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Novel theory for polynomial and rational matrices at infinity Part 2. Rational matrices

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The development reported in Part 1 of this series is continued by presenting a unifying theory for rational matrices at infinity. The cornerstone for the development is again the homogeneous form for polynomial and rational matrices. It is shown that when a rational matrix is factorized in matrix fraction description (MFD), redundancy at infinity may be introduced. This redundancy can be characterized precisely as the common divisor of the numerator and the denominator matrices at infinity. Moreover, when the numerator and the denominator matrices are completely coprime, the determinantal degree of the denominator is the McMillan degree of the rational matrix.

1. Introduction

In Part 1 (Tan and Vandewalle 1988), a new approach is presented for dealing with polynomial matrices at infinity. The main purpose of this paper is to apply the theory developed there to handle the infinite algebraic structure of rational matrices represented in matrix fraction description (MFD). It is emphasized that the rational matrices considered in this paper are of a general nature, and are not necessarily proper.

Once the headway has been made for polynomial matrices, it is relatively easy to proceed to rational matrices. In fact, any rational matrix can always be factored as a ratio of two (or more) polynomial matrices which fall into some specific classes. Similar to the finite case, this kind of factorization, or more generally, any polynomial matrix based factorizations may induce redundancy at infinity between the numerator(s) and the denominator(s). The concept of coprimeness at infinity therefore enters naturally into the context to describe the irreducibility at infinity between the numerator(s) and the denominator(s) of the factorized rational matrix. Furthermore, the concept of complete irreducibility for rational matrices can be defined on the same basis.

As we shall show, any MFD can alternatively be decomposed (or realized) into the so-called *descriptor system representation* (Rosenbrock 1974, Luenberger 1977, Verghese *et al.* 1981). Thus it can be expected that the concepts defined with respect to the descriptor system representation will be linked to those defined with respect to MFD. As an important result, we establish the following equivalence circle: a given MFD is completely irreducible if and only if any of its realizations with the order equal to the determinantal degree of the denominator of the MFD is controllable and reconstructible; further, the controllability and the reconstructibility of any realization is equivalent to the minimality *in order* of the realization. Based on this equivalence circle, a very interesting interpretation of the well-known McMillan degree of rational

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matrices can be provided. We shall show that when an MFD is completely irreducible, the determinantal degree of the denominator will be its McMillan degree. Consequently, the McMillan degree of an MFD only describes its completely irreducible part.

This paper is structured as follows. In § 2, a basic framework is set up for analysing rational matrices. A few examples are also given in order to motivate the reader. Section 3 is devoted to the development of a complete realization theory for right MFDs. A number of concepts from the descriptor system theory are reviewed briefly. The familiar equivalence between joint controllability and reconstructibility, complete irreducibility, and minimality *in order* is established in § 4. These equivalence relations lead naturally to the new interpretation of the McMillan degree. The final section contains remarks and conclusions.

2. Rational functions and rational matrices at infinity

In this section, we shall apply the ideas developed by Tan and Vandewalle (1987) to rational functions and matrices. It will be seen throughout the discussion that our theory provides a simpler and more direct way to identify the structures of rational matrices at infinity.

2.1. *Rational functions and complete irreducibility*

To be consistent with our axiomatic approach, we define rational function as follows.

Definition 1

A rational function $g(s)$ is defined to be $b(s)/a(s)$, where $a(s)$, $b(s)$ are polynomials satisfying

$$\deg a(s) = \deg b(s)$$

and $a(s) \neq 0$. The poles of $g(s)$—finite and infinite—are the zeros (finite and infinite) of $a(s)$; the zeros of $g(s)$—finite and infinite—are the zeros (finite and infinite) of $b(s)$.

Our definition of rational functions is basically guided by a simple fact concerning the conventional rational functions, i.e. when taking infinity into account, any rational function will have the same number of poles and zeros. This fact is reflected by the degree constraint in Definition 1. We shall provide an example to further clarify the idea.

Example 1

Let a rational function (in the conventional sense) be given by

$$g(s) = \frac{s^3 + s^2 + s + 1}{s + 1} \tag{1}$$

It obviously has three finite zeros and one finite pole. It also has two poles at infinity, which can be revealed by using the conformal mapping $s \to 1/\lambda$. In accordance with Definition 1, $g(s)$ should be written as

$$g(s) = \frac{s^3 + s^2 + s + 1}{0 \cdot s^3 + 0 \cdot s^2 + s + 1} \tag{2}$$

We can see that (2) shows more explicitly all the poles and zeros discussed above without applying the conformal mapping.

It should be noted from our definition that a polynomial is no longer considered to be a rational function. We should not confuse a polynomial

$$p(s) = a_0 s^n + \ldots + a_n$$

with the rational function

$$g(s) = \frac{(a_0 s^n + \ldots + a_n)}{(0 \cdot s^n + \ldots + 1)}$$

Now we proceed to introduce the concepts of irreducibility at infinity and complete irreducibility for rational functions.

If we are given an arbitrary rational function, it is likely that there is a common divisor at infinity between the numerator and the denominator polynomials. We are particularly interested in the case where no such divisors exist.

Definition 2

A rational function is said to be irreducible at infinity if its denominator and numerator polynomials are coprime at infinity.

A simple characterization of this concept follows directly from the results given by Tan and Vandewalle (1988).

Theorem 1

A rational function $g(s) = b(s)/a(s)$ is irreducible at infinity if and only if a_0, b_0 are not simultaneously zero, where a_0, b_0 are the leading coefficients of $a(s)$ and $b(s)$, respectively.

Taking into consideration the finite irreducibility we can state the following definition.

Definition 3

A rational function is said to be completely irreducible if its denominator and numerator polynomials are completely coprime.

A characterization of the complete irreducibility follows from that for the complete coprimeness stated by Tan and Vandewalle (1988). We shall not repeat it here.

2.2. *Rational matrices and complete irreducibility*

A rational matrix is simply a matrix whose entries are rational functions defined in the previous subsection. Let a rational matrix $G(s)$ $(m \times p)$ be given. We shall show that it can be represented alternatively by the matrix fraction $G(s) = N(s)D^{-1}(s)$, where $N(s)$, $D(s)$ are column-based polynomial matrices (see Tan and Vandewalle 1988 for definition). This representation will be called *right matrix fraction description* (MFD), and $D(s)$, $N(s)$ will be called the *denominator matrix* and the *numerator matrix* of $G(s)$, respectively.

To begin with, for the ith column $g_i(s)$, for $i = 1, ..., p$, of $G(s)$, we bring each element in that column on the same denominator, which is the least common multiple of the denominators of all the entries in that column. Then $G(s)$ can obviously be factorized as

$$G(s) = N(s)D^{-1}(s)$$

where $N(s)$ contains all the numerators of entries of $G(s)$, and

$$D(s) = \text{diag}\{d_1(s), d_2(s), ..., d_p(s)\}$$

in which each $d_i(s)$, for $i = 1, ..., p$, is the least multiple of all denominators of the entries in the ith column.

Conversely, if a rational matrix $G(s)$ is given by $G(s) = N(s)D^{-1}(s)$, where $N(s)$, $D(s)$ are column-based polynomial matrices, and moreover, let $d_i(s)$, $n_i(s)$ be ith columns of $D(s)$, $N(s)$, respectively, with $\deg d_i(s) = \deg n_i(s)$, then $G(s)$ is a rational matrix. Therefore, we can also define rational matrices in terms of MFD. In fact, we shall assume that all the rational matrices encountered in this paper are given in right MFD, moreover, the numerator and the denominator matrices are column-based polynomial matrices such that the ith column degree of the denominator equals the ith column degree of the numerator. For convenience, we shall call this type of MFD *column-based right MFD.*

A rational matrix defined in terms of MFD is more appropriate for introducing the concepts of poles and zeros of rational matrices.

Definition 4

Let a rational matrix $G(s)$ be given by

$$G(s) = N(s)D^{-1}(s)$$

Then its poles (finite and infinite) are defined to be the zeros of $\det D(s)$.

If $N(s)$ is a square matrix, then the zeros of $G(s)$ are defined to be those of $\det N(s, r)$. If it is a non-square matrix, then the definition of zeros will involve the concept of a generalized invariant polynomial. We shall, however, not go any further in this direction.

The above has only been concerned with the rational matrix represented in right MFD. In fact, similar statements and definitions can easily be stated for row-based left MFDs. Since the algebraic treatment for left MFDs is exactly the same as for the right ones, we shall omit the development for left MFDs for brevity. The reader should have no difficulties in producing the details for left MFDs when the treatment for right MFDs is available.

When a rational matrix $G(s)$ is represented in right MFD, say $N(s)D^{-1}(s)$, the numerator $N(s)$ and the denominator $D(s)$ may have common right factors at infinity. The absence of such factors is termed *irreducibility at infinity.* Moreover, the absence of any common divisor (finite and infinite) is called *complete irreducibility.* We now state the characterizations for each of the two concepts.

Theorem 2

Let an $m \times p$ rational matrix $G(s)$ be given by

$$G(s) = N(s)D^{-1}(s) \tag{3}$$

where $N(s)$, $D(s)$ are column-based polynomial matrices. Then $G(s)$ is irreducible at infinity if and only if

$$\text{rank} \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = p \tag{4}$$

in which D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively.

Theorem 3

Let $G(s)$ be given by (3). It is completely irreducible if and only if in addition to (4), the following is also true

$$\text{rank} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = p, \quad \text{for all finite } s$$

To illustrate our approach, let us look at the following two examples.

Example 2

Consider a simple rational matrix given by

$$G(s) = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{5}$$

It has been shown elsewhere (see, for example, Kailath 1980, Van Dooren 1979) to be a unimodular polynomial matrix, and has been found to have a pole, and a zero at infinity, by using the Smith–McMillan form.

Since the element s in $G(s)$ is considered as a rational function, we prefer to rewrite it more explicitly as

$$G(s) = \begin{bmatrix} 1 & \dfrac{s}{0 \cdot s + 1} \\ 0 & 1 \end{bmatrix} \tag{6}$$

By bringing the second column on its least common denominator, we have

$$G(s) = \begin{bmatrix} 1 & \dfrac{s}{0 \cdot s + 1} \\ 0 & \dfrac{0 \cdot s + 1}{0 \cdot s + 1} \end{bmatrix}$$

or in right MFD $G(s) = N(s)D^{-1}(s)$, where

$$N(s) = \begin{bmatrix} 1 & s \\ 0 & 0 \cdot s + 1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \cdot s + 1 \end{bmatrix}$$

This MFD is completely irreducible since

$$\text{rank} \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = 2$$

and

$$\operatorname{rank}\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0\cdot s+1 \\ 1 & s \\ 0 & 0\cdot s+1 \end{bmatrix} = 2, \quad \text{for all finite } s$$

The poles of $G(s)$ can be found from $\det D(s, r) = r$, the zeros of $G(s)$ from $\det N(s, r) = r$, which means that $G(s)$ only has one pole and one zero at infinity. The result agrees with the conventional analysis. However, the reader can see that our analysis is much simpler compared to the conventional one, which may require the computation of the inverse.

In fact, we can bring on the least common denominator of all the entries of (6), i.e.

$$G(s) = \begin{bmatrix} \dfrac{0\cdot s+1}{0\cdot s+1} & \dfrac{s}{0\cdot s+1} \\ 0 & \dfrac{0\cdot s+1}{0\cdot s+1} \end{bmatrix}$$

Now the numerator and the denominator of the MFD can be seen as

$$N(s) = \begin{bmatrix} 0\cdot s+1 & s \\ 0 & 0\cdot s+1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} 0\cdot s+1 & 0 \\ 0 & 0\cdot s+1 \end{bmatrix}$$

The poles of the MFD can be found from $\det D(s, r) = r^2$ and the zeros from $\det N(s, r) = r^2$, which would suggest two poles and two zeros at infinity. However, it can easily be checked that this MFD is not completely irreducible. Actually an infinite pole will be cancelled by an infinite zero. After the cancellation, the MFD will be completely irreducible. The number of poles and zeros thereby agrees with the preceding discussion.

The foregoing analysis seems to suggest that the McMillan degree of a rational matrix only reflects the irreducible part of the matrices. This is indeed true in general as we shall show in § 4.

Example 3

It is well recognized that for a given pencil, its infinite structure seen via the infinite Smith–McMillan form is quite different from the one seen via the infinite elementary divisors (Van Dooren 1979, Vardulakis and Karcanias 1983). Although the detailed connections between the two have been worked out, it is still not clear as to why these two descriptions of the infinity are essentially different.

We can, however, provide an easy answer to the question in the light of our theory. Note first that infinite elementary divisors can only apply to polynomial matrices, not rational matrices, while the Smith–McMillan form at infinity can only apply to rational matrices not polynomial matrices.

Let a pencil $Es - A$ be given. If it is regarded as a polynomial matrix, then its elementary divisors at infinity can be found via the homogeneous form $Es - Ar$. However, if it is regarded as a rational matrix, each of its elements is considered to

have the denominator $0 \cdot s + 1$, then we have to first rewrite it as

$$(Es - A)D^{-1}(s) \tag{7}$$

where $D(s) = I \cdot (0 \cdot s + 1)$, or $D(s, r) = I \cdot r$. We can see that all infinite elementary divisors are modified by the factor r^{-1}. The precise poles and zeros can be obtained by first cancelling the common factors in (7) to make it irreducible. We can see that the result coincides with those given by Van Dooren (1979) and Vardulakis and Karcanias (1983).

3. Rational matrices and descriptor system realization

3.1. *Basic definitions concerning descriptor systems*

The standard form for descriptor systems is given in the following system matrix form (Rosenbrock 1974)

$$\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{8}$$

where, as usual, E, A are constant matrices of dimension $n \times n$; B, C and D are constant matrices of dimensions $n \times p$, $m \times n$ and $p \times m$, respectively. Moreover, n and $f = \text{rank } E$ are defined as the dimension and the order of the system, respectively.

Two descriptor systems with, perhaps, different dimensions are defined to be equivalent if one can be transformed to the other by any combination of the following three operations.

(*a*) *Left and right multiplication*

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}\begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_1 s - A_1 & B_1 \\ -C_1 & D_1 \end{bmatrix} \tag{9}$$

where $QE = 0$ and $ER = 0$.

(*b*) *Trivial augmentation*

$$\text{from } \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{10}$$

$$\text{to } \begin{bmatrix} Es - A & 0 & B \\ 0 & I & 0 \\ -C & 0 & D \end{bmatrix} \tag{11}$$

(*c*) *Trivial deflation*

namely from (11) to (10).

The operations (*a*)–(*c*) were originally defined by Verghese *et al.* (1981) and are termed the *operations of strong equivalence transformation*. Moreover, the two systems are said to be *strongly equivalent* (*str. eq.*). It can be shown (see Verghese *et al.* 1981) that two strongly equivalent systems will have the same transfer matrix.

The importance of the descriptor system is that it can be used to *realize* any rational matrices (proper or not) in the sense defined below.

First apply a left and right multiplication (9) such that the non-singular matrix M row-compresses E, $Q = R = 0$, and $N = I$. The system matrix (8) is transformed into

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} E_\varepsilon s - A_\varepsilon & B_\varepsilon \\ -A_\pi & B_\pi \\ -C & D \end{bmatrix} \tag{12}$$

where E_ε is of full row rank. Then a given MFD $G(s) = N(s)D^{-1}(s)$ is said to be realized by (8) if

$$N(s, r)D^{-1}(s, r) = C\begin{bmatrix} E_\varepsilon s - A_\varepsilon r \\ -A_\pi \end{bmatrix}^{-1}\begin{bmatrix} B_\varepsilon \\ B_\pi \end{bmatrix} + D \tag{13}$$

The above definition is based on the row-compression. It is, however, possible to present the definition with respect to the column-compression.

3.2. *Special realization*

In this subsection, we show that when the denominator and the numerator matrices of an MFD are simply matrix polynomials, its descriptor system realization can be obtained straightforwardly.

Let an $m \times p$ rational matrix $G(s)$ be given in MFD, i.e.

$$G(s) = N(s)D^{-1}(s) \tag{14}$$

where $N(s)$, $D(s)$ are matrix polynomials with the following homogeneous forms

$$\begin{aligned} N(s, r) &= N_0 s^\alpha + N_1 s^{\alpha-1} r + \ldots + N_\alpha r^\alpha \\ D(s, r) &= D_0 s^\alpha + D_1 s^{\alpha-1} r + \ldots + D_\alpha r^\alpha \end{aligned} \tag{15}$$

We can state the following realization theorem.

Theorem 4

Any matrix polynomial MFD (14) can be realized in descriptor system (8), where

$$\begin{aligned} E &= \text{block diag}\,\{\overbrace{0, I_p, \ldots, I_p}^{\alpha+1}\} \\ A &= \begin{bmatrix} -D_0 & -D_1 & \ldots & -D_{\alpha-1} & -D_\alpha \\ I_p & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I_p & 0 \end{bmatrix} \\ B^{\mathrm{T}} &= [\overbrace{I_p \quad 0 \quad \ldots \quad 0}^{\alpha+1}] \\ C &= [N_0 \quad N_1 \quad \ldots \quad N_{\alpha-1} \quad N_\alpha] \\ D &= 0 \end{aligned} \tag{16}$$

The realization (16) will be called the *block controllable realization* for reasons given later.

Proof

First we compute

$$\Delta(s,r)=\begin{bmatrix} D_0 & D_1 & \dots & D_{\alpha-1} & D_\alpha \\ -I_p r & I_p s & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_p r & -I_p s \end{bmatrix}^{-1}\begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which turns out to be

$$\Delta(s,r)=\begin{bmatrix} I_p s^\alpha D^{-1}(s,r) \\ I_p s^{\alpha-1} r D^{-1}(s,r) \\ \vdots \\ I_p r^\alpha D^{-1}(s,r) \end{bmatrix}=\begin{bmatrix} I_p s^\alpha \\ I_p s^{\alpha-1} r \\ \vdots \\ I_p r^\alpha \end{bmatrix} D^{-1}(s,r)$$

therefore

$$C\Delta(s,r)=N(s,r)D^{-1}(s,r)=G(s,r)$$

By the definition of realization, we conclude that (16) indeed realizes the given MFD. □

Before presenting some interesting properties of the realization (16), we shall first introduce two important concepts.

We start with the matrix

$$[Es-A \quad B] \tag{17}$$

By *str. eq.* operations, (17) can be brought to

$$\begin{bmatrix} E_\varepsilon s-A_\varepsilon & B_\varepsilon \\ A_\pi & B_\pi \end{bmatrix} \tag{18}$$

with E_ε of full row rank. Similarly, the matrix

$$\begin{bmatrix} Es-A \\ -C \end{bmatrix} \tag{19}$$

can be brought, by *str. eq.* operations, to

$$\begin{bmatrix} E_\eta s-A_\eta & A_\xi \\ -C_\eta & -C_\xi \end{bmatrix} \tag{20}$$

with E_η of full column rank. We now state the following definition.

Definition 5

Descriptor system (8) is said to be finitely controllable if

$$\operatorname{rank}[Es-A \quad B]=n \tag{21}$$

for all finite s; and controllable at infinity if

$$\operatorname{rank}\begin{bmatrix} E_\varepsilon & 0 \\ A_\pi & B_\pi \end{bmatrix} = n \tag{22}$$

where the block matrix in (22) is built with the corresponding block elements in (18). It is said to be controllable if it is both finitely controllable and controllable at infinity.

Equation (8) is said to be finitely reconstructible if

$$\operatorname{rank}\begin{bmatrix} Es - A \\ -C \end{bmatrix} = n \tag{23}$$

for all finite s; and reconstructible at infinity if

$$\operatorname{rank}\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} = n \tag{24}$$

where the block matrix in (24) is built with the corresponding block elements in (20). It is said to be reconstructible if it is both finitely reconstructible and reconstructible at infinity.

The dynamical implications of the controllability and reconstructibility have been explained by Verghese *et al.* (1981). The present paper will only be concentrating on their algebraic aspects. We now proceed to discuss the properties of the realization (16).

Theorem 5

The realization (16) has the following properties.

(*a*) The order of the realization is equal to the determinantal degree of the denominator of the transfer matrix, i.e.

$$\operatorname{rank} E = \deg \det D(s, r)$$

(*b*) It is controllable.

(*c*) $\det D(s, r) = \det \Theta(s, r)$ where

$$\Theta(s, r) = \begin{bmatrix} D_0 & D_1 & \dots & D_{\alpha-1} & D_\alpha \\ -I_p r & I_p s & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -I_p r & I_p s \end{bmatrix} \tag{25}$$

Proof

The proofs for (*a*) and (*b*) are trivial and thus omitted. To prove (*c*), denote the ith block column of the matrix $\Theta(s, r)$ by F_i, for $i = 1, \dots, \alpha + 1$. Apply a column transformation $K(r)$ such that

$$\Theta(s, r)K = [\tilde{F}_1 \quad \tilde{F}_2 \quad \dots \quad \tilde{F}_{\alpha+1}] \tag{26}$$

where $\tilde{F}_i = F_i r^{i-1} + \tilde{F}_{i-1} s$, for $i = 2, \dots, \alpha + 1$, and $\tilde{F}_1 = F_1$. It is easy to see that

$$\det K = \det (I_p r) \dots \det (I_p r^\alpha)$$

From the structure of $\Theta(s, r) \cdot K$, it follows that

$$\det(\Theta(s, r) \cdot K) = \det D(s, r) \cdot \det(I_p) \ldots \det(I_p r^a)$$

Therefore $\det \Theta(s, r) = \det D(s, r)$. □

An algebraically dual procedure can apparently be pursued for the row-based left MFD. We shall, however, not go into any detail for the sake of brevity.

3.3. *Realization for column-based MFD*

It is often the case that the numerator matrix and the denominator matrix of a given right MFD are not matrix polynomials, but column-based polynomial matrices instead. Moreover, even if they are matrix polynomials, there might exist a common right divisor between them. The authors have shown (Tan and Vandewalle 1988) that after the cancellation, the matrix polynomials of the denominator and the numerator reduce to column-based polynomial matrices. Therefore, it is of great interest to consider the realization for column-based right MFDs.

3.3.1. *Construction of realization.* Let a column-based right MFD be given and k_i, for $i = 1, \ldots, p$, be the ith column degree of $N(s)$ $(m \times p)$ and $D(s)$ $(p \times p)$.

As the first step of the construction, define the matrix

$$\Gamma^{\mathrm{T}}(s, r) = \{[s^{k_i} \quad s^{k_i - 1} r \quad \ldots \quad s r^{k_i - 1} \quad r^{k_i}], \quad i = 1, \ldots, p\} \tag{27}$$

and write $D(s, r)$ and $N(s, r)$ in the form

$$\begin{aligned} D(s, r) &= D_c \Gamma(s, r) \\ N(s, r) &= N_c \Gamma(s, r) \end{aligned} \tag{28}$$

where D_c and N_c are appropriate constant matrices.

Secondly, construct the *core realization* of the form

$$\left.\begin{aligned} E_c s - A_c &= \text{block diag}\left\{\left\{\begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ -1 & s & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & s \end{bmatrix}, \begin{matrix}(k_i+1) \times (k_i+1) \\ i = 1, \ldots, p\end{matrix}\right\}\right\} \\ B_c^{\mathrm{T}} &= \text{block diag}\,\{[1 \quad 0 \quad \ldots \quad 0], 1 \times (k_i + 1), i = 1, \ldots, p\} \\ C_c &= I_n, \quad n = \sum_{i=1}^{p} k_i + p \end{aligned}\right\} \tag{29}$$

Finally, the desired realization follows from (28) and (29)

$$E = E_c, \quad A = A_c - B D_c, \quad B = B_c, \quad C = N_c \cdot C_c = N_c \tag{30}$$

3.3.2. *Proof and properties of realization.* We first prove that (30) indeed realizes the given right MFD. Then some interesting properties of the realization are shown.

Theorem 6

The descriptor system in (30) realizes the given column-based MFD.

Proof

It is easily checked that

$$(Es - A_c r + BD_c)\Gamma(s, r) = BD_c\Gamma(s, r) = BD(s, r)$$

or equivalently

$$(Es - A_c r + BD_c)^{-1} B = \Gamma(s, r) D^{-1}(s, r) \tag{31}$$

(The non-singularity of $Es - A_c r + BD_c$ will be proved later.) By premultiplying (31) with $C\ (= N_c)$ and by observing that $N(s, r) = N_c\Gamma(s, r)$, the theorem is proved. □

Theorem 7

The realization (30) has the following properties:

(*a*) $\det(Es - A_c r + BD_c) = \det D(s, r)$

(*b*) it is controllable.

These two properties can readily be checked by direct computation. We shall omit the detail. Observe that the second property guarantees the non-singularity of the matrix $Es - A_c r + BD_c$, which is needed in the proof of Theorem 6. Moreover, this property also implies that the order of the realization is equal to the determinantal degree of the denominator of the transfer matrix.

To illustrate, consider the following example.

Example 4

The rational matrix is given by

$$\begin{bmatrix} s^2 + 0 \cdot s + 0 & s + 1 \\ 0 \cdot s^2 + s + 0 & s + 0 \end{bmatrix} \begin{bmatrix} s^2 + s + 1 & 0 \cdot s + 1 \\ 0 \cdot s^2 + s + 0 & 0 \cdot s + 1 \end{bmatrix}^{-1}$$

To realize it, we first write $D(s, r)$, $N(s, r)$ as follows:

$$D(s, r) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ sr & 0 \\ r^2 & 0 \\ 0 & s \\ 0 & r \end{bmatrix}$$

$$N(s, r) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ sr & 0 \\ r^2 & 0 \\ 0 & s \\ 0 & r \end{bmatrix}$$

The core realization is constructed as

$$E_c s - A_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, the final realization turns out to be

$$Es - A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ -1 & s & 0 & 0 & 0 \\ 0 & -1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & s \end{bmatrix}$$

$$B = B_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

4. Relationships between minimality in order, complete irreducibility, joint controllability and reconstructibility

An equivalence circle will be proven in this section for joint controllability and reconstructibility, minimality in order, and complete irreducibility. This new equivalence circle thus connects the concept of complete irreducibility defined in § 2 to the well-known McMillan degree.

4.1. *Minimality in order versus joint controllability and reconstructibility*

The relation between minimality in order and joint controllability and reconstructibility has been established by Verghese *et al.* (1981). In this subsection, we shall sketch the main idea for completeness of our discussion.

Let the system matrix of a descriptor system be given by

$$\mathscr{P}(s) = \begin{bmatrix} Es - A & B \\ -C & D \end{bmatrix} \tag{32}$$

It has been shown by Verghese *et al.* (1981) that a strong equivalence transformation can bring (32) to the following fourfold decomposition:

$$\begin{bmatrix} E_{\bar{u}c}s - A_{\bar{u}c} & X & X & X & B_{\bar{u}c} \\ 0 & E_{\bar{u}\bar{c}}s - A_{\bar{u}\bar{c}} & 0 & X & 0 \\ 0 & 0 & E_{uc}s - A_{uc} & 0 & B_{uc} \\ 0 & 0 & 0 & E_{u\bar{c}}s - A_{u\bar{c}} & 0 \\ 0 & 0 & -C_{uc} & -C_{u\bar{c}} & D \end{bmatrix} \tag{33}$$

where the X denote some constant matrices; and the subscripts u, $\bar{u}$, c, $\bar{c}$ denote reconstructible, unreconstructible, controllable and uncontrollable, respectively. Consequently, the order f of the system is given by

$$f = \text{rank } E_{\bar{u}c} + \text{rank } E_{\bar{u}\bar{c}} + \text{rank } E_{uc} + \text{rank } E_{u\bar{c}} \tag{34}$$

It is therefore straightforward to see from (34) that f attains its minimal value if and only if the system is jointly controllable and reconstructible.

4.2. *Complete irreducibility versus joint controllability and reconstructibility*

We shall be making use of the realization (30) as a primary vehicle to derive the relationship. First we show that for this particular realization the desired relationship holds.

Theorem 8

The realization (30) is reconstructible if and only if the MFD $N(s)D^{-1}(s)$ is completely irreducible.

Proof

As usual, only a proof for the equivalence between the reconstructibility at infinity of the realization and the irreducibility at infinity of an MFD is given. The finite part is well known and thus omitted.

From Definition 5, the realization (30) is reconstructible at infinity if and only if the matrix

$$\begin{bmatrix} E_\eta & A_\xi \\ 0 & -C_\xi \end{bmatrix} \tag{35}$$

has full column rank, where the block elements in (35) are defined in (24). From (30), it is easily seen that (35) can be transformed, by elementary transformations, to the following matrix

$$\begin{bmatrix} D_{hc} & 0 \\ 0 & I \\ N_{hc} & 0 \end{bmatrix}$$

where D_{hc}, N_{hc} are the leading coefficient matrices of $D(s)$ and $N(s)$, respectively. The above matrix is of full column rank if and only if $D(s)$ and $N(s)$ are irreducible at infinity. □

With Theorem 8 in hand, the rest of the development follows quite straightforwardly.

Theorem 9

If a column-based MFD has one controllable and reconstructible fth-order realization, then all its fth-order realization must also be controllable and reconstructible.

Proof

Otherwise, by using the fourfold decomposition (33), this would imply that f is not the minimal order, which clearly contradicts the conclusion in the preceding subsection. □

We are now in a position to state and prove the main theorem of this subsection.

Theorem 10

A column-based right MFD is completely irreducible if and only if all of its nth-order realizations are controllable and reconstructible, where n is the determinantal degree of the denominator $D(s)$, i.e. $n = \deg \det D(s, r)$.

Proof

In view of Theorem 9, it suffices to show that the theorem holds for a special nth order realization. However, Theorem 8 indicates that the realization (30) can serve this purpose. □

It has been shown by Verghese *et al.* 1981 that the order of a descriptor system will equal the McMillan degree of the underlying transfer matrix if and only if the descriptor system is controllable and reconstructible. Combining this fact with Theorem 10 we obtain the following important result.

Theorem 11

A column-based right MFD is completely irreducible if and only if the determinantal degree of its denominator is equal to the McMillan degree of the MFD.

This theorem is very significant since it provides an alternative, yet the simplest interpretation for the well-known McMillan degree of rational matrices, viz., the McMillan degree only describes the completely irreducible part of rational matrices. It is now clear that when a rational matrix is factorized in, say an MFD, some redundancy (finite and infinite) may be introduced. We can then perform the cancellation. As a consequence, the determinantal degree of the denominator of the MFD will decrease. As soon as the MFD becomes completely irreducible the determinantal degree attains the minimal value which is precisely the McMillan degree of the MFD.

5. Conclusions

In this paper, a theory for dealing with rational matrices at infinity is presented. This theory is intuitively appealing in the sense that the algebraic structure of rational matrices at infinity is tackled in exactly the same way as the finite structure. It has also clearly been seen that Verghese's framework in handling descriptor systems gears nicely with our theory of rational matrices.

It is believed that our theory provides a simpler alternative to the existing theories, and may be conveniently applied to deal with other related problems.

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