is equivalent to saying that the given column-based MFD is irreducible at infinity by Theorem 2.9. □

3.4 Relations among Joint Reachability and Observability, Minimality in Dimension and Complete Irreducibility

We are now in the position to prove the equivalence among the three concepts, namely, the minimality in dimension of the realization, the joint reachability and observability of the realization, and the complete irreducibility of the MFD. We shall in the first subsection show the equivalence between the first two notions, which is a direct result of a fourfold canonical decomposition described by Rosenbrock (1974). This link will be used in the second subsection to show the equivalence between the latter two notions.

3.4.1 Minimality in Dimension of the Realization versus Joint Reachability and Observability

For convenience of the discussion, we shall start from the system matrix of singular system (3.1), which is defined by Rosenbrock (1970) to be

$$\mathcal{P}(s) = \begin{bmatrix} Es - A & B \\ -C & 0 \end{bmatrix} \tag{3.37}$$

We shall say that two singular systems $\mathcal{P}(s)$ and $\mathcal{P}_1(s)$ are *restricted system equivalent* (or *r.s.e.*) (Rosenbrock 1974) if their system matrices are connected by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & 0 \end{bmatrix}$$

where M, N are $n \times n$ non-singular matrices, and all parameters with subscript 1 are those for system $\mathcal{P}_1(s)$. One of the most important properties is that *r.s.e.* transformations preserve reachability and observability. It can also be used to bring (3.37) to the form

$$\mathcal{P}^*(s) = \begin{bmatrix} sI - \tilde{A} & 0 & \tilde{B} \\ 0 & I + s\tilde{J} & \tilde{B} \\ -\tilde{C} & -\tilde{C} & 0 \end{bmatrix} \tag{3.38}$$

where $\tilde{J}$ is a nilpotent (i.e., a matrix which only has zero eigenvalues); the I is used throughout to denote the unit matrices of appropriate sizes.

The decomposition (3.38) in fact separates the strict proper part from the non-proper one for a given singular system. To be specific, we can write down, from (3.38), the transfer matrix of the singular system as

$$G(s) = \tilde{C}(Is - \tilde{A})^{-1}\tilde{B} + \tilde{C}(I + s\tilde{J})^{-1}\tilde{B} \tag{3.39}$$

since $\tilde{J}$ is a nilpotent, it follows that the second term in (3.39) is indeed a polynomial matrix.

Rosenbrock (1974) has shown that the definitions of reachability and observability at infinity (or in his terminology, the absence of infinite input and output decoupling zeros) induce a fourfold decomposition of the part in (3.38) corresponding to the polynomial matrix in the transfer matrix, i.e.,

$$\begin{bmatrix} I + s\tilde{J} & \tilde{B} \\ -\tilde{C} & 0 \end{bmatrix} \tag{3.40}$$

which is similar to the famous *Kalman decomposition* for state-space systems. Since the proper part of the system in (3.38) is in the state-space form, it admits a fourfold Kalman decomposition. If we combine the resulting eightfold decomposition of (3.38), we shall arrive at the following lemma.

Lemma 3.1 *Any singular system is r.s.e. to a fourfold canonical decomposition of the form*

$$\begin{bmatrix} E_{\bar{o}r}s - A_{\bar{o}r} & X & X & X & B_{\bar{o}r} \\ 0 & E_{\bar{o}\bar{r}}s - A_{\bar{o}\bar{r}} & 0 & X & 0 \\ 0 & 0 & E_{or}s - A_{or} & X & B_{or} \\ 0 & 0 & 0 & E_{o\bar{r}}s - A_{o\bar{r}} & 0 \\ 0 & 0 & -C_{or} & -C_{o\bar{r}} & 0 \end{bmatrix} \tag{3.41}$$

where the X denote some constant matrices which we do not care; the subscripts $o, \bar{o}, r, \bar{r}$ denote observable, unobservable, reachable and unreachable, respectively. □

It can be seen in particular that the reachable and observable part of the system, i.e.,

$$\begin{bmatrix} E_{or}s - A_{or} & B_{or} \\ -C_{or} & 0 \end{bmatrix} \tag{3.42}$$

has the same transfer matrix as (3.38)

$$G(s) = C(Es - A)^{-1}B = C_{or}(E_{or}s - A_{or})^{-1}B_{or}$$

We are now ready to establish the relation between the minimality in dimension and the joint reachability and observability for singular systems.

Theorem 3.6 *A singular system (3.37) is minimal in dimension if and only if it is reachable and observable.* □

The necessity of the theorem follows directly from Lemma 3.1. To prove the sufficiency, we still need the following result stated and proved by Rosenbrock (1974)

Lemma 3.2 *Let $\mathcal{P}(s), \mathcal{P}_1(s)$ be two systems in the form (3.37), both of which are reachable and observable. Then $\mathcal{P}(s), \mathcal{P}_1(s)$ are r.s.e. if and only if they give the same transfer matrix.* □

proof of Theorem 3.6 *Sufficiency* Let $\mathcal{P}(s), \mathcal{P}_1(s)$ be two reachable and observable realizations for a given transfer matrix, but the dimension of $\mathcal{P}(s)$ were less than that of $\mathcal{P}_1(s)$. Then $\mathcal{P}(s), \mathcal{P}_1(s)$ would not be *r.s.e.*, which is impossible by Lemma 3.2. □

3.4.2 Complete Irreducibility versus Joint Reachability and Observability

The connection to be established in this subsection is the following: for a given column-based right MFD, any of its singular system realizations with the dimension equal to the determinantal degree of the denominator matrix will be jointly reachable and observable if and only if the given MFD is completely irreducible.

To begin with, we state and prove the following lemma from which our main result will follow as an easy consequence.

Lemma 3.3 *If a rational matrix in column-based right MFD has one reachable and observable nth-dimension realization, then* all *its nth-dimension realizations must also be reachable and observable.*

proof We note that the dimension n is minimal by Theorem 3.6 since the nth-dimension realization is reachable and observable. If there were one nth-dimension realization which is unreachable or unobservable, then the canonical decomposition given in Lemma 3.1 would apply, therefore there would be a realization with dimension less then n. This is impossible. □

Theorem 3.7 *A column-based right MFD is completely irreducible if and only if all its nth-dimension realizations are reachable and observable, where n is the determinantal degree of $D(s)$, i.e., $n = deg\, det D(s, r)$.*

proof In view of Lemma 3.3, it suffices to show that the theorem holds for a special nth-dimension realization. But Theorem 3.5 indicates that the realization (3.18) with the particular choice of D_h^*, D_l^* can serve this

purpose. □

This theorem can certainly be applied to the realization (3.18) with arbitrary choice of D_h, D_l, so that our remarks before Theorem 3.5 are validated.

3.5 A General Analog-computer Simulation Scheme for Transfer Matrices of Linear Multivariable Systems

It may have been observed that the realization algorithm presented in the previous section also provides a concrete analog-computer simulation scheme for any column-based right MFD which may be considered as a transfer matrix resulting from practical modelling or some equivalence transformations. The purpose of this section is to explicitly construct the scheme without using the algebraic arguments. We also discuss, among other things, the so-called *impulsive behaviour* (first identified by Verghese et al. 1981) which may occur when the underlying system fails to be proper and/or column-reduced.

3.5.1 The simulation configuration

Let us assume that a column-based MFD be given by $G(s) = N(s)D(s)^{-1}$, the task is to construct an analog-computer simulation scheme from the MFD.

On introducing a so-called partial state $z(s)$, the system can be rewritten as

$$D(s)z(s) = u(s) \tag{3.43}$$

$$y(s) = N(s)z(s) \tag{3.44}$$

As usual, the strategy is to first implement Eq. (3.43), then Eq. (3.44) will follow as a simple extension.

As we have done in Sec.3.3, the initial step taken is to rearrange $D(s)$ by separating the higher order differential part from the lower one. For convenience of the reference, (3.18) is repeated below

$$D(s) = D_h\Phi(s) + D_l\Psi(s) \tag{3.45}$$

where $\Phi(s), \Psi(s)$ are two column-based polynomial matrices with homogeneous forms $\Phi(s, r), \Psi(s, r)$, respectively.

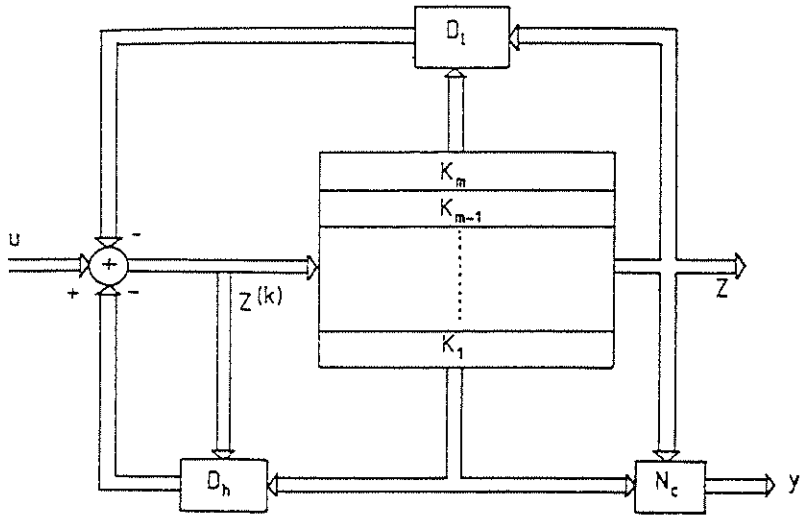


Figure 3.1: A Block Diagram of the Simulation Scheme

Substituting (3.45) into (3.43) and properly rearranging terms, we have

$$D_h\Phi(s)z(s) = -D_l\Psi(s)z(s) + u(s) \tag{3.46}$$

The particular advantage in having (3.46) is that it is in a column-decoupled form. This special form makes the rest of the construction straightforward.

As conventionally done, we temporarily assume that each $z_i^{(k_i)}$ is available (where z_i is the ith variable of the partial state z in the time domain; $z_i^{(k_i)}$ is the k_ith derivative). Then, by integrating each $z_i^{(k_i)}$ k_i times, we obtain m chains of all required lower derivatives. A simple assembling of all derivatives according to (3.46) will give the whole scheme. FIG.3.1 shows the schematic of the connections at the matrix level. Where in FIG.3.1 $z^{(k)} = [z_1^{(k_1)} \cdots z_m^{(k_1)}]$.

Whenever (3.43) is available, we simply rewrite (3.44) into the form

$$y(s) = N_c\Psi(s)z(s) \tag{3.47}$$

so that it is obtained from the existing derivatives. This part is also incorporated into FIG.3.1.

It is clear from FIG.3.1 that apart from the familiar static feedback designated by D_l, there also exists a feedback designated by D_h. Since

the latter feedback only involves derivatives, we shall call it the *differential feedback*. This term will further be justified in the next subsection, when we start to build up the dynamical representation for FIG.3.1.

The second point we wish to mention is that the number of integrators used in FIG.3.1 is evidently equal to the determinantal degree of the matrix $D(s)$. This is analogous to one of the features of the conventional scheme (though in a generalized sense).

3.5.2 A Singular System Representation of the Simulation Scheme

Due to the presence of the differential feedback, it can be conceived that the underlying system is not necessarily representable, depending on whether it is column-reduced or not, in the state-space form. It can, however, be described more generally by the singular system representation. This is shown below.

As usual, the outputs of all integrators are considered to be the set of internal variables. Denoting by x this set of internal variables, it is easily seen that when the feedbacks (static and differential) are not incorporated, the m integrator chains will be linked by

$$\dot{x} = A_c x + B_c u \tag{3.48}$$

where

$$A_c = block\ diag\{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},\ k_i \times k_i,\ i = 1, \ldots, p\}$$

$$B_c = block\ diag\{[1\ 0\ \cdots\ 0\ 0],\ 1 \times k_i,\ i = 1, \ldots, p\}$$

We now can see that (3.48) is actually the *core realization* in the realization algorithm. Since each output of the integrator is both an internal variable and the derivative of the output of the succeeding integrator (except, of course, for each of the last outputs of m chains), when considering it as an internal variable the static feedback in FIG.3.1 can be thought in terms of an additional input given by $D_l x$; while considering it as the derivative, all these derivatives together with $z^{(k)}$ will form yet another input given by $(D_h - B_c^T)\dot{x}$. We note that the term $B_c^T\dot{x}$ should be substracted out from the latter input in order to make $z^{(k)}$ available.

By including the aforementioned two additional inputs into (3.48), we have

$$\dot{x} = A_c x + B_c[-D_l x - (D_h - B_c^T)\dot{x} + u] \tag{3.49}$$

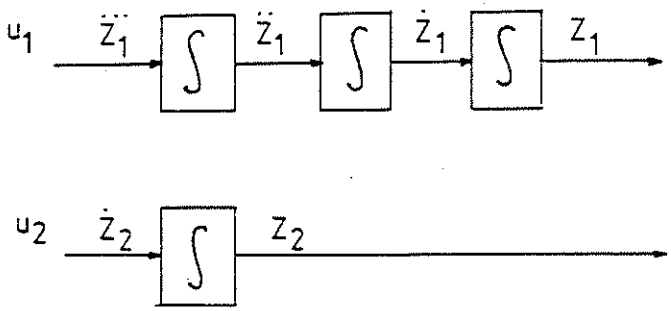


Figure 3.2: Integrator Chains of the Example to be the First Step in Setting up the Whole Scheme

rearranging terms in (3.49)

$$(I - B_c B_c^T + B_c D_h)\dot{x} = (A_c - B_c D_l)x + B_c u \tag{3.50}$$

The output equation can simply be read out from FIG.3.1

$$y = N_c x \tag{3.51}$$

We can now see that (3.50), (3.51) are precisely the realization given in the previous section. It should be noted that whenever the given right MFD is column-reduced, the matrix $E = I - B_c B_c^T + B_c D_h$ is non-singular, which implies that the underlying system is proper.

3.5.3 An Illustrative Example

We use an example in this subsection to clarify the ideas made above. The example is also helpful in exhibiting the so-called *impulsive behaviour* associated with pseudo-proper MFD's.

Example 3.2 The transfer matrix of a system is given by $G(s) = N(s)D^{-1}(s)$, where

$$N(s) = \begin{bmatrix} s^2 + 0 \cdot s + 1 & 0 \\ 0 \cdot s^2 + s + 0 & 1 \end{bmatrix}, D(s) = \begin{bmatrix} s^3 + 0 \cdot s^2 + s + 0 & s + 2 \\ 0 \cdot s^3 + 0 \cdot s^2 + s + 1 & 0 \cdot s + 1 \end{bmatrix}$$

which is clearly pseudo-proper. Since $k_1 = 3, k_2 = 1$, two chains of integrators can first be set up. This is shown in FIG.3.2.

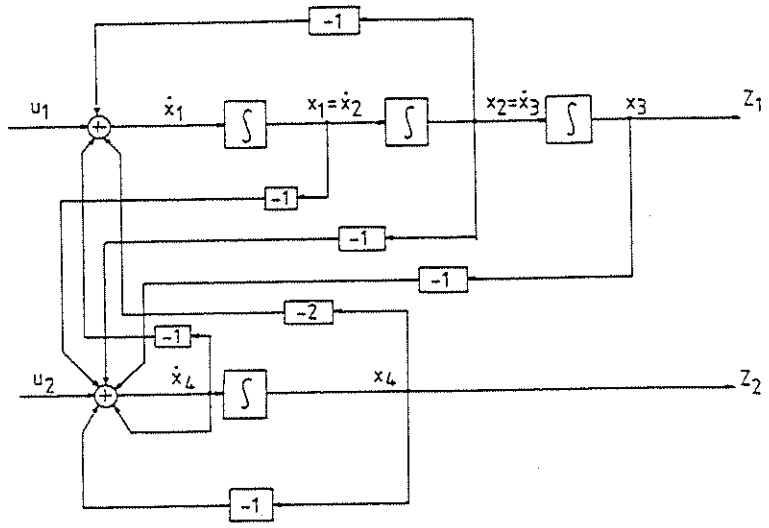


Figure 3.3: The Complete Simulation Configuration for the Example

As the second step, we separate the differential feedback from the static one

$$D(s) = D_h \Phi(s) + D_l \Psi(s) \tag{3.52}$$

where

$$D_h = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D_l = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The next step is simply to add two types of feedbacks in FIG.3.2 according to (3.52). FIG.3.3 shows the final construction. The reader may notice that an integrator-free loop is present in the configuation.

Once FIG.3.3 is obtained, the outputs can straightforwardly be generated. We shall omit this part of the drawing.

A particular kind of dynamical behaviour, known as *impulsive free-response* (first identified theoretically by Verghese et al. (1981)) can be readily seen in FIG.3.3.

For example, initially, we set $u_1 = u_2 = x_1(0) = x_2(0) = x_3(0) = 0$, but $x_4(0) \neq 0$. Since the differential feedback in the second chain in FIG.3.3 cancels the derivative of x_4, the nonzero $x_4(0)$ thus can not be balanced. Consequently, an impulse occurs in order to bring $x_4(0)$ to zero.

We note that the occurrence of the impulsive free-response is exclusively due to the set-up of the initial conditions. In fact, those initial conditions which cause the impulsive free-response had been termed *inconsistent initial conditions* prior to the work of Verghese et al. We refer the reader to Verghese et al. (1981) for a complete study of this new type of dynamical behaviour. □

3.6 Conclusions and Remarks

In this chapter we have investigated the general properties of pseudo-proper MFD's in connection with singular system. Some important realization algorithms are presented. Although only the column-based right MFD is treated, an algebraic duality will enable us to state exactly the same results for the *row-based left* MFD.

It is clear from our treatment that the relation between pseudo-proper MFD and singular system is exactly analogous to the one between proper MFD and state-space system. In the next chapter, we shall pursue this analogy further by introducing the canonical forms for singular systems.

When the attention is restricted to the pseudo-properness, we are actually led to the framework set up by Rosenbrock (1974), within which singular systems are handled. In chapter 5, we shall see that if such a assumption is dropped, we shall shift to the framework of Verghese et al. (1981). Thus, a sharp difference between these two frameworks is characterized in algebraic terms.

We shall use the pseudo-properness as a key assumption throughout this and the next chapter which include the normal properness as a special case. Chapter 5, 6 will not assume this condition, but some analogous properties can still be derived.

Chapter 4

Canonical Structure of Pseudo-proper Singular Systems

4.1 Introduction

Our attention has so far been largely focused on rational matrices represented in MFD. We have also shown that a pseudo-proper MFD can be alternatively represented (or realized) in a singular system.

As we know from the experiences with the state-space representation of proper systems, the first degree representation of the system can often provide a great deal of advantages in gaining insights into the underlying MFD.

In this chapter, we shall be exclusively concerned with the pseudo-proper singular system. We shall introduce the so-called *restricted system equivalence* (Rosenbrock 1974) for the pseudo-proper singular system. It can be seen that such an equivalence relation will induce some interesting canonical forms for reachable or observable singular systems, i.e., the so-called generalized controller canonical form, generalized observable canonical form, and so on. Moreover, we shall show that a set of indices found from the canonical forms is linked to the so-called minimal indices of a pencil constructed from the system.

We shall further apply these canonical forms to solve the observer design and the pole placement problems. Since we shall only be dealing with pseudo-proper singular systems in this chapter, the term *pseudo-proper* will frequently be dropped for convenience of the discussion.

The content of this chapter is mainly drawn from three of our papers (Tan and Vandewalle 1986c, 1986d, 1986e), and is organized as follows. Section 4.2 presents the generalized controller (observer) canonical forms for singular systems. The detailed derivations are provided. We also show that a set of indices induced from the generalized controller (observer) canonical form is a rearrangement of the set of minimal indices for the columns (rows) of the pencil $[Es - A \;\; 0 \cdot s + B]$ ($[E^T s - A^T \;\; 0 \cdot s - C^T]^T$).

In section 4.3, the generalized observer canonical form is utilized to devise observers for generating the estimates of the descriptor vector. The separation theorem is also proved in this section.

Section 4.4 is concerned with the descriptor vector feedback and the pole placement for singular systems. The generalized controller canonical form is used to investigate the possibilities of placing the poles of the closed-loop system by using the constant descriptor vector feedback. Section 4.5 contains the conclusions and remarks.

4.2 Canonical Forms for Pseudo-proper Singular Systems

4.2.1 Singular Systems and the Restricted System Equivalence

Let a singular system be given by

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4.1}$$

where E and A are real $n \times n$ matrices, B is a real $n \times p$ matrix and C is a real $m \times n$ matrix; E can be singular.

As for the state-space case, it is always convenient to transform a system to another one which is regarded to be equivalent in some sense to the original system, and which reveals directly the structure of the system. Rosenbrock (1974) has defined the following equivalence relation, the so-called *restricted system equivalence.*

Definition 4.1 *Let two singular systems be given in system matrix (Rosenbrock 1970)*

$$\mathcal{P}(s) = \begin{bmatrix} Es - A & B \\ -C & 0 \end{bmatrix}, \quad \mathcal{P}_1(s) = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & 0 \end{bmatrix} \tag{4.2}$$

Then they are said to be restricted system equivalent (or r.s.e.) if they are related by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & 0 \end{bmatrix} \tag{4.3}$$

where M and N are non-singular matrices. □

The equivalence relation defined above can obviously be interpreted as a kind of coordinate change in the descriptor vector space (the linear space where x lies). Consider two different linear descriptor vector spaces both of dimension n. E, A are two linear operators defined upon them. Then a *r.s.e.* transformation simply corresponds to an arbitrary and independent change of bases of these two linear spaces. Such a transformation is necessary because the properties of a system with respect to one basis can often better show up with respect to a different basis.

In the next subsection, we shall show that the *r.s.e.* equivalence transformation can induce some canonical forms for the reachable (or observable) singular system. These canonical forms exhibit explicitly the finite and infinite structure of the system.

4.2.2 Generalized Controller and Observer Canonical Forms

Single-variable Systems

We begin the development by first considering the canonical forms for single-variable systems. The reason for doing so is not only because the derivation is considerably simpler in this case, but also the canonical forms for such systems are uniquely defined. We shall see later that the uniqueness is no longer guaranteed for multivariable singular systems.

Definition 4.2 *A single-variable singular system (4.1) (with $m = p = 1$) is said to be in the generalized controller canonical form if*

$$E = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$B^T = [1 \;\; 0 \;\; \cdots \;\; 0] \tag{4.4}$$

where a_0 is either 1 or zero. □

We remark that the structure of C is irrelevant. Moreover, $Es - A$ in (4.4) coincides with the generalized companion form defined in Chapter 2. The case that a_0 is zero corresponds to a singular E. Not surprisingly, we have the following theorem

Theorem 4.1 *The single-variable singular system is reachable if and only if it is r.s.e. to a system in the generalized controller canonical form (4.4).*

proof

Sufficiency Since, by assumption, (4.1) is *r.s.e.* to a system in the generalized controller form, there exist non-singular matrices M and N such that $[MENs - MAN \;\; MB]$ is in the structure (4.4). Therefore

$$rank[MENs - MAN \;\; MB]$$

$$= rank \begin{bmatrix} a_0 s + a_1 & a_2 & \cdots & a_{n-1} & a_n & 1 \\ -1 & s & \cdots & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s & 0 \end{bmatrix} = n$$

and

$$rank[MEN \;\; MB]$$

$$= rank \begin{bmatrix} a_0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = n$$

Note that the *r.s.e.* transformation preserves the ranks of the matrices $[Es - A \;\; B]$ and $[E \;\; B]$. Hence the system is reachable.

Necessity Let (4.1) be reachable. Then $rank[E \;\; B] = n$. Since B is a column vector, we must have $n - 1 \leq rankE \leq n$. Therefore we can choose two non-singular matrices M_1 and N_1 such that

$$M_1 E N_1 = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where a_0 will be 1 if $rankE = n$, and zero otherwise. Now, apply N_1 along with M_1 to $[E \;\; B]$ in the following way

$$M_1[E \;\; B] \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} = [M_1 E N_1 \;\; M_1 B] = \begin{bmatrix} a_0 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{bmatrix} \tag{4.5}$$

If $a_0 = 1$, (4.1) reduces to a state-space system, it can then be obviously brought to (4.4) by applying the similarity transformations. If $a_0 = 0$, we shall surely have b_1 in (4.5) nonzero because the rank of (4.5) is n. Multiplying the first row of (4.5) by b_1^{-1}, then annihilating $b_2, \cdots, b_n$ with the first row, we obtain

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{4.6}$$

Notice that the above transformations applied to (4.5) are *r.s.e.* operations. Or equivalently, we can find two non-singular matrices M_2 and N_2 such that $[M_2 M_1 E N_1 N_2 \;\; M_2 M_1 B]$ assumes the form (4.6).

We now turn to $[Es - A \;\; B]$. Denoting $M_2 M_1$ and $N_1 N_2$ by M_3 and N_3, respectively, applying them to $[Es - A \;\; B]$ yields

$$\begin{aligned} & M_3[Es - A \;\; B] \begin{bmatrix} N_3 & 0 \\ 0 & I \end{bmatrix} \\ & = [M_3 E N_3 s - M_3 A N_3 \;\; M_3 B] = \begin{bmatrix} g_1 & g^T & 1 \\ p & Is - A_1 & 0 \end{bmatrix} \end{aligned} \tag{4.7}$$

where g and p are column $(n-1)$-vectors; g_1 is a scalar and A_1 is an $(n-1) \times (n-1)$ matrix. By observing $rank\,[p \;\; Is - A_1] = n - 1$, we know that (A_1, p) is a *controllable pair* (the controllability here is in the conventional sense, of course). Hence there exists a non-singular matrix T such that

$$\begin{aligned} & T^{-1}[p \;\; Is - A_1] \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} = [T^{-1} p \;\; Is - T^{-1} A_1 T] \\ & = \begin{bmatrix} 1 & s + c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ 0 & -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & s \end{bmatrix} \end{aligned} \tag{4.8}$$

Now construct two transformations as follows

$$\begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix}, \quad \begin{bmatrix} I & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{bmatrix}$$

apply these to (4.7)

$$\begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} g_1 & g^T & 1 \\ p & Is - A & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} g_1 & d_1 & d_2 & \cdots & d_{n-2} & d_{n-1} & 1 \\ 1 & s + c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & 0 \\ 0 & -1 & s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & s & 0 \end{bmatrix} \tag{4.9}$$

(We shall always use different Roman letters to indicate possible nonzero elements following each transformation). Observe that the above used operations are *r.s.e.* transformations. To proceed, using the first column of (4.9) to zero out $c_1, c_2, \cdots, c_{n-1}$, multiplying the first column of the resulting matrix by -1, (4.9) is converted into

$$\begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_{n-1} & r_n & 1 \\ -1 & s & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & s & 0 \end{bmatrix} \tag{4.10}$$

(4.10) is obviously *r.s.e.* to (4.1), and it assumes the structure (4.4) with $a_0 = 0$. The theorem is proven. □

We can similarly develop the *generalized observer canonical form* for observable singular systems.

Definition 4.3 *The single-variable system (4.1) is said to be in generalized observer canonical form if*

$$E = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -a_1 & 1 & \cdots & 0 \\ -a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{bmatrix} \tag{4.11}$$

$$C = [1 \ 0 \ \cdots \ 0]$$

where a_0 is either 1 or zero. □

Analogous to Theorem 4.1, we can state

Theorem 4.2 *System (4.1) is observable if and only if it is r.s.e. to a system in generalized observer canonical form (4.11).* □

This theorem can be proved either by recalling the duality or by spelling out the proof of Theorem 4.1 with proper changes.

Multivariable Singular Systems

Our present aim is to develop the *generalized controller canonical form* and the *generalized observer canonical form* for multivariable singular systems. We shall see, as for the single-variable case, that these two canonical forms are closely associated with the well-known controller and observer canonical forms for state-space systems.

We first recall that a state-space system $[Is - A \quad B]$ of order n is controllable if and only if $rank[B \ AB \ A^2B \ \cdots \ A^{n-1}B] = n$. Or write it down columnwise

$$rank[b_1, \cdots b_q, Ab_1, \cdots, Ab_q, \cdots, A^{n-1}b_1, \cdots, A^{n-1}b_q] = n \tag{4.12}$$

where q is the number of inputs of the system. Consequently, there are n linearly independent columns in (4.12). We now select from left to right in (4.12) n linearly independent columns. That is, start from $b_1, b_2, \cdots, b_q$ and then $Ab_1, Ab_2, \cdots, Ab_q$ till a linearly dependent column is found. We skip it and continue the search until n linearly independent columns are obtained. Conventionally (see, e.g., Kailath 1980) this way of searching is called *the scheme II search* .

As a standard result, any controllable state-space system is *similar* to the scheme II controller canonical form (see Kailath 1980, Luenberger 1967). Here we use a special case to show the structure of such a form

$$[Is - A \quad B] =$$

$$\begin{bmatrix} s+X & X & X & X & X & X & X & 1 & X & X \\ -1 & s & & & & & & & & X \\ & -1 & s & & & & & & & X \\ X & X & X & s+X & X & X & X & & 1 & X \\ & & & -1 & s & & & & & X \\ & & & & -1 & s & & & & X \\ & & & & & -1 & s & & & X \end{bmatrix} \tag{4.13}$$

where the X's denote possible nonzero values; the entries not shown are zeros. Intuitively one would expect that the canonical form for a reachable singular system takes the form (special case for illustration)

$$[Es-A \quad B] =$$

$$\begin{bmatrix} a_1 s+X & X & X & X & X & X & X & 1 & X & X \\ -1 & s & & & & & & & & X \\ & -1 & s & & & & & & & X \\ X & X & X & a_2 s+X & X & X & X & & 1 & X \\ & & & -1 & s & & & & & X \\ & & & & -1 & s & & & & X \\ & & & & & -1 & s & & & X \end{bmatrix} \tag{4.14}$$

where the a_i $(i=1,2)$ are either 1 or zero. This intuition is indeed true as proved in the following theorem.

Theorem 4.3 *Every reachable singular system $[Es-A \quad B]$ is r.s.e. to the form shown in (4.15).*

$$\begin{bmatrix} a_1 s+X & X & \cdots & X & X & X & X & \cdots & X & X & \cdots & X & X & \cdots & X & X & 1 & X & X & \cdots & X & X & \cdots & X \\ -1 & s & \cdots & 0 & 0 & & & & & & & & & & & & & & & & X & \cdots & X \\ \vdots & \vdots & & \vdots & \vdots & & & & & & & & & & & & & & & & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & s & & & & & & & & & & & & & & & & X & \cdots & X \\ X & X & \cdots & X & X & a_\kappa s+X & X & \cdots & X & X & \cdots & X & X & \cdots & X & X & & 1 & X & \cdots & X & X & \cdots & X \\ & & & & & -1 & s & \cdots & 0 & 0 & & & & & & & & & & & & X & \cdots & X \\ & & & & & \vdots & \vdots & & \vdots & \vdots & & & & & & & & & & & \vdots & & \vdots \\ & & & & & 0 & 0 & \cdots & -1 & s & & & & & & & & & & & & X & \cdots & X \\ & & & & & & & & & & \ddots & & & & & & & & & & & X & \cdots & X \\ X & X & \cdots & X & X & X & X & \cdots & X & X & \cdots & a_{\kappa-1} s+X & X & \cdots & X & X & & & & & 1 & X & \cdots & X \\ & & & & & & & & & & & -1 & s & \cdots & 0 & 0 & & & & & & X & \cdots & X \\ & & & & & & & & & & & \vdots & \vdots & & \vdots & \vdots & & & & & & \vdots & & \vdots \\ & & & & & & & & & & & 0 & 0 & \cdots & -1 & s & & & & & & X & \cdots & X \end{bmatrix}$$

(4.15)

where the a_i $(i=1,\ldots,\kappa)$ are either 1 or zero; the X's denote possible nonzero values and the entries not shown are zeros. (4.15) will be called the generalized controller canonical form for singular systems.

proof Suppose $rank E = \rho$. Then we can find two non-singular matrices M and N, such that

$$M[E \quad B]\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = [MEN \quad MB]$$

$$= \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1p} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b_{n-\rho,1} & b_{n-\rho,2} & \cdots & b_{n-\rho,p} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & X & X & \cdots & X \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & X & X & \cdots & X \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & X & X & \cdots & X \end{bmatrix} \tag{4.16}$$

Since $rank[E \quad B] = n$, then $rank[MEN \quad MB] = n$. Therefore the elements of MB in the first row of (4.16), namely $b_{11}, b_{12}, \cdots, b_{1p}$, should not all be identically zero. Without loss of generality, we assume that $b_{11} \neq 0$. Thus multiplying the first row by b_{11}^{-1}, then annihilating all other elements in the first column of MB we have

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & d_{12} & \cdots & d_{1p} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & d_{22} & \cdots & d_{2p} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & d_{n-\rho,2} & \cdots & d_{n-\rho,p} \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & X & \cdots & X \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & X & \cdots & X \end{bmatrix} \tag{4.17}$$

By repeating the preceding arguments to the second ... the $(n-\rho)$th row till we obtain

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & X & \cdots & X & X & \cdots & X \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & X & X & \cdots & X \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & X \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & X & \cdots & X \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & X & \cdots & X \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & X & \cdots & X \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 0 & \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & I & 0 & \tilde{B}_{22} \end{bmatrix} \tag{4.18}$$

where

$$\tilde{B}_{11} = \begin{bmatrix} 1 & X & \cdots & X \\ 0 & 1 & \cdots & X \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Now we turn to $[Es - A \;\; B]$. As is seen above, we can always find M_1 and N_1 such that

$$\begin{aligned} & M_1[Es - A \;\; B] \begin{bmatrix} N_1 & 0 \\ 0 & I \end{bmatrix} \\ = \;& [M_1 E N_1 s - M_1 A N_1 \;\; M_1 B] \\ = \;& \begin{bmatrix} -\tilde{A}_{11} & -\tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ -\tilde{A}_{21} & Is - \tilde{A}_{22} & 0 & \tilde{B}_{22} \end{bmatrix} \end{aligned} \tag{4.19}$$

We then apply another *r.s.e.* operation to column-compress $\tilde{A}_{21}$

$$\begin{aligned} & \begin{bmatrix} -\tilde{A}_{11}K & -\tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ -\tilde{A}_{21}K & Is - \tilde{A}_{22} & 0 & \tilde{B}_{22} \end{bmatrix} \\ = \;& \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & -\tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ \bar{A}_3 & 0 & Is - \tilde{A}_{22} & 0 & \tilde{B}_{22} \end{bmatrix} \end{aligned} \tag{4.20}$$

where $\bar{A}_3$ in (4.20) has full column rank. Since the rank of (4.20) is n, it is easy to see that

$$rank \begin{bmatrix} \bar{A}_3 & 0 & Is - \tilde{A}_{22} & 0 & \tilde{B}_{22} \end{bmatrix} = \rho \tag{4.21}$$

Eq. (4.21) tells that if we treat $[\bar{A}_3 \;\; \tilde{B}_{22}]$ as an input matrix (as in the state-space form), (4.21) actually defines a controllable state-space system. Hence it can be put into the controller canonical form (4.13). However, the search process akin to (4.12) should be performed in a way described below.

Let $\bar{A}_3 = [\bar{a}_1, \bar{a}_2, \cdots]$ and $\tilde{B}_{22} = [\tilde{b}_1, \tilde{b}_2, \cdots]$. Starting from $\bar{A}_3$, we select $\bar{a}_1, \bar{a}_2, \cdots$. Since $\bar{A}_3$ is of full column rank, obviously all of its columns will be selected. Then we proceed to $\tilde{b}_1, \tilde{b}_2, \cdots$ and select, from left to right, only those columns which are linearly independent on the previous vectors in the sequence. After having done this, as usual, we continue to select $\tilde{A}_{22}\bar{a}_1, \tilde{A}_{22}\bar{a}_2, \cdots, \tilde{A}_{22}\tilde{b}_1, \cdots$ and so on. When all ρ linearly independent columns have been chosen, we arrange them so that the chains related to the columns of $\tilde{B}_{22}$ precede the chains related to those of $\bar{A}_3$. This is shown as follows

$$[\tilde{b}_1, \tilde{A}_{22}\tilde{b}_1, \cdots, \tilde{b}_2, \tilde{A}_{22}\tilde{b}_2, \cdots, \bar{a}_1, \tilde{A}_{22}\bar{a}_1, \cdots, \bar{a}_2, \tilde{A}_{22}\bar{a}_2, \cdots] \tag{4.22}$$

As a consequence of (4.22), we can construct a transformation T (as in Kailath 1980, p.436, see also Luenberger 1967) such that (special case for

illustration)

$$[T^{-1}\tilde{A}_3 \;\; 0 \;\; Is - T^{-1}\tilde{A}_{22}T \;\; 0 \;\; T^{-1}\tilde{B}_{22}] =$$

$$\begin{bmatrix} X & 0 & s+d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & 0 & 0 & 1 & X \\ & & -1 & s & 0 & & & & & & & X \\ & & 0 & -1 & s & & & & & & & X \\ 1 & 0 & f_1 & f_2 & f_3 & s+f_4 & f_5 & f_6 & 0 & 0 & 0 & X \\ & & & & & -1 & s & 0 & & & & X \\ & & & & & 0 & -1 & s & & & & X \end{bmatrix} \tag{4.23}$$

For simplicity of notations, in what follows we shall, without loss of generality, work with a special case to illustrate the remaining steps of derivation. Our statements, however, apply to the general case.

Return to (4.20). Let

$$[\bar{A}_1 \;\; \bar{A}_2 \;\; -\tilde{A}_{12} \;\; \tilde{B}_{11} \;\; \tilde{B}_{12}] =$$

$$\begin{bmatrix} X & X & X & X & X & X & X & X & 1 & X & X & X \\ X & X & X & X & X & X & X & X & 0 & 1 & X & X \end{bmatrix} \tag{4.24}$$

and $[\bar{A}_3 \;\; 0 \;\; Is - \tilde{A}_{22} \;\; 0 \;\; \tilde{B}_{22}]$ has already been brought to (4.23). Use 1 in the last column of $T^{-1}\bar{A}_3$ to zero out the nonzero values of the matrix $T^{-1}\tilde{A}_{22}T$ in that row. Then do the same with 1 in the next to last column, ..., the first column of $T^{-1}\bar{A}_3$. When this is done, multiplying the first, ..., the last column of $T^{-1}\bar{A}_3$ by -1, we obtain

$$\begin{bmatrix} X & X & X & X & X & X & X & X & 1 & X & X & X \\ X & X & X & X & X & X & X & X & 0 & 1 & X & X \\ X & 0 & s+X & X & X & X & X & X & & & 1 & X \\ & & -1 & s & 0 & & & & & & & X \\ & & 0 & -1 & s & & & & & & & X \\ 1 & 0 & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & X \\ & & & & & -1 & s & 0 & & & & X \\ & & & & & 0 & -1 & s & & & & X \end{bmatrix} \tag{4.25}$$

(4.25) can then be easily transformed, by *r.s.e.* operations, to

$$\begin{bmatrix} X & X & X & X & X & X & X & X & 1 & X & X & X \\ X & X & X & X & X & X & X & X & 0 & 1 & X & X \\ & -1 & s & 0 & 0 & & & & & & & X \\ & 0 & -1 & s & 0 & & & & & & & X \\ & 0 & 0 & -1 & s & & & & & & & X \\ X & X & X & X & X & s+X & X & X & 0 & 0 & 1 & X \\ & & & & & -1 & s & 0 & & & & X \\ & & & & & 0 & -1 & s & & & & X \end{bmatrix} \tag{4.26}$$

(4.26) is of the structure (4.15). The theorem is proved. □

Like Theorem 4.1, Theorem 4.3 also has a converse.

Theorem 4.4 *If a singular system is r.s.e. to the generalized controller canonical form, then it is reachable.* □

This theorem can be proved by first converting the system into the canonical form, its reachability is obvious, then by recalling the fact that *r.s.e.* operations preserves reachability.

As in the single-variable case, the *generalized observer canonical form* can be developed as an algebraic dual of Theorem 4.3. It is stated as follows.

Theorem 4.5 *A multivariable singular system (4.1) is observable if and only if the matrix* $[E^T s - A^T \quad -C^T]^T$ *is r.s.e. to the generalized observer canonical form shown below.*

$$
\begin{bmatrix}
a_1 s + X & -1 & \cdots & 0 & X & & & & X & & & \\
X & s & \cdots & 0 & X & & & & X & & & \\
\vdots & \vdots & & \vdots & \vdots & & & & \vdots & & & \\
X & 0 & \cdots & -1 & X & & & & X & & & \\
X & 0 & \cdots & s & X & & & & X & & & \\
X & & & & a_2 s + X & -1 & \cdots & 0 & X & & & \\
X & & & & X & s & \cdots & 0 & X & & & \\
\vdots & & & & \vdots & \vdots & & \vdots & \vdots & & & \\
X & & & & X & 0 & \cdots & -1 & X & & & \\
X & & & & X & 0 & \cdots & s & X & & & \\
\vdots & & & & \vdots & & \ddots & & \vdots & & & \\
X & & & & X & & & & a_\ell s + X & -1 & \cdots & 0 \\
X & & & & X & & & & X & s & \cdots & 0 \\
\vdots & & & & \vdots & & & & \vdots & \vdots & & \vdots \\
X & & & & X & & & & X & 0 & \cdots & -1 \\
X & & & & X & & & & X & 0 & \cdots & s \\
1 & & & & 0 & & & & & & & \\
X & & & & 1 & & & & & & & \\
\vdots & & & & \vdots & & & & & & & \\
X & & & & X & & & & 1 & & & \\
X & X & \cdots & X & X & X & \cdots & X \cdots & X & X & \cdots & X \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
X & X & \cdots & X & X & X & \cdots & X \cdots & X & X & \cdots & X
\end{bmatrix}
\tag{4.27}
$$

where the a_i $(i = 1, \ldots, \ell)$ *are either 1 or zero.* □

In fact, other well-known canonical forms of state-space systems can also be generalized in the same way. However, proper care must be taken in the process of derivation.

4.2.3 Relation between the Canonical Forms and the Minimal Indices

For a given pencil of matrices, say, $Fs - L$, its minimal indices for the rows and for the columns are well defined (see, Gantmacher 1960b, Rosenbrock 1970 for the standard definitions). To be consistent with the theme of the thesis, we shall first present the definitions in terms of homogeneous form.

Let us start with the equation

$$(Fs - Lr)\theta(s, r) = 0 \tag{4.28}$$

where $\theta(s, r)$ is the homogeneous form of any polynomial solution vector $\theta(s)$. It is easy to see that all the entries of $\theta(s)$ have the same degree. Therefore, the degree of $\theta(s)$ is naturally defined to be the degree of any polynomial entry in $\theta(s)$. The following procedure will generate a sequence of polynomial vectors $\{\theta_i(s)\}$ and a sequence of non-negative numbers $\{\alpha_i\}$.

Let $\theta_1(s)$ be a solution of (4.28) with the least degree α_1 . Among all solutions of (4.28) that are linearly independent of $\theta_1(s)$, there exists a vector $\theta_2(s)$ with the least degree α_2 (two vectors $\eta(s), \xi(s)$ are said to be linearly independent if there exist no homogeneous polynomials $p(s, r), q(s, r)$ such that $\eta(s, r) \cdot p(s, r) + \xi(s, r) \cdot q(s, r) = 0$). Next, among all solutions of (4.28) that are linearly independent of both $\theta_1(s)$ and $\theta_2(s)$, there exists a vector $\theta_3(s)$ with the least degree. Proceeding this way, we obtain successive polynomial vectors $\theta_1(s), \theta_2(s), \cdots$, each linearly independent of its preceding vectors, until no more can be found. If k is the total number of vectors found above, the sequence of numbers $\{\alpha_i,\ i = 1, \ldots, k\}$ is defined as the *minimal indices for the columns* of $Fs - L$. The minimal indices for the rows are similarly defined when the equation

$$\zeta(s, r)(Fs - Lr) = 0 \tag{4.29}$$

is considered instead of (4.28).

Now turn to singular system (4.1) and consider the matrix $[Es - A \quad B]$. It is clearly a column-based polynomial matrix, but not a pencil of matrices. Define $[Es - A \quad 0 \cdot s + B]$ as the *augmented pencil of matrices* for the matrix. Its homogeneous form is easily seen to be $[Es - Ar \quad Br]$. Rosenbrock (1974) has shown a very interesting result: if (4.1) is reachable then the augmented pencil $[Es - A \quad 0 \cdot s + B]$ has no minimal indices for the rows. What we are trying to establish in the present subsection is the connection between the set of minimal indices for the columns of $[Es - A \quad 0 \cdot s + B]$ and a set of indices induced from the generalized controller canonical form developed in the previous subsection.

To begin with, we first write the generalized controller canonical form (4.14) in a compact way below

$$E_c s - A_c = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1\kappa} \\ R_{21} & R_{22} & \cdots & R_{2\kappa} \\ \vdots & \vdots & & \vdots \\ R_{\kappa 1} & R_{\kappa 2} & \cdots & R_{\kappa\kappa} \end{bmatrix} \tag{4.30}$$

where κ is defined in (4.15) as the number of diagonal blocks; and for $i = 1, \cdots, \kappa$

$$R_{ii} = E_{ii} s - A_{ii} = \begin{bmatrix} a_i s + X & X & \cdots & X & X \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

while the R_{ij} $(i \neq j)$ have all rows zero except possibly the first one. Further,

$$B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \\ \vdots \\ B_{c\kappa} \end{bmatrix} \tag{4.31}$$

where each B_{ci} $(i = 1, \ldots, \kappa)$ has the form

$$B_{ci} = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \begin{matrix} b_i = 1 \\ b_1 = b_2 = \cdots b_{i-1} = 0 \end{matrix}$$

Here we have assumed that B_c has full column rank.

Based on the canonical form, a sequence of non-negative numbers can be defined as

$$\beta_i = rank E_{ii} \quad (i = 1, \ldots, \kappa) \tag{4.32}$$

The following theorem connects the set $\{\beta_i\}$ to the set $\{\alpha_i\}$ which is the set of minimal indices for the columns of $[Es - A \;\; 0 \cdot s + B]$.

Theorem 4.6 *For a reachable system (4.1), the set $\{\beta_i\}$ is a rearrangement of the set $\{\alpha_i\}$.*

proof We first put system (4.1) into the generalized controller canonical form by using *r.s.e.* operations. Since the minimal indices are invariant under the following transformation (Gantmacher 1960b)

$$M[Es - A \quad 0 \cdot s + B]N = [E_1 s - A_1 \quad 0 \cdot s + B_1] \tag{4.33}$$

where M, N are non-singular matrices. That is to say, the pencil $[Es - A \;\; 0 \cdot s + B]$ has the same set of minimal indices for the columns as the pencil $[E_1 s - A_1 \;\; 0 \cdot s + B_1]$.

Make use of 1's in the matrix B_c to zero out all constants in those rows where 1 lies, then multiply all the 1's by -1. The resulting pencil is

$$\begin{bmatrix} Q_1 & 0 & \cdots & 0 & P_1 \\ 0 & Q_2 & \cdots & 0 & P_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & Q_\kappa & P_\kappa \end{bmatrix} \tag{4.34}$$

where

$$\begin{aligned} Q_i &= \begin{bmatrix} a_i s & 0 & \cdots & 0 & 0 \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix} \\ &= E_{ci} s - J_{ci} \end{aligned}$$

and

$$P_i = \begin{bmatrix} g_1 & g_2 & \cdots & g_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{matrix} g_i = -1 \\ g_j = 0, \quad j \neq i \end{matrix}$$

By properly rearranging the columns in (4.34) it is possible to arrive at the form

$$\begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & L_\kappa \end{bmatrix} \tag{4.35}$$

where for $i = 1, \ldots, \kappa$

$$L_i(s, \tau) = \begin{bmatrix} -\tau & a_i s & 0 & \cdots & 0 & 0 \\ 0 & -\tau & s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tau & s \end{bmatrix} = [-\tau \cdot e_1 \quad E_{ci} s - J_{ci} \tau]$$

in which $e_1^T = [1 \ 0 \ \cdots \ 0]$. Note that a_i $(i = 1, \ldots, \kappa)$ is either 0 or 1. If $a_i = 1$, then β_i is the dimension of the matrix E_{ii} (or equivalently E_{ci}). The equation

$$L_i(s, r)v_i(s, r) = 0 \tag{4.36}$$

thus has the solution

$$v_i = \begin{bmatrix} s^{\beta_i} d(s, r) \\ s^{\beta_i - 1} r d(s, r) \\ \vdots \\ s r^{\beta_i - 1} d(s, r) \\ r^{\beta_i} d(s, r) \end{bmatrix} \tag{4.37}$$

in which $d(s, r)$ is an arbitrary homogeneous polynomial. It is clear that the least degree of the solution $v_i(s, r)$ is β_i , and all the other solutions are linearly dependent on $v_i(s, r)$. We therefore conclude that β_i is a minimal index.

If $a_i = 0$, then β_i is the dimension of the matrix E_{ci} minus 1. The equation (4.36) thus has the solution

$$v_i^* = \begin{bmatrix} 0 \\ s^{\beta_i} d(s, r) \\ s^{\beta_i - 1} r d(s, r) \\ \vdots \\ s r^{\beta_i - 1} d(s, r) \\ r^{\beta_i} d(s, r) \end{bmatrix} \tag{4.38}$$

obviously, in this case β_i is again a minimal index.

Since it is easy to see from (4.35) that there exist only κ linearly independent solutions for the equation

$$[Es - Ar \quad Br]v(s, r) = 0 \tag{4.39}$$

the number of the minimal indices is precisely κ. Our assertion thus follows.

□

A simple algebraically dual result can be stated for observable systems and the row indices. We shall however omit the development.

4.3 Observer Design for Singular Systems Using the Canonical Forms

Observers are of extreme importance in those circumstances where the descriptor vector feedback is needed, but it can not be directly (or physically)

accessible. In this section the generalized observer canonical form is utilized to derive a systematic observer design procedure. Although the method presented here may not be computationally advantageous, it does provide insights into the nature of the observers.

4.3.1 (n-1)th-order Observer for Single-variable Systems

It has been shown that a given observable singular system can be put into the generalized observer canonical form by *r.s.e.* transformations, i.e., to the form

$$\begin{bmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & s & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1} & 0 & 0 & \cdots & -1 \\ a_n & 0 & 0 & \cdots & s \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \tag{4.40}$$

$$y = [1 \ 0 \ 0 \ \cdots \ 0]x$$

To ease the notation the same x is used to denote the transformed descriptor vector.

Apply further *r.s.e.* transformations to bring (4.40) into the form

$$\begin{bmatrix} d_1 & -1 & 0 & \cdots & 0 \\ d_2 & s + e_1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ d_{n-1} & e_{n-2} & 0 & \cdots & -1 \\ d_n & e_{n-1} & 0 & \cdots & s \end{bmatrix} x = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} u \tag{4.41}$$

$$y = [1 \ 0 \ 0 \ \cdots \ 0]x$$

(Remind that different Roman letters are used to indicate obvious changes resulting from the transformations). It should be noted that e_i $(i = 1, \ldots, n-1)$ in (4.41) can be set up arbitrarily.

Let x be the descriptor vector which is to be observed. Since $y = x_1$, we only have to estimate $x_2, \cdots, x_n$. This can apparently be achieved by applying the $(n-1)$-order observer shown as follows

$$\begin{bmatrix} s + e_1 & -1 & \cdots & 0 \\ e_2 & s & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n-2} & 0 & \cdots & -1 \\ e_{n-1} & 0 & \cdots & s \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_n \end{bmatrix} = - \begin{bmatrix} d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix} y + \begin{bmatrix} f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} u \tag{4.42}$$

where the $\hat{x}_i$ $(i = 2, \ldots, n)$ denote the estimates of x_i.

We now prove that (4.42) is indeed an asymptotic observer. Define $\tilde{x}_i = x_i - \hat{x}_i$ $(i = 2, \ldots, n)$. By substracting (4.42) from the last $(n-1)$ equations in (4.41) we obtain

$$\begin{bmatrix} s+e_1 & -1 & \cdots & 0 \\ e_2 & s & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e_{n-2} & 0 & \cdots & -1 \\ e_{n-1} & 0 & \cdots & s \end{bmatrix} \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \\ \vdots \\ \tilde{x}_n \end{bmatrix} = 0 \tag{4.43}$$

It has been seen from the foregoing arguments that if the system is observable, then $e_1, \cdots, e_{n-1}$ can be chosen arbitrarily. This implies that the poles of (4.43) can be assigned to any desired locations. Consequently, the error $\tilde{x}_i$ can be controlled to zero asymptotically in any exponential speed. Moreover, since the input does not play any role in the foregoing discussion, this observation remains valid for multi-input systems.

The preceding design method is obviously a generalization of the reduced order observer design method found in the theory of state-space system (Luenberger 1966). The process shows quite naturally how the $(n-1)$-order observer operates on observable singular systems.

4.3.2 Reduced-order Observer for Multivariable Systems

As we have seen in the preceding section, an observable multivariable system can be transformed, by *r.s.e.* operations, to the generalized observer canonical form (4.27). It is easy to see from the structure of (4.27) that a non-singular matrix F can be found such that all the X's in the first κ rows of FC can be eliminated (this is equivalent to a coordinate change in the output space), i.e.,

$$Fy = FCx = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ X & X & \cdots & X & X & X & \cdots & X & \cdots & X & X & \cdots & X \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ X & X & \cdots & X & X & X & \cdots & X & \cdots & X & X & \cdots & X \end{bmatrix} x \tag{4.44}$$

Denoting by $\bar{y}$ the first κ entries in the output vector Fy, we have

$$\bar{y} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} x \tag{4.45}$$

Equation (4.45) along with the upper part in (4.27) can be looked upon as consisting of κ multi-input single-output systems of the form

$$\begin{bmatrix} a_i s + X & -1 & \cdots & 0 \\ X & s & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ X & 0 & \cdots & -1 \\ X & 0 & \cdots & s \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{i,\rho_i - 1} \\ x_{i,\rho_i} \end{bmatrix} = H_i \bar{y} + B_i u \tag{4.46}$$

$$\bar{y}_i = [1 \ 0 \ \cdots \ 0] x_i, \quad (i = 1, \cdots, \kappa)$$

where H_i, B_i are appropriate constant matrices; ρ_i is the dimension of the ith generalized companion block in (4.27). Therefore, the result obtained in the previous section can be applied here. For Eq. (4.46), we can construct an asymptotic observer of dimension $\rho_i - 1$ no matter what value a_i will take (zero or 1). Consequently, for the entire system (4.46) κ observers are needed with the total dimension $n - \kappa$ (κ is the number of generalized companion blocks in (4.27). These observers will generate asymptotically $n - \kappa$ components of the descriptor vector, while the remaining κ components can be found from the part of the output vector $\bar{y}$. Moreover, the eigenvalues of all observers can be arbitrarily chosen. To summarize, we state

Theorem 4.7 *If system (4.1) is observable, an $(n-\kappa)$-order asymptotic descriptor vector observer with any desired eigenvalues can be constructed, where κ is the number of the generalized companion blocks in its generalized observer canonical form.* □

4.3.3 Joint Controller-Observer Design Scheme and the Separation Property

In this subsection, we shall briefly look at another property related to the observer constructed above. For simplicity, we limit the discussion to single-variable systems. However, the generalization to multivariable case can easily be achieved.

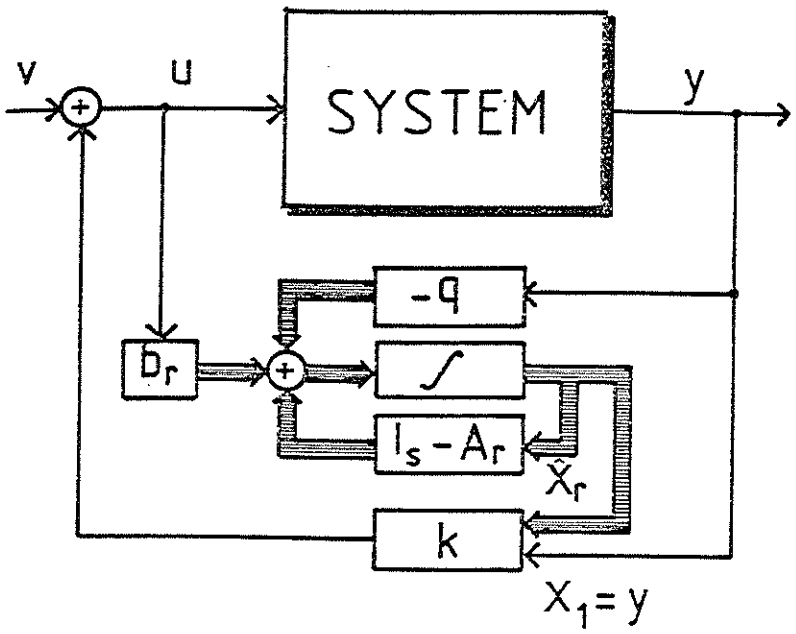


Figure 4.1: Observer and the descriptor vector feedback control scheme

Since the observer (4.42) is a state-space system, its states (the estimates of the descriptor variables) are directly accessible. These estimates plus the output of (4.40) enable us to set up a descriptor vector feedback control scheme (FIG.4.1). Note that the notation used in FIG.4.1 is consistent with the following proper partitioning of (4.41)

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} d_1 & p^T \\ q & A_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_r \end{bmatrix} + \begin{bmatrix} f_1 \\ f_r \end{bmatrix} u \tag{4.47}$$

where $x_r^T = [x_2\ x_3\ \cdots\ x_n]$, and all the matrices in (4.47) are partitioned accordingly. As usual, the descriptor vector feedback takes the form

$$u = K \begin{bmatrix} x_1 \\ x_r \end{bmatrix} + v = k_1 x_1 + k_r x_r + v \tag{4.48}$$

in which $K = [k_1 \quad k_r]$. By noting the structure of the $(n-1)$th order observer (4.42), the dynamic equation of the overall system in FIG.4.1 can

be set up easily as follows

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_r \\ \dot{\hat{x}}_r \end{bmatrix} \\ &= \begin{bmatrix} d_1 + f_1 k_1 & p^T & f_1 k_r \\ q + f_r k_1 & A_r & f_r k_r \\ q + f_r k_1 & 0 & A_r + f_r k_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_r \\ \hat{x}_r \end{bmatrix} + \begin{bmatrix} f_1 \\ f_r \\ f_r \end{bmatrix} v \end{aligned} \tag{4.49}$$

Apply a *r.s.e.* transformation to bring (4.49) to the following form

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_r \\ \dot{\hat{x}}_r \end{bmatrix} \\ &= \begin{bmatrix} d_1 + f_1 k_1 & p^T + f_1 k_r & -f_1 k_r \\ q + f_r k_1 & A_r + f_r k_r & -f_r k_r \\ 0 & 0 & A_r \end{bmatrix} \begin{bmatrix} x_1 \\ x_r \\ \hat{x}_r \end{bmatrix} + \begin{bmatrix} f_1 \\ f_r \\ 0 \end{bmatrix} v \end{aligned}$$

or equivalently

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}}_r \end{bmatrix} = \begin{bmatrix} A + BK & -Bk_r \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ \hat{x}_r \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \tag{4.50}$$

where

$$E = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},\ A = \begin{bmatrix} d_1 & p^T \\ q & A_r \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_r \end{bmatrix},\ B = \begin{bmatrix} f_1 \\ f_r \end{bmatrix}$$

(Note that in the above derivation, we have kept the same notation for $\hat{x}$ before and after the transformations). From (4.50) we immediately observe that the characteristic polynomial of the overall system is the product of $det(Es - Ar - BKr)$ and $det(Is - A_r r)$. A simple interpretation is that as far as the eigenvalues are concerned, the feedback of the vector $[x_1 \quad \hat{x}_r^T]^T$ is as perfect as the real descriptor vector x. As a consequence, the design of the descriptor vector feedback and the design of the observer can be carried out independently. This nice property, known in the state-space theory, is also called the *separation property*.

4.4 Static Descriptor Vector Feedback and Pole Placement for Singular Systems via Canonical Forms

In this section, we shall be concerned with the problem of applying constant descriptor vector feedback to place the poles of singular systems. This

problem has been considered by many authors. Cobb (1981) has, among others, made the first attempt. His approach is based on the canonical decomposition of the pencil $Es-A$ (Gantmacher 1960). Quite recently, the problem was tackled by Fletcher et al. (1986). Our solution here is more conventional, and based on the canonical forms derived in the previous section. We shall see that these canonical forms enable us to identify easily many important properties when the descriptor vector feedback is applied. They can also help to overcome certain difficulties in other approaches.

4.4.1 Descriptor Vector Feedback for Single-variable Systems

Let a reachable single-variable system be given by

$$E\dot{x} = Ax + Bu \tag{4.51}$$

and let a descriptor vector feedback

$$u = Kx + v \tag{4.52}$$

be applied to (4.51). Then the closed-loop system assumes the form

$$E\dot{x} = (A + BK)x + Bv \tag{4.53}$$

In order to maintain the solvability, it is natural to require that

$$det(Es - A - BK) \neq 0 \tag{4.54}$$

In fact, (4.54) will serve as a kind of admissibility condition for the feedback gain K. It is easy to see from (4.54) and the generalized controller canonical form that for reachable systems, there will be only one K such that (4.54) fails. Therefore, this requirement is generic at least for reachable systems.

The following theorem establishes the connection between reachability of the original system and that of the closed-loop system (4.53).

Theorem 4.8 *A single-variable singular system (4.51) is reachable if and only if for any feedback gain K, the closed-loop system (4.53) is reachable.*

proof If (4.51) is reachable, it can be brought, by *r.s.e.* transformation, to the generalized controller canonical form shown below

$$\begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{4.55}$$

Now applying the descriptor vector feedback with the feedback gain $K = [k_1, k_2, \cdots, k_n]$, we see that the closed-loop system has the form

$$[Es - (A + BK) \;\; B] .$$

$$= \begin{bmatrix} a_0 s + (a_1 - k_1) & a_2 - k_2 & \cdots & a_{n-1} - k_{n-1} & a_n - k_n & 1 \\ -1 & s & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s & 0 \end{bmatrix} \tag{4.56}$$

Eq. (4.56) shows that the closed-loop system can also be brought to the canonical form. It is therefore reachable.

The converse statement is proved similarly by first putting the closed-loop system to the canonical form, then applying an additional feedback to cancel the effect of the first one. □

In state-space systems, an important feature of the state feedback is that it can be used to place arbitrarily the poles of the system. Though in singular systems the situation is more complicated, we can still identify many significant properties of this kind by using the generalized controller canonical form.

To proceed, let (4.51) be a given singular system. Then its characteristic polynomial is defined to be

$$\psi(s, r) = det(Es - Ar) \tag{4.57}$$

Recall that the poles of the system consist of finite and infinite poles, both defined as the finite and the infinite zeros of the polynomial $\psi(s, r)$. We can state the following theorem

Theorem 4.9 *Except the leading coefficient, all the coefficients of the characteristic polynomial of a singular system can be modified arbitrarily by the linear descriptor vector feedback (4.52) if and only if the system is reachable.*

proof The theorem is actually well-known when E is non-singular. Here we only provide a proof for the case where E is singular.

Sufficiency Since the system is reachable it can be put into the generalized controller canonical form (4.55). Because of the special structure of the canonical form, we can always apply an appropriate feedback gain K to modify the coefficients $a_1, \cdots, a_n$ of $\psi(s, r)$ at will.

Necessity Let a feedback gain K be chosen such that $\psi(s,r) = r^n$. This implies that

$$rank[Es - A \quad B] = rank[Es - (A + BK) \quad B] = n \tag{4.58}$$

We now proceed to show that $rank[E \quad B] = n$. Choose a different feedback gain K such that $\psi(s,r) = rs^{n-1} + \cdots$. Since

$$n - 1 \geq rank E \geq deg\, \psi(s,1) = n - 1$$

we must have $rank E = n - 1$. This result along with (4.58) can be used to bring $[Es - A \quad B]$ into the form

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n & b_1 \\ -1 & s & \cdots & 0 & 0 & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s & b_n \end{bmatrix} \tag{4.59}$$

by *r.s.e.* transformations. We now show $b_1 \neq 0$, which clearly implies that $rank[E \quad B] = n$.

By contradiction. Suppose that $b_1 = 0$. Then (4.59) reduces to

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n & 0 \\ -1 & s & \cdots & 0 & 0 & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s & b_n \end{bmatrix} \tag{4.60}$$

We can assume $a_1 \neq 0$ in (4.60). If this is not the case, a proper feedback can be applied to make it so. Therefore, a_1 can be used to annihilate the first column and the first row in (4.60). The resulting matrix is of the form

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & Is - A_1 & f_1 \end{bmatrix} \tag{4.61}$$

where A_1 is an $(n-1) \times (n-1)$ matrix, f_1 is an $(n-1)$-vector. From (4.61), it can be readily seen that no feedback of any kind could be used to set a_1 to zero, which clearly contradicts the assumption. Thus we must have $b_1 \neq 0$. □

As a simple application of the preceding theorem we state

Corollary 4.1 *If the system (4.51) with singular E is reachable, then $n-1$ of its poles (finite and infinite) can be assigned arbitrarily by linear descriptor vector feedback; the remaining one is fixed at infinity.* □

This corollary follows directly from Theorem 4.9. Note that it allows to place $n-1$ poles of a reachable system even from the finite positions to the infinity or from the infinity to arbitrary finite positions by the descriptor vector feedback.

4.4.2 Descriptor Vector Feedback for Multivariable Systems

In this subsection, we consider the descriptor vector feedback for multivariable systems. The feedback used has the following form

$$u = Kx + v \tag{4.62}$$

We also assume that the matrix B of the system is of full column rank. This assumption is purely for convenience of the discussion.

Modified by (4.62), the closed-loop system takes the form

$$E\dot{x} = (A + BK)x + Bv \tag{4.63}$$

If the original system is reachable, it can easily be shown, by using the generalized controller canonical form, that almost any feedback gain K in (4.62) can guarantee $det(Es - A - BK) \neq 0$. However, unlike the single-variable case, there will generally be more than one feedback gain K which make $det(Es - A - BK) = 0$. The precise number is determined uniquely by the rank of E.

Secondly we can state the counterpart of Theorem 4.8 as follows

Theorem 4.10 *A multivariable singular system is reachable if and only if for any feedback gain K in (4.63), the closed-loop system is reachable.* □

The above assertion can be proved quite easily by utilizing the generalized controller canonical form. We shall omit the detail.

As far as the pole placement problem is concerned, let the characteristic polynomial of the system be

$$\begin{aligned} \psi(s,r) &= det(Es - Ar) \\ &= \psi_0 s^n + \psi_1 s^{n-1} r + \cdots + \psi_{n-f} s^f r^{n-f} + \cdots + \psi_n r^n \end{aligned} \tag{4.64}$$

where $f = rank E$ (here we assume that $f \leq n-1$). It is then readily checked that $\psi_0, \cdots, \psi_{n-f-1}$ will always be zero.

Since the descriptor vector feedback (4.62) can never change the rank of E, $\psi_0, \cdots, \psi_{n-f-1}$ will remain zero after the feedback. Hence the number of poles which can be shifted by the descriptor vector feedback (4.62) is effectively reduced to f.

As shown in the previous subsection, for single-variable systems the arbitrary modification of the coefficients of the characteristic polynomial is equivalent to the reachability of the system. Whereas for multivariable systems, the situation is slightly different, as is made clear by the following theorem and the subsequent remarks.

Theorem 4.11 *If a multivariable singular system is reachable, then the coefficients $\psi_{n-f}, \cdots, \psi_n$ of its characteristic polynomial can be arbitrarily modified by the feedback (4.62).* □

proof Since the system is reachable, the matrix $[Es-A \;\; B]$ can be brought to the generalized controller canonical form by *r.s.e.* operations. Further *r.s.e.* operations along with some proper feedback will put the canonical form into the form below

$$\begin{bmatrix}
1 & & & & & & & & 0 & \\
 & 1 & & & & & & & 0 & B_{12} \\
 & & \ddots & & & & & & \vdots & \\
 & & & 1 & & & & & 0 & \\
 & & & & \psi_{n-f} & \psi_{n-f+1} & \psi_{n-f+2} & \cdots \; \psi_{n-1} & \psi_n & 1 \\
 & & & & -1 & s & & & & 0 \\
 & & & & & -1 & s & & & 0 \quad B_{22} \\
 & & & & & & \ddots & \ddots & & \vdots \\
 & & & & & & & -1 & s & 0
\end{bmatrix} \tag{4.65}$$

where B_{12} and B_{22} are two constant matrices; the entries not shown in (4.65) are zero. It is obvious that the structure of (4.65) enables to change arbitrarily the actually values of $\psi_{n-f}, \ldots, \psi_n$. The theorem follows. □

Unlike single-variable systems, the converse of the above theorem is not necessarily true in general. This is due to the fact that reachability is more directly related to the generalized invariant polynomials than to the characteristic polynomial which is the product of the set of invariant polynomials. However, if $rank E = n - 1$, the converse of the theorem can surely be guaranteed.

The practical placement of f $(= rank E)$ poles can be achieved as described in the last corollary of the chapter.

Corollary 4.2 *If a multivariable system is reachable, then f $(= rank E)$ of its poles can be placed arbitrarily to the finite and the infinite locations by the descriptor vector feedback (4.62). The remaining $n - f$ poles are fixed at infinity.* □

4.5 Conclusions and Remarks

The main contribution in this chapter is the introduction of some canonical forms for singular systems. These canonical forms are important because they display directly the structural information of the system and enables the simple design and analysis of the system to be carried out.

Based on these canonical forms, the problems of designing observers and the feedback for pole placement are also considered. The great simplicity and advantage of using the canonical forms can easily be seen in the course of derivation.

There have been other approaches available (e.g., Al-Nasa et al. 1984) in the literature for both the observer design and the pole placement. The distinct feature of our approach, as compared to others, is that the design procedure is transparent from the very beginning. The disadvantage is obviously that the canonical form is numerically difficult to obtain, which confines the practical applicability of the theory.

Chapter 5

General Matrix Fraction Descriptions and Singular System realizations

5.1 Introduction

The matrix fraction descriptions and singular systems considered in the last two chapters have been restricted to be *pseudo-proper*. The reason to introduce this type of MFD is largely due to the fact that many physical modellings often end up with MFD's of this type. A typical example is the well-known proper ARMA model.

However, if the modelling is extended beyond the physical situations, for example the socio-economical systems, prediction systems and so forth, it then can very well be that the resulting MFD is not pseudo-proper, but *general* in the sense defined in Chapter 2.

The main purpose of this chapter is to apply the theory developed in Chapter 2 to handle general MFD's. We shall establish, among others, a link between general MFD's and singular systems, thereby develop a realization theory to realize any given general MFD into a singular system.

Owing to the underlying differences, the realized singular systems differ from those treated in the preceding chapters, and will be called *general* singular system. The issue is pretty much like viewing the same entity in two entirely different ways. One of the crucial deviations is that for singular system $E\dot{x} = Ax + Bu$, we consider *rank* of E (which will be termed the *order* of the system) rather than the *dimension* of E to be significant. In the content of this chapter, even two systems of different dimensions

could be considered equivalent! Of course, the concepts of reachability and observability can no longer apply, they are replaced by controllability and reconstructibility. As an important result, we establish the following equivalence circle: a given general MFD is completely irreducible if and only if any of its realizations with the order equal to the determinantal degree of the denominator of the MFD is controllable and reconstructible; further, the controllability and the reconstructibility of any realization is equivalent to the minimality in order of the realization. Based on this equivalence circle, a very interesting interpretation of the well-known McMillan degree of rational matrices can be found. We shall show that when a general MFD is completely irreducible, the determinantal degree of the denominator will be its McMillan degree. Consequently, the McMillan degree of a rational matrix only describes its completely irreducible part.

The concepts of controllability and reconstructibility along with the notion of strong equivalence will also induce some canonical forms for singular systems. Our treatments in Chapter 3, 4 and the present chapter will in fact highlight the difference between the approach taken by Rosenbrock (1974) and the one by Verghese et al. (1981) in dealing with singular systems.

This chapter is structured as follows. In Section 5.2, we discuss the different viewpoint towards singular systems. Once the new viewpoint is clarified, the rest of the development in this chapter parallels that of Chapter 3 and Chapter 4. We shall first present a simple realization for general matrix polynomial right MFD's in this section along with some properties. Section 5.3 is then devoted to the development of a complete realization theory for general column-based right MFD's. The familiar equivalence among joint controllability and reconstructibility, complete irreducibility, minimality *in order* is established in Section 5.4. The new canonical structures for singular systems under the so-called *strong equivalence transformation* is treated in Section 5.5. In Section 5.6, we discuss the connection between the canonical form and the minimal indices. The contents of Section 5.3, 5.5 also extends the results in Tan and Vandewalle (1986f, 1986g). The final section contains remarks and conclusions.

5.2 General MFD and the Singular System Realization

5.2.1 New Views on Singular System

We have been quite familiar with the standard form for singular systems

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.1}$$

where, as usual, $x \in R^n, u \in R^p$ and $y \in R^m$; E, A are matrices of dimension $n \times n$, B and C are matrices of dimensions $n \times p, m \times n$, respectively. Moreover, n has been defined as the dimension of the system.

It has been seen in the preceding two chapters that the dimension n of a singular system is of crucial importance in that it is the *determinantal degree* of the denominator of the underlying transfer matrix in MFD; further, n necessarily decides whether two singular systems are *restricted system equivalent*. The point is that when the dimension is of significance, the matrices $C, Es - A, B$ in the transfer matrix of (5.1), i.e.,

$$G(s) = C(Es - A)^{-1}B \tag{5.2}$$

are also considered to be *matrix polynomials*. Thus (5.2) obviously suggests the pseudo-proper framework, since the denominator is always one degree higher than the numerators.

On the other hand, there are situations where we have to face *general MFD's* not just pseudo-proper ones. Since the singular system representation (5.1) is easy to handle, it is highly preferred to realize general MFD's into singular system.

The conflict is resolved by a slightly different attitude towards singular system. The trick is simple and was originated by Verghese et al. (1981). Instead of the dimension of singular systems, we now consider $f = rank E$ (which will be called the *order* of singular systems) to be a measure. To carry out the scheme further, we shall define two singular systems with, perhaps, different dimensions to be equivalent if one can be transformed to the other by any combination of the following three operations:

1. *The left and right multiplication*

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \tag{5.3}$$

   where $QE = 0$ and $ER = 0$;

2. *the trivial augmentation*

$$from \quad \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{5.4}$$

$$to \quad \begin{bmatrix} Es - A & 0 & B \\ 0 & I & 0 \\ -C & 0 & D \end{bmatrix} \tag{5.5}$$

3. *the trivial deflation*, namely from (5.4) to (5.5).

The reader may have noted that in the system matrices in (5.3) through (5.5), an extra constant matrix D is inserted due to the effect of the transformation. Therefore we are actually considering more general singular systems than (5.1)

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{5.6}$$

The operations (1)–(3) are originally defined by Verghese et al. (1981) and are termed the *operations of strong equivalence transformation*. Moreover, the two systems will be called to be *strongly equivalent (str. eq.)*. It can be shown (see Verghese et al. 1981) that two strongly equivalent systems will have the same transfer matrix.

A scrutiny over the operations (5.3)–(5.5) reveals that the dimension of the system can be altered but not the order f $(= rank E)$. In fact, we shall see in the ensuing development that the order takes over the role of the dimension in the realization for general MFD's.

It now follows that the matrices $Es - A, B, C$ are no longer regarded as polynomial matrices. Instead, they are simply a sort of "carrier" in which polynomial matrices can be embodied. Thus a part of the infinite structure of these matrices are irrelevant, and can be modified by the strong equivalence transformation. To be distinct, the singular system of this nature will be called *general singular system*. Since in this and the next chapter we shall only be dealing with general singular systems, the adjective "general" will frequently be dropped for simplicity of notation. We should note, however, that if the infinity is not of concern it is meaningless to talk about the aforementioned viewpoints on singular system.

5.2.2 General Matrix Polynomial MFD and the Block Controllable Realization

Since general singular system differs from the singular system discussed in the last two chapters, the notion of realization in general singular system needs to be defined accordingly. Let us start with (5.6). To define the realization we first apply a left and right multiplication (5.3) such that the non-singular matrix M row-compresses E, $Q = R = 0$, and $N = I$. The system matrix of (5.6) is transformed into

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_\epsilon s - A_\epsilon & B_\epsilon \\ -A_\tau & B_\tau \\ -C & D \end{bmatrix} \tag{5.7}$$

where E_ϵ is of full row rank. Then a given general MFD $G(s) = N(s)D^{-1}(s)$ is said being realized by (5.6) if

$$N(s,\tau)D^{-1}(s,\tau) = C \begin{bmatrix} E_\epsilon s - A_\epsilon \tau \\ -A_\tau \end{bmatrix}^{-1} \begin{bmatrix} B_\epsilon \\ B_\tau \end{bmatrix} + D \tag{5.8}$$

The above definition is based on the row-compression. It is, however, possible to present the definition with respect to the column-compression. It should be noted that (5.8) also defines the *transfer matrix* of a general singular system.

Let an $m \times p$ rational matrix $G(s)$ be given by a general matrix polynomial MFD, i.e. ,

$$G(s) = N(s)D^{-1}(s) \tag{5.9}$$

where $N(s), D(s)$ are matrix polynomials with the following homogeneous forms

$$\begin{aligned} N(s,\tau) &= N_0 s^\alpha + N_1 s^{\alpha-1}\tau + \cdots + N_\alpha \tau^\alpha \\ D(s,\tau) &= D_0 s^\alpha + D_1 s^{\alpha-1}\tau + \cdots + D_\alpha \tau^\alpha \end{aligned} \tag{5.10}$$

We can state the following realization theorem.

Theorem 5.1 *Any matrix polynomial MFD (5.9) can be realized in general singular system (5.6), where*

$$\begin{aligned} E &= block\ diag\{\overbrace{0, I_p, \cdots, I_p}^{\alpha+1}\} \\ A &= \begin{bmatrix} -D_0 & -D_1 & \cdots & -D_{\alpha-1} & -D_\alpha \\ I_p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix} \\ B^T &= [\overbrace{I_p \ 0 \ \cdots \ 0}^{\alpha+1}] \\ C &= [N_0 \ N_1 \ \cdots \ N_{\alpha-1} \ N_\alpha] \\ D &= 0 \end{aligned} \tag{5.11}$$

□

The realization (5.11) will be called the *block controllable realization* for reasons given later.

proof First we compute

$$\Delta(s,r) = \begin{bmatrix} D_0 & D_1 & \cdots & D_{\alpha-1} & D_\alpha \\ -I_p r & I_p s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p r & -I_p s \end{bmatrix}^{-1} \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which turns out to be

$$\Delta(s,r) = \begin{bmatrix} I_p s^\alpha D^{-1}(s,r) \\ I_p s^{\alpha-1} r D^{-1}(s,r) \\ \vdots \\ I_p r^\alpha D^{-1}(s,r) \end{bmatrix} = \begin{bmatrix} I_p s^\alpha \\ I_p s^{\alpha-1} r \\ \vdots \\ I_p r^\alpha \end{bmatrix} D^{-1}(s,r)$$

therefore

$$C\Delta(s,r) = N(s,r)D^{-1}(s,r) = G(s,r)$$

By the definition of realization, we conclude that (5.11) indeed realizes the given MFD. □

Before presenting some interesting properties of the realization (5.11), we shall first introduce two important concepts.

We start with the matrix

$$[Es - A \quad B] \tag{5.12}$$

By *str. eq.* operations, (5.12) can be brought to

$$\begin{bmatrix} E_\epsilon s - A_\epsilon & B_\epsilon \\ A_\pi & B_\pi \end{bmatrix} \tag{5.13}$$

with E_ϵ of full row rank. Similarly, the matrix

$$\begin{bmatrix} Es - A \\ -C \end{bmatrix} \tag{5.14}$$

can be brought, by *str. eq.* operations, to

$$\begin{bmatrix} E_\eta s - A_\eta & A_\xi \\ -C_\eta & -C_\xi \end{bmatrix} \tag{5.15}$$

with E_η of full column rank. We now state the following definition.

Definition 5.1 *Singular system (5.6) is said to be finitely controllable if*

$$rank[Es - A \quad B] = n \tag{5.16}$$

for all finite s; and controllable at infinity if

$$rank \begin{bmatrix} E_\epsilon & 0 \\ A_\pi & B_\pi \end{bmatrix} = n \tag{5.17}$$

where the block matrix in (5.17) is built with the corresponding block elements in (5.13). It is said to be controllable if it is both finitely controllable and controllable at infinity.

(5.6) is said to be finitely reconstructible if

$$rank \begin{bmatrix} Es - A \\ -C \end{bmatrix} = n \tag{5.18}$$

for all finite s; and reconstructible at infinity if

$$rank \begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} = n \tag{5.19}$$

where the block matrix in (5.19) is built with the corresponding block elements in (5.15). It is said to be reconstructible if it is both finitely reconstructible and reconstructible at infinity. □

The dynamical implications of the controllability and reconstructibility have been explained by Verghese et al. (1981). The present thesis will only be concentrating on the algebraic aspects. We now proceed to discuss the properties of the realization (5.11).

Theorem 5.2 *The realization (5.11) has the properties below:*

1. *The order of the realization is equal to the determinantal degree of the denominator of the transfer matrix; i.e., $rank E = deg\, det D(s,r)$*
2. *It is controllable;*
3. *$det D(s,r) = det\, \Theta(s,r)$ where*

$$\Theta(s,r) = \begin{bmatrix} D_0 & D_1 & \cdots & D_{\alpha-1} & D_\alpha \\ -I_p r & I_p s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -I_p r & I_p s \end{bmatrix} \tag{5.20}$$

proof The proofs for (1) and (2) are trivial and thus omitted. To prove (3), denote the ith block column of the matrix $\Theta(s,r)$ by F_i $(i=1,\ldots,\alpha+1)$. Apply a column transformation $K(r)$ such that

$$\Theta(s,r)K = [\ \tilde{F}_1 \quad \tilde{F}_2 \quad \cdots \quad \tilde{F}_{\alpha+1}\ ] \tag{5.21}$$

where $\tilde{F}_i = F_i r^{i-1} + \tilde{F}_{i-1} s$ $(i=2,\ldots,\alpha+1)$ and $\tilde{F}_1 = F_1$. It is easy to see that $detK = det(I_p r)\cdots det(I_p r^\alpha)$. From the structure of $\Theta(s,r)\cdot K$, it follows that $det(\Theta(s,r)\cdot K) = detD(s,r)\cdot det(I_p)\cdots det(I_p r^\alpha)$. Therefore $det\Theta(s,r) = detD(s,r)$. □

An algebraically dual procedure can apparently be pursued for the general *left* MFD. We shall however not go into any detail for conciseness of the thesis.

5.3 Realization for General Column-based MFD

It is often the case that the numerator matrix and the denominator matrix of a given right MFD are not matrix polynomials, but column-based polynomial matrices instead. Further even if they are matrix polynomials, there might exist a common right divisor between them. We have shown in Chapter 2 that after the cancellation, the general matrix polynomial right MFD reduces to a general column-based right MFD. Therefore it is of great interest to consider the realization for general column-based right MFD's.

5.3.1 Construction of the Realization

Let a general column-based right MFD be given and k_i $(i=1,\cdots,p)$ be the ith column degree of $N(s)$ $(m\times p)$ and $D(s)$ $(p\times p)$.

As the first step of the construction, define the following matrix

$$\Gamma^T(s,r) = \{[\ s^{k_i} \quad s^{k_i-1}r \quad \cdots \quad sr^{k_i-1} \quad r^{k_i}\ ],\quad i=1,\ldots,p\} \tag{5.22}$$

and write $D(s,r)$ and $N(s,r)$ in the form

$$\begin{aligned} D(s,r) &= D_c\Gamma(s,r)\\ N(s,r) &= N_c\Gamma(s,r) \end{aligned} \tag{5.23}$$

where D_c and N_c are appropriate constant matrices.

Secondly, construct the *core realization* of the form

$$E_c s - A_c = block\ diag\{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0\\ -1 & s & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & -1 & s \end{bmatrix},\ \begin{matrix}(k_i+1)\times(k_i+1)\\ i=1,\cdots,p\end{matrix}\}$$

$$B_c^T = block\ diag\{[1\ 0\ \cdots\ 0],\ 1\times(k_i+1),\ i=1,\ldots,p\}$$

$$C_c = I_n,\quad n=\textstyle\sum_{i=1}^p k_i + p \tag{5.24}$$

Finally, the desired realization follows form (5.23) and (5.24)

$$E = E_c,\quad A = A_c - BD_c,\quad B = B_c,\quad C = N_c\cdot C_c = N_c \tag{5.25}$$

(9)

5.3.2 Proof and Properties of the Realization

In this subsection we first prove that (5.25) indeed realizes the given right MFD. Then some interesting properties of the realization are shown.

Theorem 5.3 *The singular system in (5.25) realizes the given general column-based MFD.*

proof It is easily checked that

$$(Es - A_c r + BD_c)\Gamma(s,r) = BD_c\Gamma(s,r) = BD(s,r)$$

or equivalently

$$(Es - A_c r + BD_c)^{-1}B = \Gamma(s,r)D^{-1}(s,r) \tag{5.26}$$

(The non-singularity of $Es - A_c r + BD_c$ will be proved later). By premultiplying (5.26) with C $(=N_c)$ and by observing that $N(s,r) = N_c\Gamma(s,r)$, the theorem is proved. □

Theorem 5.4 *The realization (5.25) has the properties below*

1. $det(Es - A_c r + BD_c) = detD(s,r)$;
2. *it is controllable.*

□

These two properties can readily be checked by direct computation. We shall omit the detail. Observe that the second property guarantees the non-singularity of the matrix $Es - A_c r + BD_c$, which is needed in the proof of Theorem 5.3. Moreover, this property also implies that the order of the realization is equal to the determinantal degree of the denominator of the transfer matrix.

5.3.3 Illustrative Examples

In this subsection, we provide two examples to demonstrate the preceding realization algorithm.

Example 5.1 Let a transfer function be given by

$$G(s) = \frac{b(s)}{a(s)} = \frac{2s^3 + s + 1}{0 \cdot s^3 + s^2 + 2s + 2} \tag{5.27}$$

To realize $G(s)$, we first construct

$$D(s,r) = D_c\Gamma(s,r) = [0\ 1\ 2\ 2] \begin{bmatrix} s^3 \\ s^2 r \\ s r^2 \\ r^3 \end{bmatrix}$$

$$N(s,r) = N_c\Gamma(s,r) = [2\ 0\ 1\ 1] \begin{bmatrix} s^3 \\ s^2 r \\ s r^2 \\ r^3 \end{bmatrix}$$

The core realization is obviously

$$E_c s - A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the final realization is

$$Es - A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & s \end{bmatrix}$$

$$B = B_c$$

and $C = [2\ 0\ 1\ 1]$. The reader can easily check that the transfer function of the above realization is indeed $b(s,r)/a(s,r)$. □

Example 5.2 The transfer matrix is given by

$$G(s) = \begin{bmatrix} s^2 & s+1 \\ 0 \cdot s^2 + s & s \end{bmatrix} \begin{bmatrix} s^2 + s + 1 & 0 \cdot s + 1 \\ 0 \cdot s^2 + s & 0 \cdot s + 1 \end{bmatrix}^{-1}$$

Hence

$$D(s,r) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ sr & 0 \\ r^2 & 0 \\ 0 & s \\ 0 & r \end{bmatrix}$$

$$N(s,r) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ sr & 0 \\ r^2 & 0 \\ 0 & s \\ 0 & r \end{bmatrix}$$

The core realization is constructed as

$$E_c s - A_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, the final realization turns out to be

$$Es - A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B = B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

□

We have only been concerned with general column-based right MFD's. There should be no additional difficulties in adapting the algorithm to realize general row-based left MFD's.

5.4 Relationships among Minimality in Order, Complete Irreducibility, Joint Controllability and Reconstructibility

In a way exactly analogous to the equivalence circle established in Section 3.4 a similar equivalence circle will be proven in this section for joint controllability and reconstructibility, minimality in order and complete irreducibility. This new equivalence circle thereby connects the concept of complete irreducibility defined in Chapter 2 to the well-known McMillan degree.

5.4.1 Minimality in Order versus Joint Controllability and Reconstructibility

The relation between minimality in order and joint controllability and reconstructibility has been established by Verghese et al. (1981). In this subsection, we shall sketch the main idea for completeness of our discussion.

Let the system matrix of a singular system be given by

$$\mathcal{P}(s) = \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{5.28}$$

It is shown in Verghese et al. (1981) that a strong equivalence transformation can bring (5.28) to the following fourfold decomposition

$$\begin{bmatrix} E_{\bar{u}c}s - A_{\bar{u}c} & X & X & X & B_{\bar{u}c} \\ 0 & E_{\bar{u}\bar{c}}s - A_{\bar{u}\bar{c}} & 0 & X & 0 \\ 0 & 0 & E_{uc}s - A_{uc} & 0 & B_{uc} \\ 0 & 0 & 0 & E_{u\bar{c}}s - A_{u\bar{c}} & 0 \\ 0 & 0 & -C_{uc} & -C_{u\bar{c}} & D \end{bmatrix} \tag{5.29}$$

where the X denote some constant matrices; and the subscripts $u, \bar{u}, c, \bar{c}$ denote reconstructible, unreconstructible, controllable and uncontrollable, respectively. Consequently, the order f of the system is given by

$$f = rank E_{\bar{u}c} + rank E_{\bar{u}\bar{c}} + rank E_{uc} + rank E_{u\bar{c}} \tag{5.30}$$

It is therefore straightforward to see from (5.30) that f attains its minimal value if and only if the system is jointly controllable and reconstructible.

5.4.2 Complete Irreducibility versus Joint Controllability and Reconstructibility

We shall be making use of the realization (5.25) as a primary vehicle to derive the relationship. First we show that for this particular realization the desired relationship holds.

Theorem 5.5 *The realization (5.25) is reconstructible if and only if the general MFD $N(s)D^{-1}(s)$ is completely irreducible.*

proof As usual, only a proof for the equivalence between the reconstructibility at infinity of the realization and the irreducibility at infinity of a general MFD is given. The finite part is well-known and thus omitted.

From Definition 5.1, the realization (5.25) is reconstructible at infinity if and only if the matrix

$$\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} \tag{5.31}$$

has full column rank, where the block elements in (5.31) are defined in (5.19). From (5.25), it is easily seen that (5.31) can be transformed, by elementary transformations, to the following matrix

$$\begin{bmatrix} D_{hc} & 0 \\ 0 & I \\ N_{hc} & 0 \end{bmatrix}$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. The above matrix is of full column rank if and only if $D(s)$ and $N(s)$ are irreducible at infinity. □

With Theorem 5.5 in hand, the rest of the development follows quite straightforwardly.

Theorem 5.6 *If a general column-based MFD has one controllable and reconstructible fth-order realization, then all its fth-order realization must also be controllable and reconstructible.*

proof Otherwise by using the fourfold decomposition (5.29), this would imply that f is not the minimal order, which clearly contradicts the conclusion in the preceding subsection. □

We are now in the position to state and prove the main theorem of this subsection.

Theorem 5.7 *A general column-based right MFD is completely irreducible if and only if all of its nth-order realizations are controllable and reconstuctible; where n is the determinantal degree of the denominator $D(s)$, i.e., $n = deg\, det D(s, r)$.*

proof In view of Theorem 5.6, it suffices to show that the theorem holds for a special nth order realization. But Theorem 5.5 indicates that the realization (5.25) can serve this purpose. □

It has been shown in Verghese et al. 1981 that the order of a singular system will equal the McMillan degree of the underlying transfer matrix if and only if the singular system is controllable and reconstructible. Combining this fact with Theorem 5.7 we obtain the following important result.

Theorem 5.8 *A general column-based right MFD is completely irreducible if and only if the determinantal degree of its denominator is equal to the McMillan degree of the MFD.* □

This theorem is very significant since it provides an alternative, yet the simplest interpretation for the well-known McMillan degree of rational matrices, viz., the McMillan degree only describes the completely irreducible part of rational matrices. It is now clear that when a rational matrix is factorized in, say general MFD, some redundancy (finite and infinite) may be introduced. We can then perform the cancellation. As a consequence, the determinantal degree of the denominator of the MFD will decrease. As soon as the MFD becomes completely irreducible the determinantal degree attains the minimal which is precisely the McMillan degree of the MFD.

With the above equivalence circle and the one established in Chapter 3, we can also conclude that reachability and observability, or controllability and recostructibility are the same in nature because these two pairs of concepts all describe the complete irreducibility of the underlying transfer matrix. Their seeming difference is caused by the fact that the objects to which they apply are different. In this respect, a unification of these two pairs of concepts is achieved purely algebraically. The reader should draw a comparison between the unifying approach phrased in dynamical terms (for example Cobb 1984) and ours. It turns out that our approach seems more natural and provides a better understanding of these concepts.

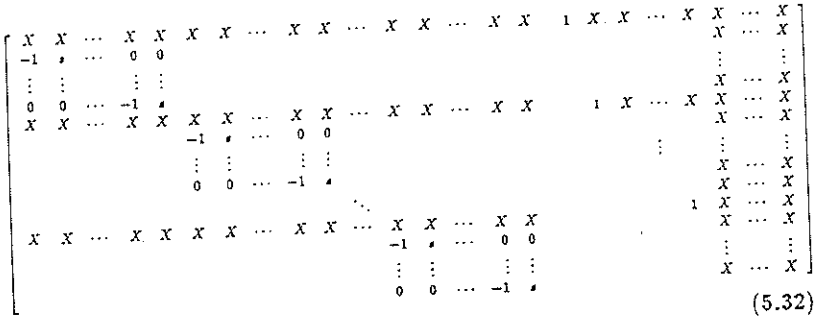
5.5 Canonical Forms for Singular Systems under Strong Equivalence Transformation

In the present framework, it is also desirable to have canonical forms for design and analysis purposes. This section will concentrate on singular system itself and derive some canonical forms by using the *str. eq.* transformation.

5.5.1 Singular Controller Canonical Form

We start the development with the following definition

Definition 5.2 *A singular system is said in singular controller canonical form if its matrix $[Es - A \;\; B]$ assumes the form shown in (5.32). Where the X denote possible nonzero values and entries not shown are zeros.* □



(5.32)

It is noted that the structures of C and D are of no concern in Definition 5.2. As can be observed, the obvious difference between the generalized controller canonical form and (5.32) is that the coefficient rows in (5.32) are all constants, this is, however, not the case for the generalized controller canonical form. The theorem below states the main result of this section.

Theorem 5.9 *A singular system is strongly equivalent to the singular controller canonical form (5.32) if and only if it is controllable.* □

The necessity of the theorem is easy to establish since the singular controller canonical form is controllable in nature (this can be shown by using Definition 5.1) and since *str. eq.* transformations can never alter controllability (Verghese et al. 1981). Here we shall concentrate on the sufficiency proof of the theorem.

We first recall, from the theory of matrix pencils, that for every regular pencil $Es - A$, there exist two non-singular matrices M and N such that (Gantmacher 1960)

$$M(Es - A)N = \begin{bmatrix} Is - J_1 & 0 \\ 0 & I - J_2 s \end{bmatrix} \tag{5.33}$$

where J_1, J_2 are nilpotent and in Jordan form, i.e., both are block diagonal matrices whose diagonal blocks have the form

$$J_d = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \tag{5.34}$$

Further, the I in (5.33) are the unit matrix of compatible sizes.

Let $Es - A$ be the pencil of a singular system. Then the system is said to have no non-dynamic variables if J_2 contains no 1×1 Jordan diagonal blocks (Verghese et al. 1981).

The following lemma is essential to the subsequent derivation.

Lemma 5.1 *If a singular system (5.6) has no non-dynamic variables, then* $rank[E \;\; B] = n$ *if and only if*

$$rank \begin{bmatrix} E_\epsilon & 0 \\ A_\pi & B_\pi \end{bmatrix} = n$$

where n is the dimension of the system; the matrices E_ϵ, A_π, B_π are defined in (5.13).

proof Using the non-singular matrices M, N in (5.33) to form a *str. eq.* transformation, the matrix $[Es - A \;\; B]$ is brought to the form

$$\begin{bmatrix} Is - J_1 & 0 & \tilde{B} \\ 0 & I - J_2 s & \tilde{B} \end{bmatrix} = \begin{bmatrix} Is - J_1 & 0 & & \tilde{B} \\ & N_1 & 0 & \bar{B}_1 \\ 0 & & \ddots & \vdots \\ & & N_\lambda & \bar{B}_\lambda \end{bmatrix} \tag{5.35}$$

where

$$[-N_i \;\; \bar{B}_i] = \begin{bmatrix} -1 & s & \cdots & 0 & 0 & \bar{b}_{i1} \\ 0 & -1 & \cdots & 0 & 0 & \bar{b}_{i2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s & \bar{b}_{i\mu_i - 1} \\ 0 & 0 & \cdots & 0 & -1 & \bar{b}_{i\mu_i} \end{bmatrix} \quad (i = 1, \ldots, \lambda) \tag{5.36}$$

In (5.36), the $\bar{b}_{ij}$ $(j = 1, \ldots, \mu_i)$ are row vectors and since J_2 contains no 1×1 Jordan blocks, we must have $\mu_i > 1$ $(i = 1, \ldots, \lambda)$.

Therefore, it can be easily seen that

$$rank[E \;\; B] = rank \begin{bmatrix} I & 0 & \tilde{B} \\ 0 & -J_2 & \bar{B} \end{bmatrix} = n$$

if and only if the matrix R is of full row rank; where

$$R = \begin{bmatrix} \bar{b}_{1\mu_1} \\ \vdots \\ \bar{b}_{\lambda\mu_\lambda} \end{bmatrix}$$

which is composed of the last rows of matrices $\bar{B}_i$ $(i = 1, \ldots, \lambda)$.

On the other hand, the block matrix in (5.17) can easily be constructed from (5.35), which assumes the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & R \end{bmatrix} \tag{5.37}$$

where K is a block diagonal matrix with the diagonal blocks of the form

$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \end{bmatrix}$$

Hence we conclude that (5.37) is of full row rank if and only if R is so. This in turn means that our assertion is true. □

We are now ready to prove the sufficiency of Theorem 5.9.

proof of Theorem 5.9

Sufficiency Let us assume that the given singular system has no non-dynamic variables. If this is not the case, it can always be made so by using *str. eq.* transformations. Then we know by Lemma 5.1 and Theorem 4.3 that if the system is controllable, it is transformable by a special *str. eq.* operations to the structure of (4.15) (we would have said *r.s.e.* and *generalized controller canonical form*. However, since we are in a different framework, these terms can no longer apply. Any misuse is likely to cause confusion).

Now suppose that the system has already been put into (4.15) we show that further *str.eq.* transformations can reduce the system to the singular controller canonical form (5.32).

To be specific, we start from the ith block row ($i = 1, \ldots, \kappa$) of the system which is assumed to be in (4.15), namely (in matrices which follow, the X's denote possible nonzero values, the entries not shown are zeros)

$$\begin{bmatrix} X & \cdots & X & a_i s + X & X & \cdots & X & X & \cdots & X & 1 & \cdots & X \\ & & & -1 & s & \cdots & 0 & & & & & & \\ & & & \vdots & \vdots & & \vdots & & & & & & \\ & & & 0 & 0 & \cdots & s & & & & & & \end{bmatrix} \tag{5.38}$$

If $a_i = 0$, we will do nothing but simply move to the next block row. Otherwise, by a trivial augmentation, (5.38) is brought to

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ X & \cdots & X & 0 & s+X & \cdots & X & X & \cdots & X & 1 & \cdots & X \\ & & & 0 & -1 & \cdots & 0 & & & & & & \\ & & & \vdots & \vdots & & \vdots & & & & & & \\ & & & 0 & 0 & \cdots & s & & & & & & \end{bmatrix} \tag{5.39}$$

Adding, in (5.39), the first row multiplied by -1 to the second one, we have

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ X & \cdots & X & -1 & s+X & \cdots & X & X & \cdots & X & 1 & \cdots & X \\ & & & 0 & -1 & \cdots & 0 & 0 & & & & & \\ & & & \vdots & \vdots & & \vdots & \vdots & & & & & \\ & & & 0 & 0 & \cdots & -1 & s & & & & & \end{bmatrix} \tag{5.40}$$

Eliminating all the X's in the second row using -1 in the same row, we arrive at

$$\begin{bmatrix} X & \cdots & X & 1 & X & \cdots & X & X & \cdots & X & 1 & \cdots & X \\ 0 & \cdots & 0 & -1 & s & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & \vdots & \vdots & & \vdots & & & & & & \\ & & & 0 & 0 & \cdots & s & & & & & & \end{bmatrix} \tag{5.41}$$

(5.41) is evidently in the desired structure. When i goes from 1 to κ, the system will be in the form (5.32). The theorem is proved. □

The reader may also realize that in an algebraically dual way, the *singular observer canonical form* can be derived for reconstructible singular systems. The details are omitted.

5.6 Relation between the Minimal Indices and the Canonical Form

In this section, we shall establish a link between the canonical form developed in the preceding section and the minimal indices of some singular pencils. The discussion is restricted to the singular controller canonical form and the minimal indices for the columns.

For convenience of the discussion, the singular controller canonical form (5.32) is written in a more compact way below

$$Es - A = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1\kappa} \\ R_{21} & R_{22} & \cdots & R_{2\kappa} \\ \vdots & \vdots & & \vdots \\ R_{\kappa 1} & R_{\kappa 2} & \cdots & R_{\kappa\kappa} \end{bmatrix} \tag{5.42}$$

where κ is defined in (5.32) as the number of the diagonal blocks; and

$$R_{ii} = E_i s - A_i = \begin{bmatrix} X & X & \cdots & X & X \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

$$(i = 1, \ldots, \kappa)$$

the R_{ij} $(i \neq j)$ matrices have all rows zero except possibly the first one;

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_\kappa \end{bmatrix} \tag{5.43}$$

in which

$$B_i = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \begin{matrix} b_i = 1 \\ b_1 = b_2 = \cdots = b_{i-1} = 0 \end{matrix}$$

Again we assume that B is of full column rank.

(5.42) allows to define a set of non-negative numbers $\{\gamma_i\}$ as

$$\gamma_i = rank E_i \quad (i = 1, \ldots, \kappa) \tag{5.44}$$

The minimal indices $\{\alpha_i\}$ for the columns of the pencil $[Es - Ar \;\; Br]$ have been defined in Sec.4.1.3. The following theorem relates the set $\{\gamma_i\}$ to the set of minimal indices.

Theorem 5.10 *If a controllable singular system has no non-dynamic variables, then the set $\{\gamma_i\}$ is a rearrangement of the set $\{\alpha_i\}$ which is the set of minimal indices for the columns of $[Es - Ar \;\; Br]$.*

proof The proof is similar to that for Theorem 4.6. Here we only provide a sketch.

Start with the pencil $[Es - Ar \;\; Br]$. Since *str. eq.* transformation can not change the minimal indices for the columns we can assume that $[Es - A \;\; B]$ is in singular controller canonical form. Further elementary transformations can bring $[Es - Ar \;\; Br]$ to

$$\begin{bmatrix} L_1(s,r) & 0 & \cdots & 0 \\ 0 & L_2(s,r) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & L_\kappa(s,r) \end{bmatrix} \tag{5.45}$$

where for $i = 1, \ldots, \kappa$

$$L_i(s,r) = \begin{bmatrix} -r & 0 & 0 & \cdots & 0 & 0 \\ 0 & -r & s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -r & s \end{bmatrix}$$

so that the equation

$$L_i(s,r)v_i(s) = 0 \tag{5.46}$$

admits the solution

$$v_i(s,r) = \begin{bmatrix} 0 \\ s^{\gamma_i} d(s,r) \\ s^{\gamma_i - 1} r d(s,r) \\ \vdots \\ s r^{\gamma_i - 1} d(s,r) \\ r^{\gamma_i} d(s,r) \end{bmatrix} \tag{5.47}$$

in which $d(s,r)$ is an arbitrary homogeneous polynomial. Obviously, γ_i is a minimal index for the columns.

Since there only exist κ linearly independent solutions for the equation

$$[Es - Ar \;\; Br]v(s,r) = 0$$

the number of minimal indices is precisely κ. □

Notice that the condition in Theorem 5.10 is only sufficient. In fact, if $[Es - A \;\; B]$ does have non-dynamic variables, we shall have no way to deduce the result.

5.7 Conclusions and Remarks

In this chapter, a theory for dealing with general MFD is presented. It has been seen clearly that Verghese's framework in handling singular systems gears nicely with our theory of general rational matrices. The development parallels that for pseudo-proper MFD, and the treatments extend to the topics like the equivalence circle, realization and canonical forms, etc.

Our theory presented in the past chapters unifies the two major approaches in singular systems in the sense that the two approaches are tackling the exactly the same problem, i.e., the redundancy (finite and infinite) for the underlying transfer matrices in MFD, but the objects they address are different: one is pseudo-proper MFD, another general MFD. Moreover, our theory also unifies the treatment of the infinity with that for the finite points, thus puts the two on an equal footing.

Chapter 6

Other Aspects of General Singular Systems

6.1 Introduction

The development in the last chapter has largely focused on the fundamental problems of realizations and a few basic concepts such as irreducibility, controllability, etc. The present chapter extends the established theory to include two specific topics, i.e., the inversion and decoupling of general singular systems. These two problems which will be treated in detail in this chapter are mutually independent and each displays some special features of general singular systems. Our main objective is to show the usefulness of the techniques developed in the last chapter.

The results in this chapter have been drawn from two of our papers (Tan and Vandewalle 1987e, 1987f), and are organized as follows. In Section 6.2 the problem of inverting square general singular systems is considered. We shall present a method to construct an inverse directly from the system itself. Some of the important properties of such an inverse are also discussed. Although the attention will be concentrated only on square systems, we believe that the result has some general implications and render itself very well to further investigation. Section 6.3 discusses the so-called *complete decoupling problem*. We shall show that singular system naturally arises when the static and differential state feedback is applied to a state-space system. A very interesting result is that for any controllable state-space system, there always exists a static and differential state feedback such that the closed-loop system is completely decoupled. Section 6.4. contains some remarks and conclusions. Since the present chapter will only be con-

cerned with general singular systems, for simplicity of the notation, we shall frequently drop the adjective "general" when no confusion can arise.

6.2 Inversion of Singular Systems

The problem of inverting the dynamical systems has been under investigation for some time (Sain and Massey 1969, Massey and Sain 1968, Silverman 1969, Emre et al. 1974, Rosenbrock et al. 1977). Although there have been many algorithms available for inverting time-invariant state-space systems, (for example, Silverman 1969, Moylan 1977) their common deficiency is that the resulting inverse does not remain in state-space representation. This is simply due to the fact that the state-space representation can only accommodate proper dynamical systems, while inverses of proper dynamical systems are not necessarily proper.

In this section, we shall present a new inversion method for singular systems which include the state-space systems as a special case. Instead of generalizing the existing inversion algorithms to singular systems (one of these generalizations is given in Lewis 1983), we shall explore the flexibility of the representation of singular systems. It will be seen that the construction leads to a simple inverse which is itself a singular system.

6.2.1 Construction of an Inverse System

Let a singular system be given by

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{6.1}$$

where $x \in R^n$; $u, y \in R^m$; E, A are square matrices of order n, and B, C, D are matrices of appropriate sizes. (6.1) will be called a square system and such a square system is said to be *invertible* if the determinant of its transfer matrix

$$G(s) = C(Es - A)^{-1}B + D \tag{6.2}$$

does not vanish identically. Since in the present chapter we are not concerned with the redundancy (finite and infinite), the homogeneous form (e.g., (5.8)) of the transfer matrix is not used here. The reader should notice, however, that the entries of (6.2) are general rational functions.

The following construction based upon (6.1) will result in a singular system. It will be shown later that if (6.1) is invertible, then such a construction will yield an inverse of it.

As the first step of the construction, the second equation in (6.1) is alternatively expressed as

$$y = [C \quad D - I]\begin{bmatrix} x \\ u \end{bmatrix} + u \tag{6.3}$$

This is equivalent to a change of the descriptor vector. The effect of such a change on the first equation in (6.1) is obviously

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} B \\ I_m \end{bmatrix} u \tag{6.4}$$

where I_m is the unit matrix of order m, the 0's are zero matrices of compatible dimensions.

Define $z^T = [x^T \quad u^T]^T$, rewrite (6.3) in the following form

$$u = [-C \quad I - D]z + y \tag{6.5}$$

substituting (6.5) into (6.4), simple manipulations give

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\dot{z} = \begin{bmatrix} A - BC & B - BD \\ -C & -D \end{bmatrix} z + \begin{bmatrix} B \\ I_m \end{bmatrix} y \tag{6.6}$$

We now show that the invertibility of system (6.1) is in fact equivalent to the solvability of the system (6.6), (6.5). First we state a lemma.

Lemma 6.1 *A square singular system (6.1) is invertible if and only if the determinant of its system matrix is not identically zero.* □

Remark This lemma is a simple generalization of a result given by Wang and Davison (1973).

proof Since $Es - A$ is regular, the determinant of $\mathcal{P}(s)$ can be readily seen as

$$det\mathcal{P}(s) = det(Es - A) \cdot det[C(Es - A)^{-1}B + D]$$

Therefore, $det\mathcal{P}(s)$ is zero if and only if (6.1) is not invertible by definition. □

The following result follows directly from Lemma 6.1

Theorem 6.1 *System (6.1) is invertible if and only if the singular system (6.6), (6.5) is solvable, or equivalently, the matrix*

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} s - \begin{bmatrix} A - BC & B - BD \\ -C & -D \end{bmatrix} \tag{6.7}$$

is regular.

proof Adding to the first block row in (6.7) the second one premultiplied by $-B$, the resulting matrix is precisely the system matrix of (6.1). Note that such a transformation will not affect the determinant of (6.7). The theorem is proved by recalling Lemma 6.1. □

Finally we verify that if (6.1) is invertible, (6.6) and (6.5) will indeed be an inverse of it. The verification is carried out straightforwardly.

$$\begin{aligned}
\tilde{G}(s) &= [-C \quad I-D]\begin{bmatrix} Es-A+BC & BD-B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} B \\ I \end{bmatrix} + I \\
&= [-C \quad I-D]\begin{bmatrix} Es-A & -B \\ C & D \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} + I \\
&= [-C \quad I-D]\begin{bmatrix} Es-A & -B \\ 0 & G(s) \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix} + I \\
&= -C(Es-A)^{-1}BG^{-1}(s) + (I-D)G^{-1}(s) + I \\
&= G^{-1}(s)
\end{aligned}$$

This is what we have to show.

6.2.2 Properties of the Inverse System

Apart from the great simplicity, other features of the inverse system (6.6) and (6.5) are that no assumption of controllability and reconstructibility is required; also the order of the inverse is the same as the original system (6.1). Moreover, when (6.1) is controllable and reconstructible, the inverse will also be controllable and reconstructible. Thus the inverse has the minimal order whenever the original system does. This property is proved below.

Theorem 6.2 *The inverse system (6.6), (6.5) is controllable and reconstructible if and only if (6.1) is controllable and reconstructible.*

proof We only give a proof for the reconstructible part, the controllable part follows in a similar but simpler way.

Start with the following observable matrix

$$\begin{bmatrix} Es-A+BC & BD-B \\ C & D \\ -C & I-D \end{bmatrix} \tag{6.8}$$

and adding to the first block row the second one premultiplied by $-B$ (note that this operation is a *str. eq.* operation), (6.8) is brought to

$$\begin{bmatrix} Es-A & -B \\ C & D \\ -C & I-D \end{bmatrix} \tag{6.9}$$

Column-compressing E by a *str. eq.* operation, (6.9) is put into

$$\begin{bmatrix} E_\eta s-A_\eta & A_\xi & -B \\ C_\eta & C_\xi & D \\ -C_\eta & -C_\xi & I-D \end{bmatrix} \tag{6.10}$$

where $E_\eta, A_\eta, A_\xi, C_\eta, C_\xi$ are defined in (5.15). E_η has full column rank.

For (6.6), (6.5) to be reconstructible, (6.8) (or equivalently (6.9)) and the matrix

$$\begin{bmatrix} E_\eta & A_\xi & -B \\ 0 & C_\xi & D \\ 0 & -C_\xi & I-D \end{bmatrix}$$

have to be of full column rank. They will be so if and only if the original system is reconstructible. □

Since it is the *order* rather than the dimension of a general singular system that is of significance, for some cases the dimension of the inverse (6.6), (6.5) can be reduced by applying *str. eq.* operations. In the next section we shall show by an example how the reduction can be carried out.

6.2.3 Illustrative Examples

In this subsection we provide two examples to demonstrate our inversion method.

Example 6.1 Let a system be given by

$$\dot{x} = -x+u \tag{6.11}$$
$$y = x \tag{6.12}$$

which is apparently a single-variable system. Its transfer function is $G(s) = 1/(s+1)$. To construct an inverse, modify (6.11) in the following way

$$y = [1 \quad -1]z + u \tag{6.13}$$

where $z^T = [x \quad u]^T$, and the corresponding modification of (6.11) is easily seen to be

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{z} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}z + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u \tag{6.14}$$

the substitution of (6.13) into (6.14) yields

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y \qquad (6.15)$$

$$u = [-1 \quad 1] z + y$$

The reader may wish to show that the transfer function of (6.15) is $s+1$.

□

Example 6.2 Now we consider the following 2×2 square system

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u \qquad (6.16)$$

$$y = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} u \qquad (6.17)$$

The transfer matrix of this system is

$$G(s) = \begin{bmatrix} s+1 & s \\ s & s-1 \end{bmatrix}$$

which is clearly invertible.

Again modifying (6.16) we have

$$y = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} z + u \qquad (6.18)$$

(6.16) is also modified to be

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \qquad (6.19)$$

substitute (6.18) into (6.19)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} y$$

$$u = \begin{bmatrix} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} + y \qquad (6.20)$$

The transfer matrix of (6.20) is

$$\bar{G}(s) = \begin{bmatrix} -s+1 & s \\ s & -s-1 \end{bmatrix}$$

Next, (6.20) can be transformed by *str. eq.* operation to the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{q} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} q + \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} y$$

$$u = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix} q + y \qquad (6.21)$$

A simple deflation will bring (6.21) to

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{p} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} p + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} y \qquad (6.22)$$

$$u = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} p + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} y$$

The system (6.22) obviously has the same transfer matrix as (6.20). □

To end the section, we note that the attention has been focused only on square systems. In fact, for non-square case we can pursue a similar line of thought to derive some results for the left and the right inversions of the system. This however will not be done in the present thesis.

6.3 A Complete Decoupling of State-space Systems by Static and Differential Feedback

The problem of decoupling a linear multivariable state-space system using state feedback has been a subject of extensive investigation for quite a long time (see Hautus and Heymann 1980 for a concise survey). Although the emphasis in the literature has been on the use of linear static state feedback alone to decouple systems, our primary objective in this section is to show that when the state feedback extends to include the differentiation of the state, some stronger results concerning the decoupling can be obtained.

The discussion in this section will again be confined to square systems whose transfer matrix are non-singular. The extension to the non-square case is excluded due to the degree of complication.

6.3.1 Linear Static and Differential State Feedback

Let a state-space system be given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6.23}$$

where $x \in R^n, u \in R^m, y \in R^m$; the matrices A, B, C have the appropriate dimensions. The transfer matrix of the system is obviously of the form

$$G(s) = C(Is - A)^{-1}B \tag{6.24}$$

which is a square rational matrix. It will be assumed throughout this section that $G(s)$ is non-singular. We note that this condition implies that the matrix C has full row rank, and the matrix B has full column rank.

We now apply to the system a control law of the form

$$u = K\dot{x} + Fx + Hv \tag{6.25}$$

where $v \in R^m$ is a new input vector; K, F, H are constant matrices of compatible dimensions; and H is non-singular. We shall call (6.25) a *static and differential state feedback*. The effect of such a feedback on (6.23) can be seen when (6.25) is substituted for u in (6.23), which results in the following singular system

$$\begin{aligned} (I - BK)\dot{x} &= (A + BF)x + BHv \\ y &= Cx \end{aligned} \tag{6.26}$$

It then follows directly that the transfer matrix from the new input v to y is given by

$$\tilde{G}(s) = C[(I - BK)s - (A + BF)]^{-1}BH \tag{6.27}$$

It should be noted that some choices of matrices K, F may cause the pencil $Q(s) = (I - BK)s - (A + BF)$ in (6.27) to be singular. Therefore, we have to further restrict the matrices K and F to ensure that $Q(s)$ is always non-singular, or equivalently, the singular system (6.26) is always solvable.

If system (6.23) is rewritten in the system matrix form (Rosenbrock 1970)

$$\mathcal{P}(s) = \begin{bmatrix} Is - A & B \\ -C & 0 \end{bmatrix} \tag{6.28}$$

the effect of the static and differential state feedback (6.25) on (6.23) can be considered as the following operation applied to $\mathcal{P}(s)$

$$\begin{aligned} \tilde{\mathcal{P}}(s) &= \begin{bmatrix} Is - A & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Ks - F & H \end{bmatrix} \\ &= \begin{bmatrix} (I - BK)s - (A + BF) & BH \\ -C & 0 \end{bmatrix} \end{aligned} \tag{6.29}$$

The corresponding system represented by $\tilde{\mathcal{P}}(s)$ is easily seen to be (6.27). Moreover, in view of (6.29), the effect of the feedback can further be treated as a kind of column elementary operation performed on the system matrix $\mathcal{P}(s)$, but restricted by (6.29). The exact nature of these operations is, of course, determined by the constant matrices K, F and H.

6.3.2 Complete Decoupling by Static and Differential State Feedback

To begin the development, we first define the notion of complete decoupling for state-space systems.

Definition 6.1 *A system is said to be completely decoupled if its transfer matrix is an identity matrix.* □

We now state the main result of this section.

Theorem 6.3 *For any square controllable state-space system (6.23) with non-singular transfer matrix, there exists a static and differential state feedback (6.25) such that the closed-loop system is completely decoupled.* □

First we need to prove the following lemma.

Lemma 6.2 *The transfer matrix of any controllable system (6.23) admits a right MFD factorization of the form*

$$G(s) = N(s)D^{-1}(s) \tag{6.30}$$

where $D(s)$ is some non-singular polynomial matrix and $N(s)$ is a polynomial matrix defined by

$$N(s) = CT(s)$$

where

$$T^T(s) = \text{block diag}\{[s^{k_i - 1} \quad \cdots \quad s \quad 1], \quad i = 1, \ldots, m\}$$

k_i $(i = 1, \ldots, m)$ is the ith controllability index (see, e.g., Kailath 1980 for the definition).

proof Apply a similarity transformation to bring (6.28) to its controller canonical form (see, e.g., Kailath 1980 for a standard derivation)

$$\mathcal{P}_c(s) = \begin{bmatrix} Is - A_{cc} & B_{cc} \\ -C & 0 \end{bmatrix} \tag{6.31}$$

in which C remains unchanged and

$$[Is - A_{cc} \quad B_{cc}] = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} & B_{c1} \\ R_{21} & R_{22} & \cdots & R_{2m} & B_{c2} \\ \vdots & \vdots & & \vdots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} & B_{cm} \end{bmatrix} \tag{6.32}$$

where the R_{ii} $(i = 1, \ldots, m)$ are companion matrices

$$R_{ii} = \begin{bmatrix} s + \rho_1 & \rho_2 & \cdots & \rho_{k_i - 1} & \rho_{k_i} \\ -1 & s & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & s \end{bmatrix}$$

and the R_{ij} $(i \neq j)$ have all rows zero except possibly the first one; each B_{ci} is a $k_i \times m$ matrix $(i = 1, \ldots, m)$ of the following form

$$B_{ci} = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{matrix} b_i = 1 \\ b_1 = b_2 = \cdots = b_{i-1} = 0 \end{matrix}$$

The structure of B_{cc} enables to find a non-singular matrix H such that

$$\tilde{B}^T = (B_{cc}H)^T = block\ diag\{[1 \ 0 \ \cdots \ 0], \ 1 \times k_i, i = 1, \ldots, m\} \tag{6.33}$$

Now construct a square polynomial matrix $\bar{D}(s)$ such that

$$\tilde{B}\bar{D}(s) = (Is - A_{cc})T(s) \tag{6.34}$$

It can then be easily shown, by applying elementary operations, that $det\bar{D}(s) = det(Is - A_{cc})$. We therefore conclude that $\bar{D}(s)$ is non-singular, so that (6.34) can be rearranged into (note that $\tilde{B} = B_{cc}H$)

$$(Is - A_{cc})^{-1}B_{cc} = T(s)\bar{D}^{-1}(s)H^{-1} \tag{6.35}$$

Premultiplying the matrix C on both sides of (6.35) and defining $D(s) = H\bar{D}(s)$ yield

$$C(Is - A_{cc})^{-1}B_{cc} = N(s)D^{-1}(s) \tag{6.36}$$

The left hand side of (6.36) is certainly the transfer matrix $G(s)$. Hence the lemma is proved. □

Notice that since $G(s)$ is non-singular, $N(s)$ in (6.30) must also be non-singular.

We now come to the proof of our main theorem.

proof of Theorem 6.3

We shall apply a sequence of elementary operations defined by the feedback (6.25) to bring (6.28) to such a form that its transfer matrix is a unit matrix.

To begin with, (6.28) is put into the controller canonical form (6.31) by similarity transformations. Since the given system is controllable, this can always be accomplished.

Then we make use of 1's in the matrix B_{cc} in (6.31) to change the top row of the ith block row of $Is - A_{cc}$, i.e., $[R_{i1} \ R_{i2} \ \cdots \ R_{im}]$ such that the top row of the ith block row equals the ith row of the matrix $C(i = 1, \ldots, m)$ (we remark that such changes should be done from the bottom to the top of the block rows of the matrix $Is - A_{cc}$). Equivalently, there exist matrices K and F such that

$$\begin{bmatrix} \tilde{E}s - \tilde{A} & B_{cc} \\ -C & 0 \end{bmatrix} = \begin{bmatrix} Is - A_{cc} & B_{cc} \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Ks - F & I \end{bmatrix} = \begin{bmatrix} M_1(s) \\ M_2(s) \\ \vdots \\ M_m(s) \end{bmatrix} \tag{6.37}$$

where $\tilde{E} = I - B_{cc}K$, $\tilde{A} = A_{cc} + B_{cc}F$ and

$$M_i = \begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{i\tau_i} & c_{i,\tau_i+1} & \cdots & c_{i,\tau_{i+1}-1} & c_{i,\tau_{i+1}} & \cdots & c_{in} \\ & & & -1 & s & \cdots & 0 & 0 & & \\ & & & \vdots & \vdots & & \vdots & \vdots & & \\ & & & 0 & 0 & \cdots & -1 & s & & \end{bmatrix}$$

in the above $k_0 = 1, \tau_i = \sum_{j=0}^{i} k_j$ and the c_{ij}'s are the elements of C.

We now show that the transfer matrix of (6.37) is indeed the unit matrix. To this end, first observe that the pencil $\tilde{E}s - \tilde{A}$ can easily be brought by elementary operations to the form

$$\begin{bmatrix} N(s) & 0 \\ 0 & I \end{bmatrix}$$

where $N(s)$ is defined in (6.30). Since it follows from Lemma 6.2 that $N(s)$ must be non-singular, we then know that $\tilde{E}s - \tilde{A}$ must be non-singular as well.

If the matrix H is so chosen that $\tilde{B} = B_{cc}H$ is given by (6.33), it is readily seen that the following equation holds

$$\tilde{B}N(s) = (\tilde{E}s - \tilde{A})T(s) \tag{6.38}$$

Rearranging (6.38) and substituting $B_{cc}H$ for $\tilde{B}$ yield

$$(\tilde{E}s - \tilde{A})^{-1}B_{cc}H = T(s)N^{-1}(s) \tag{6.39}$$

Premultiplying both sides of (6.39) by C, we arrive at

$$\tilde{G}(s) = C(\tilde{E}s - \tilde{A})^{-1}B_{cc}H = CT(s)N^{-1}(s) = N(s)N^{-1}(s) = I$$

This completes the proof. □

It is known that a static state feedback can always be implemented as a precompensator for the system (Hautus and Heymann 1983). We can check quite straightforwardly that this is also true for the static and differential state feedback. From this point of view, what has been done above can in fact be regarded as a construction of the inverse of the original system, this inverse turns out to be of the form

$$G^{-1}(s) = [I - (Ks + F)(Is - A)^{-1}B]^{-1}H \tag{6.40}$$

Therefore, the theorem also provides an alternative way of constructing the inverse of state-space systems.

To conclude the section we remark that Theorem 6.3 has only proven for controllable systems. Our conjecture is, however, that the result should hold without the controllability assumption. This more general statement still needs to be proved.

6.4 Conclusions and Remarks

In this section, we have considered two specific topics related to general singular systems. Our treatments have indicated that singular systems are very convenient models for a variety of applications. Also the theory developed in this last chapter serves to clarify the nature of these problems.

Chapter 7

Conclusions

The thesis has presented a relatively complete and novel algebraic theory for dealing with the infinite structure of non-causal linear multivariable systems. The merit of the theory (we believe) lies in that it provides an unambiguous way of characterizing the possible cancellations which may occur at infinity, when the system is in any polynomial matrix based factorizations (e.g., MFD, singular system representation, etc.)

Previous effort has largely been concentrated on the finite cancellations of systems in factorized form (Kalman 1960, Rosenbrock 1970 etc. are representitive), because the properness of systems is an essential assumption.

Rosenbrock (1974) has pioneered the study of the redundancy at infinity for systems which are not necessarily proper. The systems that he considered, however, are only pseudo-proper, as we have seen in the last chapters. This is also the reason why there exist a number of *difficulties* in his framework. These are really not the difficulties but rather the attempt to fit general MFD into the pseudo-proper framework. This is clearly impossible.

The recent works of Verghese et al. (1978, 1981, 1979) are of fundamental importance in the sense that they have built up an entirely new framework for singular systems, which goes beyond pseudo-properness. Their theory for rational matrices, however, fails to identify a complete set of modes of the systems in factorized forms. The reason is, apart from the elaborations given in the previous chapters, that they took the analytical approaches in dealing with the problem. These approaches essentially imply that only those modes which are not under cancellation could be displayed explicitly. Therefore, the modes of MFD's they identified is less than those based on our definition. It is believed that our treatment is complete in the sense that all hidden cancellations for a rational matrix in factorized

form are identified clearly and characterized by the complete irreducibility and the joint controllability and reconstructibility (or joint reachability and observability in case pseudo-properness is considered).

As important consequences of our theory, a unification between Rosenbrock's and Verghese's frameworks are achieved. It is now clear that both of these frameworks address the same aspect of the system (i.e., complete irreducibility in general), the nature of the systems under investigation is, however, quite different. One is concerned with pseudo-proper MFD, another general MFD. Since general MFD is the basis for defining the Smith-McMillan form and the McMillan degree, it is then easy to understand why Verghese's approach fits neatly with these well-known concepts. As a matter of fact, when adding all the modes which may be ignored due to the pseudo-properness constraint, we can readily establish a definitive relation between the dimension of singular systems on the one hand, and the McMillan degree along with the decoupling zeros (defined by Rosenbrock 1974) on the other.

Parts of the thesis have also been devoted to other aspects of singular systems. The derived canonical forms are among some of the interesting results obtained. There is also a clear distinction between the two canonical forms in the aforementioned two frameworks. In view of these canonical forms, we can make several important observations. First, the canonical form for a controllable (reachable) singular systems coincides (up to a coordinate change in the input space) with the structure which realizes a general (pseudo-proper) MFD. This (we think) is by no means a coincidence. Canonical forms, after all, are to make the underlying rational matrix transparent. Secondly, the derivation of the singular controller canonical form is based on that for the generalized controller canonical form. This, however, is no more than an algebraic trick, and does not imply any link between controllability and reachability. When a singular system is given, it must either fall into Rosenbrock's framework or Verghese's one but never both. Before taking any steps to analyze it, we have to choose (depending on the need) to enter one of the two frames. An important idea which should be kept in mind is that controllability and reachability can never be applied to the same system without changing the frame. We feel that the previous attempts (for example Cobb 1984) to unifying the two frameworks are somehow misleading, because the attention is not concentrated on the fundamental differences between the two, but rather on the the definitions of the concepts. It is felt in particular that by revealing the fundamental differences of the two frameworks, we have offered a more satisfying unification.

The developed canonical forms can be utilized to fulfill some design tasks, e.g., the observer design, pole placement, which also constitute a

part of the thesis. There are still some strange phenomena to be cleared up in these aspects. It has been shown that in Rosenbrock's frame, the observer design and the pole placement design are quite straightforward, and involve no additional complications, as compared with the same problems for state-space systems. This, however, is not the case in Verghese's frame. Since the *str. eq.* transformation involves a deflation and an augmentation procedure, the effect of the descriptor feedback to a system is completely different from that to its equivalent version after, for example, a trivial deflation (the reader can easily construct an example to verify this statement). The reason is simply because that the non-dynamic variables, although insignificant for a system itself, can have non-trivial effect whenever they are fedback to the system. This is of course a very disappointing observation, since it essentially implies that even though the feedback can cause some desirable behaviours for a system in one form, we have no obvious way to infer that there also exists a feedback for a strong equivalent version of the same system, which causes the same behaviour. Therefore, as far as Verghese's framework is concerned, the feedback is somehow awkward to be coped with.

Other contributions of the thesis include two techniques, namely an inversion method, and a decoupling technique. These are among the numerous examples which show the usefulness of singular systems. The point which we hope to have been clarified is that singular system representation is as powerful to describe a non-causal systems as state-space representation to describe causal ones. It is even more useful in the sense that when the problems like inversion are under consideration, the underlying non-causality should demand a more general representation than just the state-space one.

Finally, we would like to conclude the thesis by mentioning some open problems.

(1). As is discussed above, the major problem associated with singular system representing general MFD is that the feedback is not a natural operation to be applied. Since in the control context it is more significant to examine the effect of the feedback on the system, we think that there must still be efforts spent on finding a feedback oriented equivalence.

(2). Exactly like the relationship between the similarity transformation and the unimodular equivalence (Wolovich 1974, Kailath 1980), we feel that there also exists an analog relationship between strong equivalence and the operations defined to manipulate a general MFD, or in case of pseudo-properness a relationship between the restricted system equivalence and the operations defined for pseudo-proper MFD's. The details, however, remain

to be worked out.

(3). In this thesis, the procedure to develop the canonical forms is based on algorithmic arguments, therefore is not very satisfactory. The careful reader may have found that the coefficient rows of the forms are somewhat arbitrary. We actually expect that these canonical forms to be unique (like, for example, the normal controller canonical form which is determined by a structured basis, hence is unique). There is certainly a great need to re-develop these canonical forms on the basis of , for example, the generalized invariant polynomials.

(4). Extensions to a number of results in the thesis are believed to be possible. For instance, the inversion and decoupling techniques can be extended to include the non-square systems. Moreover, it is believed that singular system can provide a natural framework in dealing with the minimal factorization problem (Van Dooren 1979, 1984), and other related problems (those discussed in Autoulas 1986).

In general, we think that the theory for non-causal systems are far from complete (compared with the theory for causal systems). A great deal of effort still has to be made in resolving a number of important issues before such systems can be confidently analyzed and synthesized in practice.

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